Minicourse on BV functions

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Literature

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σ-algebras and positive measures

Let X be a topological space.

- A collection $\mathcal{E}$ of subsets of X is a σ-algebra if $\emptyset \in \mathcal{E}$, $X \setminus E \in \mathcal{E}$ whenever $E \in \mathcal{E}$, and for any sequence $(E_h) \subset \mathcal{E}$, we have $\bigcup_{h \in \mathbb{N}} E_h \in \mathcal{E}$.
- We say that $\mu : \mathcal{E} \to [0, \infty]$ is a positive measure if $\mu(\emptyset) = 0$ and for any sequence (E_h) of pairwise disjoint elements of $\mathcal{E}$,

$$\mu\left(\bigcup_{h \in \mathbb{N}} E_h\right) = \sum_{h \in \mathbb{N}} \mu(E_h).$$

- We say that $M \subset X$ is μ-negligible if there exists $E \in \mathcal{E}$ such that $M \subset E$ and $\mu(E) = 0$. The expression "almost everywhere", or "a.e.", means outside a negligible set. The measure μ extends to the collection of μ-measurable sets, i.e. those that can be presented as $E \cup M$ with $E \in \mathcal{E}$ and M μ-negligible.

Vector measures

- Let $N \in \mathbb{N}$. We say that $\mu\colon \mathcal{E} \to \mathbb{R}^N$ is a vector measure if $\mu(\emptyset) = 0$ and for any sequence (E_h) of pairwise disjoint elements of $\mathcal{E}$
$$\mu\left(\bigcup_{h\in\mathbb{N}} E_h\right) = \sum_{h\in\mathbb{N}} \mu(E_h).$$
- If μ is a vector measure on $(X, \mathcal{E})$, for any given $E \in \mathcal{E}$ we define the *total variation measure* $|\mu|(E)$ as
$$\sup\left\{\sum_{h\in\mathbb{N}} |\mu(E_h)| :\ E_h \in \mathcal{E} \text{ pairwise disjoint, } E = \bigcup_{h\in\mathbb{N}} E_h\right\}.$$

  We can show that $|\mu|$ is then a finite positive measure on $(X, \mathcal{E})$, that is, $|\mu|(X) < \infty$.

Borel and Radon measures

- We denote by $\mathcal{B}(X)$ the σ-algebra of Borel subsets of X, i.e. the smallest σ-algebra containing the open subsets of X.
- A positive measure on $(X, \mathcal{B}(X))$ is called a Borel measure. If it is finite on compact sets, it is called a positive Radon measure.
- A vector Radon measure is an $\mathbb{R}^N$-valued set function that is a vector measure on $(K, \mathcal{B}(K))$ for every compact set $K \subset X$. We say that it is a finite Radon measure if it is a vector measure on $(X, \mathcal{B}(X))$.

Lebesgue and Hausdorff measures

We will denote by $\mathcal{L}^n$ the n-dimensional Lebesgue measure in $\mathbb{R}^n$. Sometimes we write $|A|$ instead of $\mathcal{L}^n(A)$ for $A \subset \mathbb{R}^n$.

We denote by ω_k the volume of the unit ball in $\mathbb{R}^k$.

Definition

Let $k \in [0, \infty)$ and let $A \subset \mathbb{R}^n$. The k-dimensional Hausdorff measure of A is given by

$$\mathcal{H}^k(A) := \lim_{\delta \to 0} \mathcal{H}^k_\delta(A),$$

where for any $0 < \delta \leq \infty$,

$$\mathcal{H}^k_\delta(A) := \frac{\omega_k}{2^k} \inf \left\{ \sum_{i \in \mathbb{N}} [\mathrm{diam}(E_i)]^k \, : \, \mathrm{diam}(E_i) < \delta, \, A \subset \bigcup_{i \in \mathbb{N}} E_i \right\}$$

with the convention $\mathrm{diam}(\emptyset) = 0$.

Besicovitch differentiation theorem

Theorem

Let μ be a positive Radon measure in an open set Ω, and let ν be an $\mathbb{R}^N$-valued Radon measure in Ω. Then for μ-a.e. $x \in \Omega$ the limit

$$f(x) := \lim_{r \to 0} \frac{\nu(B(x,r))}{\mu(B(x,r))}$$

exists in $\mathbb{R}^N$ and ν can be presented by the Lebesgue-Radon-Nikodym decomposition $\nu = f\,\mu + \nu^s$, where $\nu^s = \nu \llcorner E$ for some $E \subset \Omega$ with $\mu(E) = 0$.

Definition of BV functions

The symbol Ω will always denote an open set in $\mathbb{R}^n$.

Definition

Let $u \in L^1(\Omega)$. We say that u is a function of bounded variation in Ω if the distributional derivative of u is representable by a finite Radon measure in Ω, i.e.

$$\int_\Omega u \frac{\partial \psi}{\partial x_i}\, dx = -\int_\Omega \psi \, dD_i u \qquad \forall \psi \in C_c^\infty(\Omega), \quad i = 1, \dots, n$$

for some $\mathbb{R}^n$-valued Radon measure $Du = (D_1 u, \dots, D_n u)$ in Ω. The vector space of all functions of bounded variation is denoted by $\mathrm{BV}(\Omega)$.

We can always write $Du = \sigma |Du|$, where $|Du|$ is a positive Radon measure and $\sigma = (\sigma_1, \dots, \sigma_n)$ with $|\sigma(x)| = 1$ for $|Du|$-a.e. $x \in \Omega$.

Examples of BV functions

- $W^{1,1}(\Omega) \subset \mathrm{BV}(\Omega)$, since for $u \in W^{1,1}(\Omega)$ we have $Du = \nabla u \, \mathcal{L}^n$.
- On the other hand, for the Heaviside function $\chi_{(0,\infty)} \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R})$ we have $Du = \delta_0$, and $u \notin W^{1,1}_{\mathrm{loc}}(\mathbb{R})$.
- We say that $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ if $u \in \mathrm{BV}(\Omega')$ for every $\Omega' \Subset \Omega$, i.e. every open Ω' with $\overline{\Omega'}$ compact and contained in Ω.

Mollification, part I

Let $\rho \in C_c^\infty(\mathbb{R}^n)$ with $\rho(x) \geq 0$ and $\rho(-x) = \rho(x)$ for all $x \in \mathbb{R}^n$, $\operatorname{supp} \rho \subset B(0,1)$, and

$$\int_{\mathbb{R}^n} \rho(x)\, dx = 1.$$

Choose $\varepsilon > 0$. Let $\rho_\varepsilon(x) := \varepsilon^{-n} \rho(x/\varepsilon)$, and

$$\Omega_\varepsilon := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\}.$$

Then for any $u \in L^1(\Omega)$, we define for any $x \in \Omega_\varepsilon$

$$u * \rho_\varepsilon(x) := \int_\Omega u(y)\rho_\varepsilon(x-y)\, dy = \varepsilon^{-n} \int_\Omega u(y) \rho\left(\frac{x-y}{\varepsilon}\right)\, dy.$$

Mollification, part II

Similarly, for any vector Radon measure $\mu = (\mu_1, \ldots, \mu_N)$ on Ω, we define

$$\mu * \rho_\varepsilon(x) = \int_\Omega \rho_\varepsilon(x-y)\, d\mu(y), \qquad x \in \Omega_\varepsilon.$$

We can show that $\mu * \rho_\varepsilon \in C^\infty(\Omega_\varepsilon)$ and

$$\nabla(\mu * \rho_\varepsilon) = \mu * \nabla\rho_\varepsilon.$$

If $v \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$, then $\nabla(v * \rho_\varepsilon) = \nabla v * \rho_\varepsilon$.
Also, given any $v \in L^1(\Omega)$ and a vector Radon measure μ on Ω, from Fubini's theorem it follows easily that

$$\int_\Omega (\mu * \rho_\varepsilon) v \, d\mathcal{L}^n = \int_\Omega v * \rho_\varepsilon \, d\mu$$

if either μ is concentrated in Ω_ε or $v = 0$ $\mathcal{L}^n$-a.e. outside Ω_ε.

Mollification of BV functions

- If $u \in \mathrm{BV}(\Omega)$, we have

$$\nabla(u * \rho_\varepsilon) = Du * \rho_\varepsilon \qquad \text{in } \Omega_\varepsilon.$$

  To see this, let $\psi \in C_c^\infty(\Omega)$ and $\varepsilon \in (0, \mathrm{dist}(\mathrm{supp}\,\psi, \partial\Omega))$. Then

$$\int_\Omega (u * \rho_\varepsilon)\nabla\psi \, d\mathcal{L}^n = \int_\Omega u(\rho_\varepsilon * \nabla\psi)\, d\mathcal{L}^n = \int_\Omega u\nabla(\rho_\varepsilon * \psi)\, d\mathcal{L}^n$$
$$= -\int_\Omega \rho_\varepsilon * \psi \, dDu = -\int_\Omega \psi \, Du * \rho_\varepsilon \, d\mathcal{L}^n.$$

- If $u \in \mathrm{BV}(\mathbb{R}^n)$ and $Du = 0$, u is constant, which we can see as follows. For any ε, $u * \rho_\varepsilon \in C^\infty(\mathbb{R}^n)$ and $\nabla(u * \rho_\varepsilon) = Du * \rho_\varepsilon = 0$. Thus $u * \rho_\varepsilon$ is constant for every $\varepsilon > 0$, and since $u * \rho_\varepsilon \to u$ in $L^1(\mathbb{R}^n)$ as $\varepsilon \to 0$, we must have that u is constant.

Lipschitz test functions

Let $u \in \mathrm{BV}(\Omega)$ and $\psi \in \mathrm{Lip}_c(\Omega)$. Then for small enough $\varepsilon > 0$ we have $\psi * \rho_\varepsilon \in C_c^\infty(\Omega)$, and thus

$$\int_\Omega u \, \frac{\partial(\psi * \rho_\varepsilon)}{\partial x_i} \, d\mathcal{L}^n = -\int_\Omega \psi * \rho_\varepsilon \, dD_i u, \quad i = 1, \ldots, n.$$

As $\varepsilon \to 0$, we have $\psi * \rho_\varepsilon \to \psi$ uniformly and

$$\frac{\partial(\psi * \rho_\varepsilon)}{\partial x_i} = \frac{\partial \psi}{\partial x_i} * \rho_\varepsilon \to \frac{\partial \psi}{\partial x_i}$$

almost everywhere, so by the Lebesgue dominated convergence theorem we get

$$\int_\Omega u \, \frac{\partial \psi}{\partial x_i} \, d\mathcal{L}^n = -\int_\Omega \psi \, dD_i u, \quad i = 1, \ldots, n.$$

That is, we can also use Lipschitz functions as test functions in the definition of BV.

