A New Landscape of Gravitational Instantons

Bernardo Araneda

University of Edinburgh

Penrose at 95 and Twistors at 60

Oxford, Sep 2026

Outline

I. Black holes & instantons

II. Kähler geometry

III. Infinite-dimensional structures

Twistor theory

A. Self-dual	3 parallel complex structures	✓	
B. Hermitian	1 integrable complex structure	?	(cf. Tim's talk)
C. Toric	no (obvious) complex structure	?	(cf. Maciej's talk)

This talk is about B. and C.

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- Gravitational Instantons: globally smooth, non-compact & complete, finite L^2-energy (AF, ALF, ALE,...), non-hyperkähler
- Euclidean BH uniqueness? [Gibbons 1979, Lapedes 1980, Yau 1982]

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- **2026**: $n = 4$ case explicitly constructed by Teo

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This talk

- Construction of this instanton
- Bigger picture
- Connection with twistor theory (at the very end)

II. Kähler geometry

Solution-generating techniques for the Einstein equations

- Based on isometries $\rightsquigarrow$ harmonic maps
- Based on complex structures $\rightsquigarrow$ Kähler geometry

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- Lorentzian: $SU(2,1)/S(U(2)\times U(1))$ [Kinnersely], [Breitenlohner, Maison, Gibbons]
- "Hidden symmetries": isometries of g_X

$$SL(3,\mathbb{R}) \quad \Rightarrow \quad \text{new solutions}$$

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- Explicit description [LeBrun '91], [Tod '20], [BA '23]

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- **But: given u, how do we choose a monopole W?**

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$$\begin{tikzcd}
& \text{Conformal Kähler} \arrow[loop above] \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
\text{Hyperkähler} \arrow[loop below] \arrow[rr, leftrightarrow] & & \text{Scalar-flat Kähler} \arrow[loop below]
\end{tikzcd}$$

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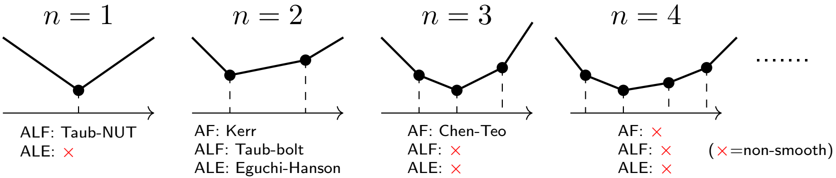

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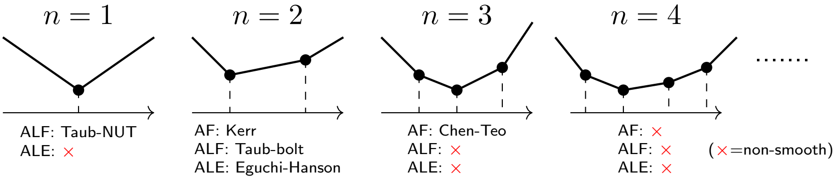

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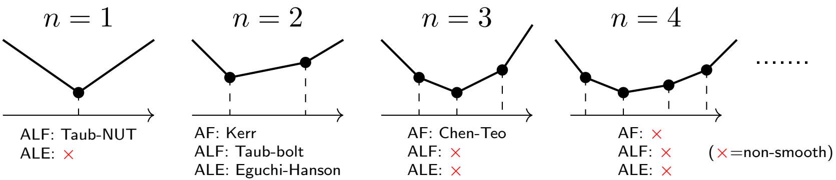

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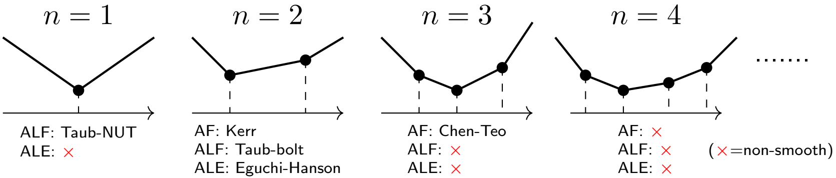

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- $n > 3$? **open**

Extremal gravitational instantons

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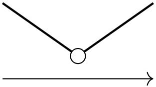

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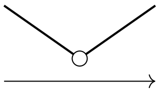
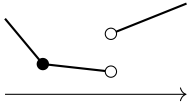


- Instanton is α-surface in twistor space of (X, g_X)
- General case? [BA & A. Colling, wip]



III. Infinite-dimensional structures

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- $\mathfrak{t}^2 = span(K_1, K_2) \rightsquigarrow$ maybe apply $SL(3,\mathbb{R})$ wrt to a different KVF?
- But can take $c_1K_1 + c_2K_2$, then change c_1, c_2, then repeat indefinitely...

Integrability

- Go gack to $X = SL(3)/GL(2)$, with

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  with $\rho = \det g(K_i, K_j)$ (and $dz = \star_2 d\rho$, $x = (\rho, z)$)
- This is an integrable system! Lax family of flat connections

$$d\mathcal{A}(x,t) - \mathcal{A}(x,t) \wedge \mathcal{A}(x,t) = 0, \qquad \mathcal{A}|_{t=0} = A$$

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- But: need to solve Riemann-Hilbert problem

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- Avoids RH factorisation, but still difficult to work with...

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- Avoids RH factorisation, but still difficult to work with...
- But we have Geroch $\widehat{SL(3)}$ $\Leftrightarrow$ Belinski-Zakharov $\Leftrightarrow$ twistors?

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Remarks:

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- Reconstruct solution from P via $z \to \gamma$ and holomorphic splitting
- This characterises the P's that give Hermitian structures
- Prove Conjecture that solutions are always solitons? (cf. [Chen 2026])

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Thank you!

Happy Birthday Roger!