A Leibniz rule

Lemma

If $\phi \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ *and* $u \in \mathrm{BV}_{loc}(\Omega)$, *we have* $u\phi \in \mathrm{BV}_{loc}(\Omega)$ *with* $D(u\phi) = \phi\, Du + u\nabla\phi\,\mathcal{L}^n$.

Proof.

Clearly $u\phi \in L^1_{\mathrm{loc}}(\Omega)$. We have for any $\psi \in C_c^\infty(\Omega)$ and $i = 1, \ldots, n$

$$\begin{aligned}\int_\Omega u\,\phi\,\frac{\partial\psi}{\partial x_i}\,dx &= \int_\Omega u\,\frac{\partial(\phi\psi)}{\partial x_i}\,dx - \int_\Omega u\,\frac{\partial\phi}{\partial x_i}\,\psi\,dx \\ &= -\int_\Omega \phi\,\psi\,dD_i u - \int_\Omega u\,\frac{\partial\phi}{\partial x_i}\,\psi\,dx,\end{aligned}$$

since $\phi\psi \in \mathrm{Lip}_c(\Omega)$. □

The variation, part I

Definition

Let $u \in L^1_{\mathrm{loc}}(\Omega)$. We define the *variation* of u in Ω by

$$V(u,\Omega) := \sup\left\{ \int_\Omega u \operatorname{div}\psi \, d\mathcal{L}^n : \ \psi \in [C^1_c(\Omega)]^n, \ |\psi| \le 1 \right\}.$$

Theorem

Let $u \in L^1(\Omega)$. Then $u \in \mathrm{BV}(\Omega)$ if and only if $V(u,\Omega) < \infty$. In addition, $V(u,\Omega) = |Du|(\Omega)$.

Proof. Let $u \in \mathrm{BV}(\Omega)$. Then for any $\psi \in [C^1_c(\Omega)]^n$ with $|\psi| \le 1$,

$$\int_\Omega u \operatorname{div}\psi \, d\mathcal{L}^n = -\sum_{i=1}^n \int_\Omega \psi_i \, dD_i u = -\sum_{i=1}^n \int_\Omega \psi_i \sigma_i \, d|Du|$$

with $|\sigma| = 1$ $|Du|$-almost everywhere, so that $V(u,\Omega) \le |Du|(\Omega)$.

The variation, part II

Assume then that $V(u,\Omega) < \infty$. By homogeneity we have

$$\left| \int_\Omega u \operatorname{div} \psi \, d\mathcal{L}^n \right| \le V(u,\Omega) \|\psi\|_{L^\infty(\Omega)} \qquad \forall \psi \in C_c^1(\Omega).$$

Since $C_c^1(\Omega)$ is dense in $C_c(\Omega)$ and thus in $C_0(\Omega)$ (which is just the closure of $C_c(\Omega)$, in the $\|\cdot\|_{L^\infty(\Omega)}$-norm) we can find a continuous linear functional L on $C_0(\Omega)$ coinciding with

$$\psi \mapsto \int_\Omega u \operatorname{div} \psi \, d\mathcal{L}^n$$

on $C_c^1(\Omega)$ and satisfying $\|L\| \le V(u,\Omega)$. Then the Riesz representation theorem says that there exists an $\mathbb{R}^n$-valued Radon measure $\mu = (\mu_1, \ldots, \mu_n)$ with $|\mu|(\Omega) = \|L\|$ and

$$L(\psi) = \sum_{i=1}^n \int_\Omega \psi_i \, d\mu_i \qquad \forall \psi \in [C_0(\Omega)]^n.$$

The variation, part III

Hence we have

$$\int_\Omega u \operatorname{div} \psi \, d\mathcal{L}^n = \sum_{i=1}^n \int_\Omega \psi_i \, d\mu_i \qquad \forall \psi \in [C_c^1(\Omega)]^n,$$

so that $u \in \mathrm{BV}(\Omega)$, $Du = -\mu$, and

$$|Du|(\Omega) = |\mu|(\Omega) = \|L\| \leq V(u, \Omega).$$

$\square$

Lower semicontinuity

Lemma

If $u_h \to u$ in $L^1_{loc}(\Omega)$, then $V(u,\Omega) \le \liminf_{h\to\infty} V(u_h,\Omega)$.

Proof.

For any $\psi \in [C^1_c(\Omega)]^n$, we have

$$\int_\Omega u \,\mathrm{div}\,\psi \, d\mathcal{L}^n = \lim_{h\to\infty} \int_\Omega u_h \,\mathrm{div}\,\psi \, d\mathcal{L}^n \le \liminf_{h\to\infty} V(u_h,\Omega).$$

Taking the supremum over such ψ we obtain the result. □

The BV norm

- The BV norm is defined as
$$\|u\|_{\mathrm{BV}(\Omega)} := \int_\Omega |u| \, d\mathcal{L}^n + |Du|(\Omega).$$
- If $u \in W^{1,1}(\Omega)$, then $|Du|(\Omega) = \|\nabla u\|_{L^1(\Omega)}$, so that $\|u\|_{\mathrm{BV}(\Omega)} = \|u\|_{W^{1,1}(\Omega)}$.
- Smooth functions are not dense in $\mathrm{BV}(\Omega)$, since the Sobolev space $W^{1,1}(\Omega) \subsetneq \mathrm{BV}(\Omega)$ is complete.

Approximation by smooth functions

While smooth functions are not dense in $\mathrm{BV}(\Omega)$, we have the following.

Theorem

Let $u \in \mathrm{BV}(\Omega)$. Then there exists a sequence $(u_h) \subset C^\infty(\Omega)$ with $u_h \to u$ in $L^1(\Omega)$ and

$$\lim_{h\to\infty} \int_\Omega |\nabla u_h| \, d\mathcal{L}^n = |Du|(\Omega).$$

Proof. Fix $\delta > 0$. Note that by lower semicontinuity, for any sequence $(u_h) \subset C^\infty(\Omega)$ with $u_h \to u$ in $L^1(\Omega)$, we have

$$|Du|(\Omega) \leq \liminf_{h\to\infty} \int_\Omega |\nabla u_h| \, d\mathcal{L}^n.$$

Thus we need to construct a function $v_\delta \in C^\infty(\Omega)$ such that

$$\int_\Omega |u - v_\delta| \, d\mathcal{L}^n < \delta, \qquad \int_\Omega |\nabla v_\delta| \, d\mathcal{L}^n < |Du|(\Omega) + \delta.$$

Approximation by smooth functions, proof I

Define $\Omega_0 := \emptyset$ and

$$\Omega_k := \{x \in \Omega \cap B(0,k) : \operatorname{dist}(x, \partial\Omega) > 1/k\}, \quad k \in \mathbb{N}.$$

Define $V_k := \Omega_{k+1} \setminus \overline{\Omega_{k-1}}$, $k \in \mathbb{N}$. Then $\bigcup_{k\in\mathbb{N}} V_k = \Omega$.

Then pick a partition of unity $\varphi_k \in C_c^\infty(V_k)$, with $0 \le \varphi_k \le 1$ and $\sum_{k\in\mathbb{N}} \varphi_k \equiv 1$ in Ω. For every $k \in \mathbb{N}$ there exists $\varepsilon_k > 0$ such that $\operatorname{supp}((u\varphi_k) * \rho_{\varepsilon_k}) \subset V_k$ and

$$\int_\Omega \left[|(u\varphi_k) * \rho_{\varepsilon_k} - u\varphi_k| + |(u\nabla\varphi_k) * \rho_{\varepsilon_k} - u\nabla\varphi_k| \right] d\mathcal{L}^n < 2^{-k}\delta.$$

Define $v_\delta := \sum_{k\in\mathbb{N}} (u\varphi_k) * \rho_{\varepsilon_k}$, so that $v_\delta \in C^\infty(\Omega)$ and

$$\int_\Omega |v_\delta - u| \, d\mathcal{L}^n \le \sum_{k\in\mathbb{N}} \int_\Omega |(u\varphi_k) * \rho_{\varepsilon_k} - u\varphi_k| \, d\mathcal{L}^n < \delta.$$

Approximation by smooth functions, proof II

We also have

$$\begin{aligned}\nabla v_\delta &= \sum_{k\in\mathbb{N}} \nabla\left((u\varphi_k) * \rho_{\varepsilon_k}\right) = \sum_{k\in\mathbb{N}} (D(u\varphi_k)) * \rho_{\varepsilon_k} \\ &= \sum_{k\in\mathbb{N}} (\varphi_k Du) * \rho_{\varepsilon_k} + \sum_{k\in\mathbb{N}} (u\nabla\varphi_k) * \rho_{\varepsilon_k} \\ &= \sum_{k\in\mathbb{N}} (\varphi_k Du) * \rho_{\varepsilon_k} + \sum_{k\in\mathbb{N}} \left[(u\nabla\varphi_k) * \rho_{\varepsilon_k} - u\nabla\varphi_k\right].\end{aligned}$$

$$\begin{aligned}\implies \int_\Omega |\nabla v_\delta|\, d\mathcal{L}^n &< \sum_{k\in\mathbb{N}} \int_\Omega (\varphi_k |Du|) * \rho_{\varepsilon_k}\, d\mathcal{L}^n + \delta \\ &= \sum_{k\in\mathbb{N}} \int_\Omega \varphi_k\, d|Du| + \delta \\ &= |Du|(\Omega) + \delta.\end{aligned}$$

□

Weak* convergence of BV functions, part I

Definition

Let $u, u_h \in \mathrm{BV}(\Omega)$. We say that (u_h) weakly* converges to u in $\mathrm{BV}(\Omega)$ if $u_h \to u$ in $L^1(\Omega)$ and $Du_h \overset{*}{\rightharpoonup} Du$ in Ω, i.e.

$$\lim_{h\to\infty} \int_\Omega \psi \, dDu_h = \int_\Omega \psi \, dDu \qquad \forall \psi \in C_0(\Omega).$$

Here $C_0(\Omega)$ is the completion of $C_c(\Omega)$ in the sup norm.

Theorem

Let $u, u_h \in \mathrm{BV}(\Omega)$. Then u_h weakly converges to u in $\mathrm{BV}(\Omega)$ if and only if $u_h \to u$ in $L^1(\Omega)$ and (u_h) is a bounded sequence in $\mathrm{BV}(\Omega)$, i.e.*

$$\sup_{h\in\mathbb{N}} \left\{ \int_\Omega |u_h| \, d\mathcal{L}^n + |Du_h|(\Omega) \right\} < \infty.$$

Proof.

"$\Leftarrow$": By the weak* compactness of Radon measures, for any subsequence $h(k)$ we have a further subsequence (not relabeled) such that $Du_{h(k)} \overset{*}{\rightharpoonup} \mu$ in Ω for a Radon measure μ. We need to show that $\mu = Du$. We have for every $k \in \mathbb{N}$

$$\int_\Omega u_{h(k)} \frac{\partial \psi}{\partial x_i}\, dx = -\int_\Omega \psi \, dD_i u_{h(k)} \quad \forall \psi \in C_c^\infty(\Omega),\ i = 1, \dots, n.$$

By letting $k \to \infty$, we obtain

$$\int_\Omega u \frac{\partial \psi}{\partial x_i}\, dx = -\int_\Omega \psi \, d\mu_i \quad \forall \psi \in C_c^\infty(\Omega),\ i = 1, \dots, n,$$

so that $\mu = Du$. Since this was true for any subsequence $h(k)$, we must have $Du_h \overset{*}{\rightharpoonup} Du$.

Weak* convergence of BV functions, part III

"$\Rightarrow$": The measures Du_h are bounded linear functionals on $C_0(\Omega)$, and for any $\psi \in C_0(\Omega)$,

$$\sup_{h\in\mathbb{N}} \left| \int_\Omega \psi \, dDu_h \right| < \infty,$$

since

$$\int_\Omega \psi \, dDu_h \to \int_\Omega \psi \, dDu.$$

Thus the Banach-Steinhaus theorem gives $\sup_{h\in\mathbb{N}} |Du_h|(\Omega) < \infty$.

Strict convergence of BV functions

Definition

Let $u, u_h \in \mathrm{BV}(\Omega)$. We say that u_h strictly converges to u in $\mathrm{BV}(\Omega)$ if $u_h \to u$ in $L^1(\Omega)$ and $|Du_h|(\Omega) \to |Du|(\Omega)$ as $h \to \infty$.

- Strict convergence of BV functions always implies weak* convergence, by our characterization of the latter.
- However, the converse does not hold: $\sin(hx)/h$ weakly* converges in $\mathrm{BV}((0, 2\pi))$ to 0 as $h \to \infty$, but does not converge strictly because $|Du_h|((0, 2\pi)) = 4$ for each h.
- We showed previously that for every $u \in \mathrm{BV}(\Omega)$ there exists $(u_h) \subset C^\infty(\Omega)$ with $u_h \to u$ strictly in $\mathrm{BV}(\Omega)$.

The area formula

Let $k, n \in \mathbb{N}$ with $k \leq n$. For a differentiable mapping $f : \mathbb{R}^k \to \mathbb{R}^n$, denote by df_x the $n \times k$-matrix whose rows are the gradient vectors of the components of f at the point $x \in \mathbb{R}^k$. Also, define the Jacobian by

$$\mathbf{J}_k df_x := \sqrt{\det(df_x^* \circ df_x)}.$$

Theorem

Let $f : \mathbb{R}^k \to \mathbb{R}^n$ be a one-to-one Lipschitz function. Then for any Borel measurable nonnegative function g we have

$$\int_{\mathbb{R}^n} g(f^{-1}(y))\, d\mathcal{H}^k(y) = \int_{\mathbb{R}^k} g(x) \mathbf{J}_k df_x \, dx.$$

BV extension domains

Definition

A bounded open set $\Omega \subset \mathbb{R}^n$ is a BV extension domain if for any open set $A \supset \overline{\Omega}$ there exists a linear and bounded extension operator $T \colon \mathrm{BV}(\Omega) \to \mathrm{BV}(\mathbb{R}^n)$ satisfying

- $Tu = 0$ in $\mathbb{R}^n \setminus A$ for any $u \in \mathrm{BV}(\Omega)$,
- $|DTu|(\partial\Omega) = 0$ for any $u \in \mathrm{BV}(\Omega)$.

Theorem

A bounded open set $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary is a BV extension domain.

Proof omitted.

Lemma

Let $u \in \mathrm{BV}(\Omega)$ and let $K \subset \Omega$ be a compact set. Then

$$\int_K |u * \rho_\varepsilon - u| \, d\mathcal{L}^n \le \varepsilon |Du|(\Omega) \qquad \forall \varepsilon \in (0, \mathrm{dist}(K, \partial\Omega)).$$

Proof. We find $(u_h) \subset C^\infty(\Omega)$ with $u_h \to u$ in $L^1(\Omega)$ and $|Du_h|(\Omega) \to |Du|(\Omega)$. Thus we can in fact assume $u \in C^\infty(\Omega)$. Pick $x \in K$ and $y \in B(0,1)$, and denote $v(t) := u(x - \varepsilon t y)$, so that

$$u(x - \varepsilon y) - u(x) = \int_0^1 v'(t) \, dt = -\varepsilon \int_0^1 \langle \nabla u(x - \varepsilon t y), y \rangle \, dt.$$

Compactness in BV part II

By Fubini we get

$$\int_K |u(x-\varepsilon y)-u(x)|\,dx \leq \varepsilon \int_0^1 \int_K |\nabla u(x-\varepsilon ty)|\,dx\,dt \leq \varepsilon |Du|(\Omega)$$

Multiplying by $\rho(y)$ and integrating we obtain, by again using Fubini

$$\int_K \left(\int_{\mathbb{R}^n} |u(x-\varepsilon y)-u(x)|\rho(y)\,dy \right)\,dx \leq \varepsilon |Du|(\Omega).$$

Thus

$$\begin{aligned}\int_K |u * \rho_\varepsilon(x) - u(x)|\,dx &= \int_K \left| \int_{\mathbb{R}^n} [u(x-\varepsilon y)-u(x)]\rho(y)\,dy \right| dx \\ &\leq \varepsilon |Du|(\Omega).\end{aligned}$$

□

Compactness in BV part III

Theorem

Let (u_h) *be a norm-bounded sequence in* $\mathrm{BV}(\mathbb{R}^n)$*, i.e.*

$$\sup_{h\in\mathbb{N}}\left\{\int_{\mathbb{R}^n}|u_h|\,d\mathcal{L}^n+|Du_h|(\mathbb{R}^n)\right\}<\infty.$$

Then for some subsequence we have $u_{h(k)}\to u\in\mathrm{BV}(\mathbb{R}^n)$ *locally weakly* in* $\mathrm{BV}(\mathbb{R}^n)$ *as* $k\to\infty$*.*

Proof. Fix $\varepsilon>0$ and for each $h\in\mathbb{N}$, let $u_{h,\varepsilon}:=u_h*\rho_\varepsilon$. Then

$$\|u_{h,\varepsilon}\|_{L^\infty(\mathbb{R}^n)}\le\|u_h\|_{L^1(\mathbb{R}^n)}\|\rho_\varepsilon\|_{L^\infty(\mathbb{R}^n)}$$

and since $\nabla u_{h,\varepsilon}=u_h*\nabla\rho_\varepsilon$,

$$\|\nabla u_{h,\varepsilon}\|_{L^\infty(\mathbb{R}^n)}\le\|u_h\|_{L^1(\mathbb{R}^n)}\|\nabla\rho_\varepsilon\|_{L^\infty(\mathbb{R}^n)}.$$

Compactness in BV part IV

Thus with ε fixed, $(u_{h,\varepsilon})$ is an equibounded and equicontinuous sequence. Fix a bounded set $U \subset \mathbb{R}^n$. By Arzelà-Ascoli we can find a subsequence converging uniformly on U. By a diagonal argument we find a subsequence $h(k)$ such that $u_{h(k),\varepsilon}$ converges uniformly on U for any $\varepsilon = 1/p$, $p \in \mathbb{N}$.

Thus we have

$$\begin{aligned}
\limsup_{k,k'\to\infty} \int_U |u_{h(k)} - u_{h(k')}|\, d\mathcal{L}^n \leq \limsup_{k,k'\to\infty} \int_U |u_{h(k)} - u_{h(k),1/p}|\, d\mathcal{L}^n \\
+ \limsup_{k,k'\to\infty} \int_U |u_{h(k),1/p} - u_{h(k'),1/p}|\, d\mathcal{L}^n \\
+ \limsup_{k,k'\to\infty} \int_U |u_{h(k'),1/p} - u_{h(k')}|\, d\mathcal{L}^n \\
\leq \frac{2}{p} \sup_{h\in\mathbb{N}} |Du_h|(\mathbb{R}^n).
\end{aligned}$$

Compactness in BV part V

Since we can take $p \in \mathbb{N}$ arbitrarily large, we have

$$\lim_{k,k' \to \infty} \int_U |u_{h(k)} - u_{h(k')}| \, d\mathcal{L}^n = 0,$$

so that $u_{h(k)}$ is a Cauchy sequence in $L^1(U)$ and necessarily converges in $L^1(U)$ to some function u. By the lower semicontinuity of the variation, we have $u \in \mathrm{BV}(U)$, and by our previous characterization of weak* convergence in BV we have that $u_{h(k)}$ weakly* converges to u in $\mathrm{BV}(U)$.

Finally, by another diagonal argument we find a subsequence $h(k)$ (not relabeled) for which this convergence takes place in every bounded open $U \subset \mathbb{R}^n$. $\square$

Compactness in BV part VI

Corollary

Let $\Omega \subset \mathbb{R}^n$ be a bounded BV extension domain, and let $(u_h) \subset \mathrm{BV}(\Omega)$ be a norm-bounded sequence. Then for some subsequence we have $u_{h(k)} \to u \in \mathrm{BV}(\Omega)$ weakly in $\mathrm{BV}(\Omega)$.*

Proof.

Extend each function u_h to $Tu_h \in \mathrm{BV}(\mathbb{R}^n)$. Then by the previous theorem, for a subsequence we have $Tu_{h(k)} \to u \in \mathrm{BV}(\mathbb{R}^n)$ locally weakly* in $\mathrm{BV}(\mathbb{R}^n)$, in particular $Tu_{h(k)} \to u$ weakly* in $\mathrm{BV}(\Omega)$, since Ω is bounded. Thus $u_{h(k)} \to u$ weakly* in $\mathrm{BV}(\Omega)$. □

Sets of finite perimeter

As before, Ω will always denote an open set in $\mathbb{R}^n$.

We denote by χ_E the characteristic function of a set $E \subset \mathbb{R}^n$, i.e. the function that takes the value 1 in the set E and the value 0 outside it.

Definition

Let $E \subset \mathbb{R}^n$ be a $\mathcal{L}^n$-measurable set. The *perimeter* of E in Ω is the variation of χ_E in Ω, i.e.

$$P(E, \Omega) := \sup \left\{ \int_E \operatorname{div} \psi \, d\mathcal{L}^n : \quad \psi \in [C_c^1(\Omega)]^n, \quad |\psi| \leq 1 \right\}.$$

We say that E is of finite perimeter in Ω if $P(E, \Omega) < \infty$.

Sets with smooth boundary

Example

If an open set E has a C^1-boundary inside Ω and $\mathcal{H}^{n-1}(\partial E \cap \Omega) < \infty$, then by the Gauss-Green theorem

$$\int_E \operatorname{div} \psi \, d\mathcal{L}^n = -\int_{\partial E \cap \Omega} \langle \nu_E, \psi \rangle \, d\mathcal{H}^{n-1} \quad \forall \psi \in [C_c^1(\Omega)]^n, \qquad (1)$$

where ν_E is the inner unit normal of E, so that $P(E, \Omega) \leq \mathcal{H}^{n-1}(\partial E \cap \Omega)$ — in fact by picking suitable ψ, we can show that equality holds.

Basic properties

- For any set E that is of finite perimeter in Ω, the distributional derivative $D\chi_E$ is an $\mathbb{R}^n$-valued Radon measure in Ω, with polar decomposition $D\chi_E = \nu_E |D\chi_E|$, so that

$$\int_E \operatorname{div} \psi \, d\mathcal{L}^n = - \int_\Omega \langle \psi, \nu_E \rangle \, d|D\chi_E| \quad \forall \psi \in [C_c^1(\Omega)]^n.$$

  Here $|\nu_E| = 1$ $|D\chi_E|$-a.e. and $|D\chi_E|(\Omega) = P(E, \Omega)$. Then we can define $P(E, B)$ to be the same as $|D\chi_E|(B)$ for any Borel set $B \subset \Omega$.
- Thus $\chi_E \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$, but not necessarily $\chi_E \in \mathrm{BV}(\Omega)$ since we might not have $\chi_E \in L^1(\Omega)$.
- Moreover, $P(E, \Omega) = P(\mathbb{R}^n \setminus E, \Omega)$.

Algebra property

We also have the following algebra property.

Lemma

Given sets E, F of finite perimeter in Ω, we have

$$P(E \cup F, \Omega) + P(E \cap F, \Omega) \leq P(E, \Omega) + P(F, \Omega).$$

Proof. We find sequences $u_h, v_h \in C^\infty(\Omega)$ with $u_h \to \chi_E$ in $L^1(\Omega)$, $v_h \to \chi_F$ in $L^1(\Omega)$, $0 \leq u_h, v_h \leq 1$, and

$$\lim_{h\to\infty} \int_\Omega |\nabla u_h| \, d\mathcal{L}^n = P(E, \Omega), \quad \lim_{h\to\infty} \int_\Omega |\nabla v_h| \, d\mathcal{L}^n = P(F, \Omega).$$

Algebra property continued

Then $u_h v_h \to \chi_{E\cap F}$ and $u_h + v_h - u_h v_h \to \chi_{E\cup F}$ in $L^1_{\text{loc}}(\Omega)$, and thus by the lower semicontinuity of the perimeter

$$\begin{aligned}
&P(E\cap F,\Omega) + P(E\cup F,\Omega)\\
&\le \liminf_{h\to\infty}\left(\int_\Omega |\nabla(u_h v_h)|\,d\mathcal{L}^n + \int_\Omega |\nabla(u_h+v_h-u_h v_h)|\,d\mathcal{L}^n\right)\\
&\le \liminf_{h\to\infty}\int_\Omega |\nabla u_h|\,(|v_h| + |1-v_h|) + |\nabla v_h|\,(|u_h| + |1-u_h|)\,d\mathcal{L}^n\\
&= \liminf_{h\to\infty}\left(\int_\Omega |\nabla u_h|\,d\mathcal{L}^n + \int_\Omega |\nabla v_h|\,d\mathcal{L}^n\right)\\
&= P(E,\Omega) + P(F,\Omega).
\end{aligned}$$

□

Coarea formula for BV

Theorem

For any $u \in \mathrm{BV}(\Omega)$*, denoting* $E_t := \{x \in \Omega : \ u(x) > t\}$*,* $t \in \mathbb{R}$*, we have*

$$|Du|(\Omega) = \int_{-\infty}^{\infty} P(E_t, \Omega)\, dt.$$

Coarea formula, proof part I

Proof. First we prove the result for $u \in C^\infty(\Omega)$. By the classical coarea formula we have

$$\int_\Omega \mathbf{C}_1 du \, d\mathcal{L}^n = \int_{-\infty}^{\infty} \mathcal{H}^{n-1}(\Omega \cap u^{-1}(t)) \, dt,$$

where $\mathbf{C}_1 du = |\nabla u|$. By Sard's theorem we know that $\{\nabla u = 0\} \cap u^{-1}(t) = \emptyset$ for a.e. $t \in \mathbb{R}$. For these values of t we have that the boundary $\partial E_t = u^{-1}(t)$ is smooth and so

$$\mathcal{H}^{n-1}(\Omega \cap u^{-1}(t)) = \mathcal{H}^{n-1}(\Omega \cap \partial E_t) = P(E_t, \Omega)$$

by (1). Thus

$$\int_\Omega |\nabla u| \, d\mathcal{L}^n = \int_{-\infty}^{\infty} P(E_t, \Omega) \, dt.$$

Coarea formula, proof part II

Let then $u \in \mathrm{BV}(\Omega)$. We prove the inequality "$\geq$". Take a sequence $(u_h) \subset C^\infty(\Omega)$ with $u_h \to u$ strictly in $\mathrm{BV}(\Omega)$. Define $E_t^h := \{x \in \Omega : \ u_h(x) > t\}$. Then

$$|u_h(x) - u(x)| = \int_{\min\{u_h(x),u(x)\}}^{\max\{u_h(x),u(x)\}} dt = \int_{-\infty}^{\infty} |\chi_{E_t^h}(x) - \chi_{E_t}(x)| \, dt,$$

so that by Fubini

$$\int_\Omega |u_h(x) - u(x)| \, dx = \int_{-\infty}^{\infty} \int_\Omega |\chi_{E_t^h}(x) - \chi_{E_t}(x)| \, dx \, dt.$$

Thus by picking a subsequence (not relabeled) we get $\chi_{E_t^h} \to \chi_{E_t}$ in $L^1(\Omega)$ as $h \to \infty$, for a.e. $t \in \mathbb{R}$.

Coarea formula, proof part III

By the lower semicontinuity of the perimeter and Fatou's lemma we have

$$\begin{aligned}
\int_{-\infty}^{\infty} P(E_t, \Omega)\, dt &\leq \int_{-\infty}^{\infty} \liminf_{h\to\infty} P(E_t^h, \Omega)\, dt \\
&\leq \liminf_{h\to\infty} \int_{-\infty}^{\infty} P(E_t^h, \Omega)\, dt \\
&= \liminf_{h\to\infty} |Du_h|(\Omega) \\
&= |Du|(\Omega).
\end{aligned}$$

Coarea formula, proof part IV

Finally, we prove the inequality "$\leq$". We can see that for any $x \in \Omega$,

$$u(x) = \int_0^\infty \chi_{E_t}(x)\,dt - \int_{-\infty}^0 (1 - \chi_{E_t}(x))\,dt.$$

Then, given any $\psi \in [C_c^1(\Omega)]^n$ with $|\psi| \leq 1$, we estimate

$$\begin{aligned}
&\int_\Omega u(x)\,\mathrm{div}\,\psi(x)\,dx \\
&= \int_\Omega \left(\int_0^\infty \chi_{E_t}(x)\,dt - \int_{-\infty}^0 (1 - \chi_{E_t}(x))\,dt \right) \mathrm{div}\,\psi(x)\,dx \\
&= \int_0^\infty \int_\Omega \chi_{E_t}(x)\,\mathrm{div}\,\psi(x)\,dx\,dt - \int_{-\infty}^0 \int_\Omega (1 - \chi_{E_t}(x))\,\mathrm{div}\,\psi(x)\,dx\,dt \\
&= \int_{-\infty}^\infty \int_\Omega \chi_{E_t}(x)\,\mathrm{div}\,\psi(x)\,dx\,dt \leq \int_{-\infty}^\infty P(E_t, \Omega)\,dt.
\end{aligned}$$

□

Coarea formula, consequences I

Let $u \in \mathrm{BV}(\Omega)$. If we define

$$\mu(B) := \int_{-\infty}^{\infty} |D\chi_{E_t}|(B)\, dt$$

for any Borel set $B \subset \Omega$, it is straightforward to check that μ is a positive Borel measure. Since $|Du|$ and μ agree on open subsets of Ω, we have

$$|Du|(B) = \int_{-\infty}^{\infty} |D\chi_{E_t}|(B)\, dt$$

for any Borel set $B \subset \Omega$. We also have

$$Du(B) = \int_{-\infty}^{\infty} D\chi_{E_t}(B)\, dt$$

for any Borel set $B \subset \Omega$, which we see as follows.

Coarea formula, consequences II

We use Fubini and then the fact that $D\chi_{E_t}$ is a finite Radon measure for a.e. $t \in \mathbb{R}$ to obtain for any $\psi \in C_c^\infty(\Omega)$

$$\begin{aligned}
\int_\Omega \psi \, dDu &= -\int_\Omega u(x) \nabla\psi(x)\, dx \\
&= -\int_\Omega \left(\int_0^\infty \chi_{E_t}(x)\, dt \right) \nabla\psi(x)\, dx \\
&\qquad + \int_\Omega \left(\int_{-\infty}^0 (1 - \chi_{E_t}(x))\, dt \right) \nabla\psi(x)\, dx \\
&= -\int_0^\infty \left(\int_\Omega \chi_{E_t}(x) \nabla\psi(x)\, dx \right) dt \\
&\qquad + \int_{-\infty}^0 \left(\int_\Omega (1 - \chi_{E_t}(x)) \nabla\psi(x)\, dx \right) dt \\
&= \int_{-\infty}^\infty \left(\int_\Omega \psi \, dD\chi_{E_t} \right) dt.
\end{aligned}$$

Sobolev inequality

Theorem

For any $u \in \mathrm{BV}(\mathbb{R}^n)$ *we have*

$$\|u\|_{L^{n/(n-1)}(\mathbb{R}^n)} \leq C_S |Du|(\mathbb{R}^n)$$

for some constant $C_S = C_S(n)$.

From now on, let us assume that $n \geq 2$. In the one-dimensional setting, easier proofs and stronger results are available, but we will not consider these.

Sobolev inequality, proof

Proof.

Pick functions $(u_h) \subset C^\infty(\mathbb{R}^n)$ with $u_h \to u$ in $L^1(\mathbb{R}^n)$, $u_h(x) \to u(x)$ for $\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n$ and $|Du_h|(\mathbb{R}^n) \to |Du|(\mathbb{R}^n)$. Then by Fatou's lemma and the Gagliardo-Nirenberg-Sobolev inequality, we have

$$\begin{aligned}
\left(\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}}\, d\mathcal{L}^n\right)^{\frac{n-1}{n}} &= \left(\int_{\mathbb{R}^n} \liminf_{h\to\infty} |u_h|^{\frac{n}{n-1}}\, d\mathcal{L}^n\right)^{\frac{n-1}{n}} \\
&\leq \liminf_{h\to\infty} \left(\int_{\mathbb{R}^n} |u_h|^{\frac{n}{n-1}}\, d\mathcal{L}^n\right)^{\frac{n-1}{n}} \\
&\leq C_S \liminf_{h\to\infty} |Du_h|(\mathbb{R}^n) \\
&= C_S |Du|(\mathbb{R}^n).
\end{aligned}$$

□

Isoperimetric inequality

Theorem

For any bounded $\mathcal{L}^n$-measurable set $E \subset \mathbb{R}^n$, we have

$$\mathcal{L}^n(E)^{(n-1)/n} \leq C_S P(E, \mathbb{R}^n).$$

Proof.

Choose $u = \chi_E$ in the Sobolev inequality. □

Poincaré inequality

Theorem

For any ball $B(x,r)$ *and any* $u \in \mathrm{BV}(B(x,r))$ *we have*

$$\left(\int_{B(x,r)} |u - u_{B(x,r)}|^{n/(n-1)} \, d\mathcal{L}^n \right)^{(n-1)/n} \leq C_P |Du|(B(x,r))$$

for some $C_P = C_P(n)$*, where*

$$u_{B(x,r)} := \fint_{B(x,r)} u \, d\mathcal{L}^n := \frac{1}{\mathcal{L}^n(B(x,r))} \int_{B(x,r)} u \, d\mathcal{L}^n.$$

Proof.

Follows from the usual Poincaré inequality for Sobolev functions. □

Relative isoperimetric inequality

Theorem

For any ball $B(x,r)$ and any $\mathcal{L}^n$-measurable set $E \subset \mathbb{R}^n$, we have

$$\min\{|B(x,r)\cap E|, |B(x,r)\setminus E|\}^{(n-1)/n} \le 2C_P P(E, B(x,r)).$$

Proof. We have

$$\int_{B(x,r)} |\chi_E - (\chi_E)_{B(x,r)}|^{n/(n-1)}\, d\mathcal{L}^n$$
$$= |B(x,r)\cap E| \left(\frac{|B(x,r)\setminus E|}{|B(x,r)|}\right)^{\frac{n}{n-1}}$$
$$+ |B(x,r)\setminus E| \left(\frac{|B(x,r)\cap E|}{|B(x,r)|}\right)^{\frac{n}{n-1}}.$$

Relative isoperimetric inequality, proof

If $|B(x,r)\cap E| \le |B(x,r)\setminus E|$, then by the Poincaré inequality

$$\begin{aligned} C_P P(E, B(x,r)) &\ge \left(\int_{B(x,r)} |\chi_E - (\chi_E)_{B(x,r)}|^{\frac{n}{n-1}}\, d\mathcal{L}^n\right)^{\frac{n-1}{n}} \\ &\ge \left(\frac{|B(x,r)\setminus E|}{|B(x,r)|}\right) |B(x,r)\cap E|^{(n-1)/n} \\ &\ge \frac{1}{2}\min\{|B(x,r)\cap E|,\, |B(x,r)\setminus E|\}^{(n-1)/n}. \end{aligned}$$

The case $|B(x,r)\cap E| \ge |B(x,r)\setminus E|$ is handled analogously.

□

Localization

Lemma

Let E be a set of finite perimeter in $\mathbb{R}^n$, and let $x \in \mathbb{R}^n$. Then for a.e. $r > 0$, we have

$$P(E \cap B(x,r), \mathbb{R}^n) \leq P(E, \overline{B}(x,r)) + m'(r),$$

where $m(r) := |E \cap B(x,r)|$.

In particular, if $P(E, \partial B(x,r)) = 0$, then

$$P(E, B(x,r)) + P(E \cap B(x,r), \partial B(x,r)) \leq P(E, B(x,r)) + m'(r),$$

and so

$$P\left(E \cap B(x,r), \partial B(x,r)\right) \leq m'(r). \tag{2}$$

Localization proof part I

Proof. We can assume that $x = 0$. Fix $r > 0$ such that the derivative $m'(r)$ exists; note that the derivative of a monotone function on the real line exists almost everywhere.

For $\sigma > 0$, set

$$\gamma_\sigma(t) := \begin{cases} 1 & \text{if } t \leq r, \\ 1 + \frac{r-t}{\sigma} & \text{if } r \leq t \leq r + \sigma, \\ 0 & \text{if } t \geq r + \sigma. \end{cases}$$

Localization proof part II

Then define $v_\sigma(y) := \chi_E(y)\gamma_\sigma(|y|)$, $y \in \mathbb{R}^n$. By the Leibniz rule

$$Dv_\sigma = \gamma_\sigma(|y|)\, D\chi_E + \chi_E(y)\gamma_\sigma'(|y|)\frac{y}{|y|}\,\mathcal{L}^n$$

and thus

$$|Dv_\sigma|(\mathbb{R}^n) \leq |D\chi_E|(B(x, r+\sigma)) + \sigma^{-1}\int_{B(x,r+\sigma)\setminus B(x,r)} \chi_E\, d\mathcal{L}^n,$$

and since $v_\sigma \to \chi_{E\cap B(x,r)}$ in $L^1(\mathbb{R}^n)$ as $\sigma \to 0$, by lower semicontinuity

$$\begin{aligned} P(E\cap B(x,r), \mathbb{R}^n) &\leq \liminf_{\sigma\to 0} |Dv_\sigma|(\mathbb{R}^n) \\ &\leq |D\chi_E|(\overline{B}(x,r)) + m'(r). \end{aligned}$$

□

Reduced boundary

Let $E \subset \mathbb{R}^n$ be a set of locally finite perimeter in $\mathbb{R}^n$.

Definition

We define the *reduced boundary* $\mathcal{F}E$ as the set of points $x \in \operatorname{supp}|D\chi_E|$ such that the limit

$$\nu_E(x) := \lim_{r \to 0} \frac{D\chi_E(B(x,r))}{|D\chi_E|(B(x,r))}$$

exists and satisfies $|\nu_E(x)| = 1$.

- According to the polar decomposition of vector measures (based on the Besicovitch differentiation theorem), we have $|D\chi_E|(\mathbb{R}^n \setminus \mathcal{F}E) = 0$ and $D\chi_E = \nu_E |D\chi_E|$.
- If $x \in \mathcal{F}E$, then

$$\mathcal{L}^n(B(x,r) \cap E) > 0, \quad \mathcal{L}^n(B(x,r) \setminus E) > 0$$

for all $r > 0$.

Note that for $x \in \mathcal{F}E$,

$$\begin{aligned}
&\frac{1}{2|D\chi_E|(B(x,r))} \int_{B(x,r)} |\nu_E(y) - \nu_E(x)|^2 \, d|D\chi_E|(y) \\
&\qquad = 1 - \frac{2}{2|D\chi_E|(B(x,r))} \int_{B(x,r)} \langle \nu_E(y), \nu_E(x) \rangle \; d|D\chi_E|(y) \\
&\qquad = 1 - \left\langle \frac{D\chi_E(B(x,r))}{|D\chi_E|(B(x,r))}, \nu_E(x) \right\rangle \to 0
\end{aligned}$$

as $r \to 0$, since $x \in \mathcal{F}E$. Thus

$$\lim_{r \to 0} \frac{1}{|D\chi_E|(B(x,r))} \int_{B(x,r)} |\nu_E(y) - \nu_E(x)| \, d|D\chi_E|(y) = 0.$$

A priori estimates on perimeter and volume part I

Lemma

Let E be a set of finite perimeter in $\mathbb{R}^n$ and let $x \in \mathcal{F}E$. Then there exist $r_0 > 0$, and constants $\alpha, \beta > 0$ depending only on n, such that

$$P(E, B(x,r)) \leq \alpha r^{n-1} \qquad \forall r \in (0, r_0),$$

$$\min\{|B(x,r) \cap E|, |B(x,r) \setminus E|\} \geq \beta r^n \qquad \forall r \in (0, r_0).$$

Proof. By the fact that $x \in \mathcal{F}E$, we can choose $r_0 > 0$ such that

$$|D\chi_E|(B(x,r)) \leq 2|D\chi_E(B(x,r))| \qquad \forall r \in (0, 2r_0). \tag{3}$$

A priori estimates on perimeter and volume part II

By the algebra property of sets of finite perimeter, $E \cap B(x,r)$ is of finite perimeter in $\mathbb{R}^n$. In general, for any bounded set F of finite perimeter in $\mathbb{R}^n$, we have (recall that ρ_ε denote standard mollifiers)

$$
\begin{aligned}
D\chi_F(\mathbb{R}^n) &= \int_{\mathbb{R}^n} D\chi_F * \rho_\varepsilon \, d\mathcal{L}^n \\
&= \int_{\mathbb{R}^n} \nabla(\chi_F * \rho_\varepsilon) \, d\mathcal{L}^n = 0.
\end{aligned}
$$

We also note that for a.e. $r > 0$, we have $P(E, \partial B(x,r)) = 0$. In total, for a.e. $r \in (0, 2r_0)$ we have

$$
\begin{aligned}
P(E, B(x,r)) &\leq 2|D\chi_E(B(x,r))| \qquad \text{by (3)} \\
&= 2|D\chi_{E\cap B(x,r)}(B(x,r))| \\
&= 2|D\chi_{E\cap B(x,r)}(\partial B(x,r))| \quad \text{since } D\chi_{E\cap B(x,r)}(\mathbb{R}^n) = 0 \\
&\leq 2P(E \cap B(x,r), \partial B(x,r)) \\
&\leq 2m'(r) \qquad \text{by (2).}
\end{aligned}
$$

A priori estimates on perimeter and volume part III

Thus for any $r \in (0, r_0)$,

$$P(E, B(x, r)) \leq \frac{1}{r} \int_r^{2r} P(E, B(x, t))\, dt \leq \frac{2m(2r)}{r} \leq 2^{n+1}\omega_n r^{n-1}.$$

This gives the first estimate. Then, by using the isoperimetric inequality as well as the localization lemma,

$$\begin{aligned} m(r)^{1-1/n} &= |E \cap B(x, r)|^{1-1/n} \\ &\leq C_S P(E \cap B(x, r), \mathbb{R}^n) \\ &\leq C_S \left[P(E, B(x, r)) + m'(r) \right] \\ &\leq C_S (3m'(r)) \end{aligned}$$

for a.e. $r \in (0, r_0)$, so that $(m^{1/n})'(r) \geq 1/(3nC_S)$ for a.e. $r \in (0, r_0)$, so that $m(r) \geq r^n/(3nC_S)^n$ for all $r \in (0, r_0)$. Finally, we can run the same argument with $|B(x, r) \setminus E|$ instead of $|B(x, r) \cap E|$. $\square$

De Giorgi structure theorem

Let $E \subset \mathbb{R}^n$ be a set of finite perimeter in $\mathbb{R}^n$. For each $x \in \mathcal{F}E$, define the hyperplane

$$H(x) := \{y \in \mathbb{R}^n : \langle \nu_E(x), y - x \rangle = 0\}$$

and the half-spaces

$$H^+(x) := \{y \in \mathbb{R}^n : \langle \nu_E(x), y - x \rangle \geq 0\},$$

$$H^-(x) := \{y \in \mathbb{R}^n : \langle \nu_E(x), y - x \rangle \leq 0\}.$$

With $x \in \mathcal{F}E$ fixed, define also

$$E_r := \{(y - x)/r + x : y \in E\}.$$

Theorem

Let $x \in \mathcal{F}E$. Then $\chi_{E_r} \to \chi_{H^+(x)}$ in $L^1_{loc}(\mathbb{R}^n)$ as $r \searrow 0$.

Blow-up proof part I

Proof. First of all, we may assume $x = 0$ and $\nu_E(0) = e_n$. Then $E_r = \{y/r : y \in E\}$, and furthermore let $\psi_s(y) := \psi(sy)$, $s > 0$. For any $\psi \in [C_c^1(\mathbb{R}^n)]^n$, we have

$$\int_{\mathbb{R}^n} \chi_{E_r} \operatorname{div} \psi \, d\mathcal{L}^n = \frac{1}{r^{n-1}} \int_{\mathbb{R}^n} \chi_E \operatorname{div} \psi_{r^{-1}} \, d\mathcal{L}^n. \tag{4}$$

Fix any $M > 0$. Noting that $\psi \in [C_c^1(B(0, M))]^n$ if and only if $\psi_{r^{-1}} \in [C_c^1(B(0, rM))]^n$ and taking the supremum (with $|\psi| \leq 1$), we obtain

$$P(E_r, B(0, M)) = \frac{P(E, (B(0, rM)))}{r^{n-1}}. \tag{5}$$

Blow-up proof part II

Thus by the a priori estimate on perimeter, we obtain

$$P(E_r, B(0,M)) \le \alpha \frac{(rM)^{n-1}}{r^{n-1}} = \alpha M^{n-1}$$

for sufficiently small $r > 0$.

Take an arbitrary sequence $r_h \searrow 0$. By the above, $P(E_{r_h}, B(0,M))$ is a bounded sequence. By compactness, we find a subsequence (not relabeled) such that $\chi_{E_{r_h}} \to v \in \mathrm{BV}(B(0,M))$ weakly* in $\mathrm{BV}(B(0,M))$. We can also assume that $\chi_{E_{r_h}}(x) \to v(x)$ for $\mathcal{L}^n$-a.e. $x \in B(0,M)$, so that $v = \chi_F$ for some set F.

This can be done for every $M > 0$, and so by a diagonal argument, we have for some set F of locally finite perimeter in $\mathbb{R}^n$ that $\chi_{E_{r_h}} \to \chi_F$ locally weakly* in $\mathrm{BV}(\mathbb{R}^n)$.

Blow-up proof part III

We also obtain from (4) for any $r > 0$

$$\int_{\mathbb{R}^n} \langle \psi, \nu_{E_r} \rangle \, d|D\chi_{E_r}| = \frac{1}{r^{n-1}} \int_{\mathbb{R}^n} \langle \psi_{r^{-1}}, \nu_E \rangle \, d|D\chi_E|,$$

so that for any $M > 0$

$$\int_{B(0,M)} \nu_{E_r} \, d|D\chi_{E_r}| = \frac{1}{r^{n-1}} \int_{B(0,rM)} \nu_E \, d|D\chi_E|.$$

Thus

$$\begin{aligned} &\frac{1}{|D\chi_{E_{r_h}}|(B(0,M))} \int_{B(0,M)} \nu_{E_{r_h}} \, d|D\chi_{E_{r_h}}| \\ &\quad = \frac{1}{|D\chi_E|(B(0,r_h M))} \int_{B(0,r_h M)} \nu_E \, d|D\chi_E| \to \nu_E(0) = e_n \end{aligned}$$

as $h \to \infty$.

Blow-up proof part IV

Moreover, if $|D\chi_F|(\partial B(0,M)) = 0$ (which is true for a.e. $M > 0$) then by the fact that $D\chi_{E_{r_h}} \xrightarrow{*} D\chi_F$ locally in $\mathbb{R}^n$,

$$\begin{aligned} |D\chi_F|(B(0,M)) &\le \liminf_{h\to\infty} |D\chi_{E_{r_h}}|(B(0,M)) \\ &\le \limsup_{h\to\infty} |D\chi_{E_{r_h}}|(B(0,M)) \\ &= \limsup_{h\to\infty} \int_{B(0,M)} \langle e_n, \nu_{E_{r_h}} \rangle \, d|D\chi_{E_{r_h}}| \\ &= \int_{B(0,M)} \langle e_n, \nu_F \rangle \, d|D\chi_F|. \end{aligned}$$

Since $|\nu_F| = 1$ $|D\chi_F|$-almost everywhere, we must have $\nu_F = e_n$ $|D\chi_F|$-almost everywhere in $B(0,M)$, and then also

$$|D\chi_F|(B(0,M)) = \lim_{h\to\infty} |D\chi_{E_{r_h}}|(B(0,M)). \tag{6}$$

Blow-up proof part V

The above is true for a.e. $M > 0$. Thus $D\chi_F = e_n|D\chi_F|$, and so by mollifying we obtain

$$\nabla(\chi_F * \rho_\varepsilon) = (D\chi_F) * \rho_\varepsilon = (|D\chi_F| * \rho_\varepsilon)e_n,$$

so that $\chi_F * \rho_\varepsilon(y) = \gamma_\varepsilon(y_n)$ for all $y \in \mathbb{R}^n$, for some increasing $\gamma_\varepsilon : \mathbb{R} \to \mathbb{R}$.

Blow-up proof part VI

Letting $\varepsilon \to 0$, we obtain $\chi_F(y) = \gamma(y_n)$ for $\mathcal{L}^n$-a.e. $y \in \mathbb{R}^n$, for some increasing $\gamma : \mathbb{R} \to \mathbb{R}$. But $\chi_F(y) \in \{0, 1\}$ for all $y \in \mathbb{R}^n$, so necessarily $F = \{y \in \mathbb{R}^n : \ y_n \geq a\}$ for some $a \in \overline{\mathbb{R}}$. Suppose $a > 0$. Since $\chi_{E_{r_h}} \to \chi_F$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$, we have

$$
\begin{aligned}
0 &= \int_{B(0,a)} \chi_F \, d\mathcal{L}^n = \lim_{h\to\infty} \int_{B(0,a)} \chi_{E_{r_h}} \, d\mathcal{L}^n \\
&= \lim_{h\to\infty} \frac{1}{r_h^n} \int_{B(0,r_h a)} \chi_E \, d\mathcal{L}^n > 0
\end{aligned}
$$

by our a priori estimate on volume, giving a contradiction. Similarly we conclude that $a < 0$ is impossible (and also $|a| = \infty$). Thus $a = 0$ and

$$
F = \{y \in \mathbb{R}^n : \ y_n \geq 0\} = H^+(0).
$$

□

Consequences of blow-up

Corollary

For every $x \in \mathcal{F}E$,

$$\lim_{r\to 0} \frac{\mathcal{L}^n(B(x,r)\cap H^-(x)\cap E)}{r^n} = 0 \tag{7}$$

and

$$\lim_{r\to 0} \frac{\mathcal{L}^n(B(x,r)\cap H^+(x)\setminus E)}{r^n} = 0 \tag{8}$$

and

$$\lim_{r\to 0} \frac{|D\chi_E|(B(x,r))}{\omega_{n-1} r^{n-1}} = 1. \tag{9}$$

Consequences of blow-up, proof part II

Proof.

$$\frac{\mathcal{L}^n(B(x,r)\cap H^-(x)\cap E)}{r^n} = \mathcal{L}^n(B(x,1)\cap H^-(x)\cap E_r)$$
$$\to \mathcal{L}^n(B(x,1)\cap H^-(x)\cap H^+(x)) = 0 \qquad \text{as } r\to 0.$$

(8) is proved similarly. Since

$$|D\chi_{H^+(x)}|(\partial B(x,1)) = \mathcal{H}^{n-1}(H(x)\cap \partial B(x,1)) = 0,$$

we also have by (5) and (6)

$$\begin{aligned}\lim_{r\to 0}\frac{|D\chi_E|(B(x,r))}{r^{n-1}} &= \lim_{r\to 0}|D\chi_{E_r}|(B(x,1))\\ &= |D\chi_{H^+(x)}|(B(x,1))\\ &= \mathcal{H}^{n-1}(H(x)\cap B(x,1)) = \omega_{n-1}.\end{aligned}$$

□

The 5-covering theorem

Given any ball $B = B(x, r)$, denote $5B := B(x, 5r)$.

Theorem

Let $\mathcal{F}$ be any collection of open balls in $\mathbb{R}^n$ with

$$\sup\{\operatorname{diam} B : B \in \mathcal{F}\} < \infty.$$

Then there exists a countable family of disjoint balls $\mathcal{G} \subset \mathcal{F}$ such that

$$\bigcup_{B \in \mathcal{F}} B \subset \bigcup_{B \in \mathcal{G}} 5B.$$

Lemma

There exists $C = C(n) > 0$ such that for any Borel set $A \subset \mathcal{F}E$, we have

$$\mathcal{H}^{n-1}(A) \leq C|D\chi_E|(A).$$

Proof. Fix $\varepsilon > 0$. By (9) we have for any $x \in \mathcal{F}E$

$$\lim_{r\to 0} \frac{|D\chi_E|(B(x,r))}{\omega_{n-1} r^{n-1}} = 1.$$

Since $|D\chi_E|$ is a Radon measure, we can find an open set $U \supset A$ such that

$$|D\chi_E|(U) \leq |D\chi_E|(A) + \varepsilon.$$

Hausdorff measure and perimeter, part II

Consider the covering of the set A by balls

$$\Big\{B(x,r):\ x\in A,\ B(x,r)\subset U,\ r<\varepsilon/10,\\ |D\chi_E|(B(x,r))\geq \frac{\omega_{n-1}r^{n-1}}{2}\Big\}.$$

By the 5-covering theorem we can pick from this covering a countable disjoint collection $\{B(x_i,r_i)\}_{i\in\mathbb{N}}$ such that $A\subset\bigcup_{i\in\mathbb{N}}B(x_i,5r_i)$. Since $\operatorname{diam}(B(x_i,5r_i))\leq\varepsilon$, we have

$$\begin{aligned}\mathcal{H}^{n-1}_{\varepsilon}(A)&\leq \omega_{n-1}\sum_{i\in\mathbb{N}}(5r_i)^{n-1}\\ &\leq 2\times 5^{n-1}\sum_{i\in\mathbb{N}}|D\chi_E|(B(x_i,r_i))\\ &\leq C|D\chi_E|(U)\leq C(|D\chi_E|(A)+\varepsilon)\end{aligned}$$

with $C=C(n)$. Letting $\varepsilon\to 0$, we obtain $\mathcal{H}^{n-1}(A)\leq C|D\chi_E|(A)$.

□

Hausdorff measure and perimeter, part III

Lemma

There exists $C = C(n) > 0$ *such that for any Borel set* $A \subset \mathcal{F}E$*, we have*

$$|D\chi_E|(A) \leq C\mathcal{H}^{n-1}(A). \tag{10}$$

Proof. Fix $\tau > 1$. For $i \in \mathbb{N}$, define

$$A_i := \left\{ x \in A : \frac{|D\chi_E|(B(x,r))}{\omega_{n-1} r^{n-1}} < \tau \quad \forall r \in (0, 1/i) \right\}.$$

The sequence (A_i) is increasing, and its union is A due to (9).

Fix $i \in \mathbb{N}$, and let $\{D_j\}_{j\in\mathbb{N}}$ be sets covering A_i with diameter less than $1/i$, which intersect A_i at least at a point x_j, and which satisfy

$$\sum_{j\in\mathbb{N}} \omega_{n-1} r_j^{n-1} \leq \mathcal{H}^{n-1}_{1/i}(A_i) + 1/i$$

with $r_j := \mathrm{diam}(D_j)/2$.

The balls $B(x_j, 2r_j)$ still cover A_i, hence

$$\begin{aligned} |D\chi_E|(A_i) &\leq \sum_{j\in\mathbb{N}} |D\chi_E|(B(x_j, 2r_j)) \leq \tau \sum_{j\in\mathbb{N}} \omega_{n-1}(2r_j)^{n-1} \\ &\leq \tau 2^{n-1} \left(\mathcal{H}^{n-1}(A_i) + 1/i\right). \end{aligned}$$

By letting $i \to \infty$ and $\tau \searrow 1$ we obtain $|D\chi_E|(A) \leq 2^{n-1}\mathcal{H}^{n-1}(A)$.

□

Rectifiability of the reduced boundary part I

Theorem

Let E be a set of finite perimeter in $\mathbb{R}^n$. Then the reduced boundary $\mathcal{F}E$ is countably $n-1$-rectifiable and $|D\chi_E| = \mathcal{H}^{n-1} \llcorner \mathcal{F}E$.

Proof. By Egorov's theorem, we can find disjoint compact sets $F_i \subset \mathcal{F}E$, $i \in \mathbb{N}$, with

$$|D\chi_E| \left(\mathcal{F}E \setminus \bigcup_{i \in \mathbb{N}} F_i \right) = 0$$

and such that the convergences (7), (8), (9) are uniform in each set F_i.

Rectifiability of the reduced boundary part II

Choose unit vectors $\nu_1, \ldots, \nu_N$ such that for any $\nu \in \partial B(0,1)$, we have $|\nu - \nu_j| < 1/4$ for some $j = 1, \ldots, N$. Partition the sets F_i further into sets F_i^j, $i \in \mathbb{N}$, $j = 1, \ldots, N$, such that for any $z \in F_i^j$, we have $|\nu_E(z) - \nu_j| < 1/4$. Relabel these sets K_i, $i \in \mathbb{N}$.

Fix $i \in \mathbb{N}$; we may as well take $i = 1$. Pick j such that $|\nu_E(z) - \nu_j| < 1/4$ for all $z \in K_1$. There exists $\delta > 0$ such that if $z \in K_1$ and $r < 2\delta$,

$$\mathcal{L}^n(B(z,r) \cap H^-(z) \cap E) < \frac{\omega_n r^n}{4^{2n+1}}$$

and

$$\mathcal{L}^n(B(z,r) \cap H^+(z) \cap E) > \frac{1}{4}\omega_n r^n.$$

For $\nu \in \mathbb{R}^n$, denote by P_ν the orthogonal projection onto the line spanned by ν, and by $P_\nu^\perp$ the orthogonal projection onto the $n-1$-plane with normal ν, denoted also by $\nu^\perp$.

Rectifiability of the reduced boundary part III

Take $x, y \in K_1$ and $d(x, y) < \delta$. Suppose that we had $y \in H^-(x)$ and

$$|P_{\nu_j}(y - x)| \geq |P^{\perp}_{\nu_j}(y - x)|.$$

Then

$$|P_{\nu_j}(y - x)| \geq |y - x|/2$$

and so

$$|P_{\nu_{E(x)}}(y - x)| \geq |y - x|/4.$$

This implies that

$$B(y, |x - y|/4) \subset B(x, 2|x - y|) \cap H^-(x).$$

so that

$$B(y, |x - y|/4) \cap E \subset B(x, 2|x - y|) \cap H^-(x) \cap E.$$

Rectifiability of the reduced boundary part IV

But since $y \in K_1$,

$$\mathcal{L}^n(B(y, |x-y|/4) \cap E) > \frac{1}{4}\omega_n \left(\frac{|x-y|}{4}\right)^n = \frac{\omega_n |x-y|^n}{4^{n+1}}$$

and similarly since $x \in K_1$,

$$\mathcal{L}^n(B(x, 2|x-y|) \cap H^-(x) \cap E) < \frac{\omega_n (2|x-y|)^n}{4^{2n+1}} \leq \frac{\omega_n |x-y|^n}{4^{n+1}}.$$

This is a contradiction. Thus

$$|P_{\nu_j}(y-x)| \leq |P_{\nu_j}^{\perp}(y-x)|$$

for all $x, y \in K_1$ with $|x-y| < \delta$.

Thus for any $z \in K_1$, $B(z, \delta/2) \cap K_1$ is the graph of a 1-Lipschitz map with domain in the $n-1$-plane $\nu_j^{\perp}$. This can be extended into a 1-Lipschitz graph S_1 defined on the whole of $\nu_j^{\perp}$.

Rectifiability of the reduced boundary part V

The set K_1 can be partitioned into finitely many sets contained in balls $B(z, \delta/2)$, $z \in K_1$, and the same can be done for each set K_i. Relabel the resulting sets H_i, $i \in \mathbb{N}$, so that each H_i is covered by a Lipschitz graph S_i.

Note that

$$\mathcal{H}^{n-1}\left(\mathcal{F}E \setminus \bigcup_{i\in\mathbb{N}} H_i\right) \leq C|D\chi_E|\left(\mathcal{F}E \setminus \bigcup_{i\in\mathbb{N}} H_i\right) = 0.$$

Thus the reduced boundary $\mathcal{F}E$ is countably $n-1$-rectifiable.

Let us show that $|D\chi_E| = \mathcal{H}^{n-1} \llcorner \mathcal{F}E$. For this it is enough to show that for any $i \in \mathbb{N}$, $|D\chi_E| \llcorner H_i = \mathcal{H}^{n-1} \llcorner H_i$. Again we can take $i = 1$.

Rectifiability of the reduced boundary part VI

We have

$$\lim_{r\to 0}\frac{\mathcal{H}^{n-1}(S_1\cap B(x,r))}{\omega_{n-1}r^{n-1}}=1$$

for $\mathcal{H}^{n-1}$-a.e. $x\in S_1$, since S_1 is a Lipschitz graph. Thus by (9),

$$\lim_{r\to 0}\frac{|D\chi_E|(B(x,r))}{\mathcal{H}^{n-1}(S_1\cap B(x,r))}=1$$

for $\mathcal{H}^{n-1}$-almost every $x\in H_1\subset\mathcal{F}E$. By (10) we know that

$$|D\chi_E|\llcorner H_1\ll\mathcal{H}^{n-1}\llcorner H_1\ll\mathcal{H}^{n-1}\llcorner S_1.$$

Thus by the Besicovitch differentiation theorem,

$$|D\chi_E|\llcorner H_1=(\mathcal{H}^{n-1}\llcorner S_1)\llcorner H_1=\mathcal{H}^{n-1}\llcorner H_1.$$

$\square$

Enlarged rationals

To illustrate how big the topological boundary of a set of finite perimeter can be compared to the reduced boundary, consider the following.

Example

Let (q_h) be an enumeration of $\mathbb{Q}^2$. Define

$$E := \bigcup_{h=1}^{\infty} B(q_h, 2^{-h}).$$

Then by subadditivity and lower semicontinuity

$$P(E, \mathbb{R}^2) \leq \sum_{h=1}^{\infty} P(B(q_h, 2^{-h}), \mathbb{R}^2) \leq 2\pi \sum_{h=1}^{\infty} 2^{-h} = 2\pi.$$

Thus E is of finite perimeter in $\mathbb{R}^2$. On the other hand, E is dense in $\mathbb{R}^2$, so that $\partial E = \mathbb{R}^2 \setminus E$. Thus $\mathcal{L}^2(\partial E) = \infty$.

The measure theoretic boundary part I

Definition

Let $E \subset \mathbb{R}^n$ be an $\mathcal{L}^n$-measurable set. We define the measure theoretic boundary $\partial^* E$ as the set of points $x \in \mathbb{R}^n$ for which

$$\limsup_{r \to 0} \frac{\mathcal{L}^n(B(x,r) \cap E)}{\mathcal{L}^n(B(x,r))} > 0$$

and

$$\limsup_{r \to 0} \frac{\mathcal{L}^n(B(x,r) \setminus E)}{\mathcal{L}^n(B(x,r))} > 0.$$

For a set of finite perimeter E, clearly $\mathcal{F}E \subset \partial^* E$. Moreover, we can show that $\partial^* E$ is a Borel set.

Theorem

Let E be a set of finite perimeter in $\mathbb{R}^n$. Then $\mathcal{H}^{n-1}(\partial^* E \setminus \mathcal{F}E) = 0$.

Proof. Consider a point $x \in \mathbb{R}^n$ where

$$\lim_{r \to 0} \frac{|D\chi_E|(B(x,r))}{r^{n-1}} = 0.$$

By the relative isoperimetric inequality, we then have

$$\frac{\min\{|B(x,r) \cap E|, |B(x,r) \setminus E|\}}{r^n} \leq \left(\frac{2C_P|D\chi_E|(B(x,r))}{r^{n-1}}\right)^{n/(n-1)}$$
$$\to 0 \quad \text{as } r \to 0.$$

Thus by continuity, either

$$\frac{|B(x,r)\cap E|}{r^n} \to 0 \qquad \text{or} \qquad \frac{|B(x,r)\setminus E|}{r^n} \to 0$$

as $r \to 0$. Thus $x \notin \partial^* E$. In conclusion, if $x \in \partial^* E$, then

$$\limsup_{r\to 0} \frac{|D\chi_E|(B(x,r))}{r^{n-1}} > 0.$$

Then by using a similar covering argument as before, from the fact that $|D\chi_E|(\partial^* E \setminus \mathcal{F}E) = 0$ we obtain that $\mathcal{H}^{n-1}(\partial^* E \setminus \mathcal{F}E) = 0$.

$\square$

Gauss-Green formula

We conclude that for any set E of finite perimeter in $\mathbb{R}^n$, we have

$$|D\chi_E| = \mathcal{H}^{n-1} \llcorner \mathcal{F}E = \mathcal{H}^{n-1} \llcorner \partial^* E$$

and thus

$$D\chi_E = \nu_E \mathcal{H}^{n-1} \llcorner \partial^* E$$

with $|\nu_E(x)| = 1$ for $\mathcal{H}^{n-1}$-a.e. $x \in \partial^* E$. Thus we obtain the following generalization of the Gauss-Green formula.

Theorem

For any set E that is of finite perimeter in Ω, we have

$$\int_E \operatorname{div} \psi \, d\mathcal{L}^n = -\int_{\partial^* E} \langle \psi, \nu_E \rangle \, d\mathcal{H}^{n-1} \quad \forall \psi \in [C_c^1(\Omega)]^n.$$

Approximate limits

Let $u \in \mathrm{BV}(\mathbb{R}^n)$. Define the lower and upper approximate limits of u for any $x \in \mathbb{R}^n$ by

$$u^\wedge(x) := \sup\left\{t \in \mathbb{R} : \lim_{r \to 0} \frac{\mathcal{L}^n(B(x,r) \cap \{u < t\})}{\mathcal{L}^n(B(x,r))} = 0\right\}$$

and

$$u^\vee(x) := \inf\left\{t \in \mathbb{R} : \lim_{r \to 0} \frac{\mathcal{L}^n(B(x,r) \cap \{u > t\})}{\mathcal{L}^n(B(x,r))} = 0\right\}.$$

Then define the approximate jump set of u by

$$S_u := \left\{x \in \mathbb{R}^n : u^\wedge(x) < u^\vee(x)\right\}.$$

We can show that $u^\wedge$ and $u^\vee$ are Borel measurable functions and that S_u is a Borel set.

Rectifiability of the jump set part I

Theorem

Let $u \in \mathrm{BV}(\mathbb{R}^n)$. Then the jump set S_u is countably $n-1$-rectifiable.

Proof. Let $x \in S_u$. Then for any $u^\wedge(x) < t < u^\vee(x)$, we have

$$\limsup_{r \to 0^+} \frac{\mathcal{L}^n(B(x,r) \cap \{u > t\})}{\mathcal{L}^n(B(x,r))} > 0$$

and

$$\limsup_{r \to 0^+} \frac{\mathcal{L}^n(B(x,r) \cap \{u < t\})}{\mathcal{L}^n(B(x,r))} > 0.$$

Thus $x \in \partial^* \{u > t\}$.

Rectifiability of the jump set part II

By the coarea formula, we can choose $D \subset \mathbb{R}$ to be a countable, dense set such that $\{u > t\}$ is of finite perimeter in $\mathbb{R}^n$ for every $t \in D$. We know that each reduced boundary $\mathcal{F}\{u > t\}$, and thus each measure theoretic boundary $\partial^*\{u > t\}$, $t \in D$, is a countably $n - 1$-rectifiable set. Thus

$$S_u \subset \bigcup_{t \in D} \partial^*\{u > t\}$$

is also countably $n - 1$-rectifiable. $\square$

Decomposition of the variation measure

Let $u \in \mathrm{BV}(\mathbb{R}^n)$. By the Besicovitch differentiation theorem, we have $|Du| = a\,\mathcal{L}^n + |Du|^s$, where $a \in L^1(\mathbb{R}^n)$ and $|Du|^s$ is singular with respect to $\mathcal{L}^n$.

Then we can further write $|Du|^s = |Du|^c + |Du|^j$, where $|Du|^c := |Du|^s \llcorner (\mathbb{R}^n \setminus S_u)$ is the *Cantor part*, and $|Du|^j := |Du|^s \llcorner S_u$ is the *jump part*.

Theorem

For $u \in \mathrm{BV}(\mathbb{R}^n)$, we have the decomposition

$$|Du| = a\,\mathcal{L}^n + |Du|^c + (u^\vee - u^\wedge)\,\mathcal{H}^{n-1} \llcorner S_u.$$

Moreover, for any Borel set $A \subset \mathbb{R}^n \setminus S_u$ that is σ-finite with respect to $\mathcal{H}^{n-1}$ (i.e. can be presented as a countable union of sets of finite $\mathcal{H}^{n-1}$-measure), we have $|Du|(A) = 0$.

Decomposition of the variation measure proof part I

Proof. We have already seen that if $x \in S_u$ and $u^\wedge(x) < t < u^\vee(x)$, then $x \in \partial^*\{u > t\}$. On the other hand, if $x \in \partial^*\{u > t\}$ for some $t \in \mathbb{R}$, then

$$\limsup_{r \to 0^+} \frac{\mathcal{L}^n(B(x,r) \cap \{u > t\})}{\mathcal{L}^n(B(x,r))} > 0,$$

whence $u^\vee(x) \geq t$, and

$$\limsup_{r \to 0^+} \frac{\mathcal{L}^n(B(x,r) \setminus \{u \leq t\})}{\mathcal{L}^n(B(x,r))} > 0,$$

whence $u^\wedge(x) \leq t$. In conclusion, $t \in [u^\wedge(x), u^\vee(x)]$.

Decomposition of the variation measure proof part II

All in all, we have

$$\begin{aligned}
&\left\{(x,t) \in \mathbb{R}^n \times \mathbb{R} : \; u^\wedge(x) < t < u^\vee(x)\right\} \\
&\qquad \subset \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : \; x \in \partial^* \{u > t\}\} \\
&\qquad\qquad \subset \left\{(x,t) \in \mathbb{R}^n \times \mathbb{R} : \; u^\wedge(x) \le t \le u^\vee(x)\right\}.
\end{aligned}$$

Thus for any Borel set $A \subset S_u$, we have by using the coarea formula and Fubini's theorem

$$\begin{aligned}
|Du|(A) &= \int_{-\infty}^{\infty} P(\{u > t\}, A)\, dt = \int_{-\infty}^{\infty} \mathcal{H}^{n-1}(\partial^*\{u > t\} \cap A)\, dt \\
&= \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} \chi_{\partial^*\{u>t\}}(x)\, d(\mathcal{H}^{n-1} \llcorner A)(x)\, dt \\
&= \int_{\mathbb{R}^n} \int_{-\infty}^{\infty} \chi_{(u^\wedge(x), u^\vee(x))}(t)\, dt\, d(\mathcal{H}^{n-1} \llcorner A)(x) \\
&= \int_A (u^\vee - u^\wedge)\, d\mathcal{H}^{n-1}.
\end{aligned}$$

Decomposition of the variation measure proof part III

We conclude that $|Du|^j = |Du| \llcorner S_u = (u^\vee - u^\wedge)\mathcal{H}^{n-1} \llcorner S_u$.

Finally, suppose that a Borel set $A \subset \mathbb{R}^n \setminus S_u$ is σ-finite with respect to $\mathcal{H}^{n-1}$. By using the coarea formula and Fubini's theorem as we did above,

$$|Du|(A) = \int_A (u^\vee - u^\wedge)\, d\mathcal{H}^{n-1} = 0$$

since $u^\wedge(x) = u^\vee(x)$ for any $x \in A$. $\square$

On measurability, part I

Let $E \subset \mathbb{R}^n$ be an $\mathcal{L}^n$-measurable set. To show that $\partial^* E$ is a Borel set, we note that for each $i \in \mathbb{N}$, the functions

$$f_i(x) := \frac{\mathcal{L}^n(B(x, 2^{-i}) \cap E)}{\mathcal{L}^n(B(x, 2^{-i}))}, \qquad g_i(x) := \frac{\mathcal{L}^n(B(x, 2^{-i}) \setminus E)}{\mathcal{L}^n(B(x, 2^{-i}))}$$

are continuous. Thus $\limsup_{i \to \infty} f_i$ and $\limsup_{i \to \infty} g_i$ are Borel measurable functions, and so

$$\partial^* E = \left\{ x \in \mathbb{R}^n : \limsup_{i \to \infty} f_i(x) > 0 \ \text{ and } \ \limsup_{i \to \infty} g_i(x) > 0 \right\}$$

is a Borel set.

On measurability, part II

To show that $u^\wedge$ and $u^\vee$ are Borel measurable functions, note that for any $t \in \mathbb{R}$,

$$\{x \in \mathbb{R}^n : u^\wedge(x) \geq t\} = \bigcap_{i=1}^{\infty} \left\{ x \in \mathbb{R}^n : \lim_{r \to 0} \frac{\mathcal{L}^n(B(x,r) \cap \{u < t - 1/i\})}{\mathcal{L}^n(B(x,r))} = 0 \right\}.$$

Here the functions

$$x \mapsto \frac{\mathcal{L}^n(B(x,r) \cap \{u < s\})}{\mathcal{L}^n(B(x,r))}$$

are continuous for any $s \in \mathbb{R}$ and fixed $r > 0$, and so

$$x \mapsto \limsup_{r \to 0} \frac{\mathcal{L}^n(B(x,r) \cap \{u < s\})}{\mathcal{L}^n(B(x,r))}$$

is a Borel measurable function. Hence $\{x \in \mathbb{R}^n : u^\wedge(x) \geq t\}$ is a Borel set. Borel measurability of $u^\vee$ is proved analogously.

On measurability, part III

By the Borel measurability of $u^\wedge$ and $u^\vee$, we have that

$$S_u = \left\{x \in \mathbb{R}^n : u^\wedge(x) < u^\vee(x)\right\}$$
$$= \bigcup_{t \in \mathbb{Q}} \left\{x \in \mathbb{R}^n : u^\wedge(x) < t\right\} \cap \left\{x \in \mathbb{R}^n : u^\vee(x) > t\right\}$$

is a Borel set. Then we can show that also

$$\left\{(x, t) \in \mathbb{R}^n \times \mathbb{R} : u^\wedge(x) < t < u^\vee(x)\right\}$$

is a Borel set in $\mathbb{R}^n \times \mathbb{R}$, justifying our previous use of Fubini's theorem.