Memory, Infrared Entanglement, and the Idealization of Scattering from Infinity

Robert M. Wald

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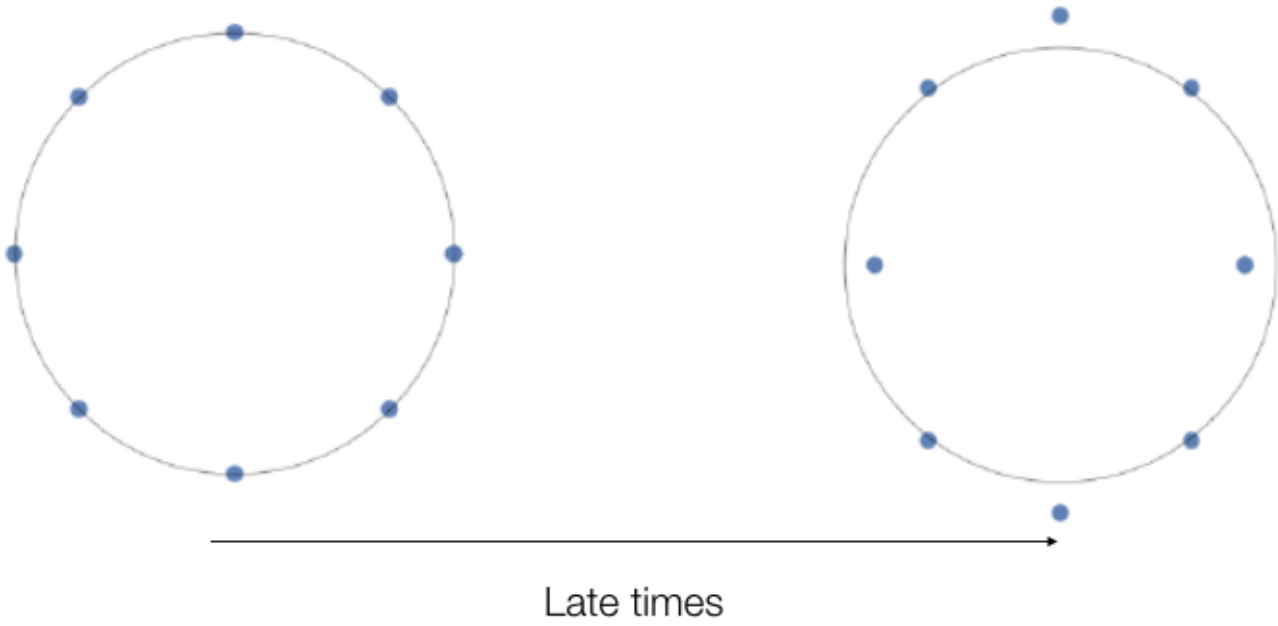
The Idealization of Scattering from Infinity

Although all experiments are done over finite regions of space and time, to describe scattering phenomena, it is very convenient to take idealized limits of fields/particles coming in from and going out to null or timelike infinity. In this limit, the particles/fields can be treated as "free" and the incoming and outgoing states thereby have a simple description.

This is not problematic in classical theory, but in quantum field theory with massless fields in 4 spacetime dimensions, one encounters infrared divergences. These infrared divergences are usually treated a nuisance with no observable consequences, and methods have been devised to evade them in the calculation of various quantities of interest such as inclusive cross-sections. However, in this talk, I will show that there are decoherence effects associated with entanglement with "soft particles" that may be negligible for laboratory experiments but are infinite in the idealized limit of scattering from infinity.

The Gravitational Memory Effect

In an asymptotically flat spacetime, the memory effect is the permanent relative displacement at $O(1/r)$ of an array of test particles after the passage of a burst of gravitational radiation.



Memory and Radiation

Introduce coordinates (u, r, x^A) where $u = t - r$ and x^A denote angular coordinates. We are interested in behavior in the limit as $r \to \infty$ at fixed u (i.e., future null infinity).
In d spacetime dimensions, radiation falls off as $1/r^{d/2-1}$, whereas Coulomb fields fall off as $1/r^{d-3}$. It is a generic feature of massless fields in even dimensions that there is a "memory effect": after a burst of radiation, the field at Coulombic order at late retarded times does not return to its early time value.
In 4 spacetime dimensions, radiative and Coulombic order are both $1/r$ and the memory effect is inextricably intertwined with radiation. This is the origin of infrared divergences in quantum field theory.

Gravitational and Electromagnetic Memory

Let $h_{\mu\nu}$ denote the deviation of the metric from $\eta_{\mu\nu}$ at $O(1/r)$. Then the memory tensor Δ_{AB} is given by

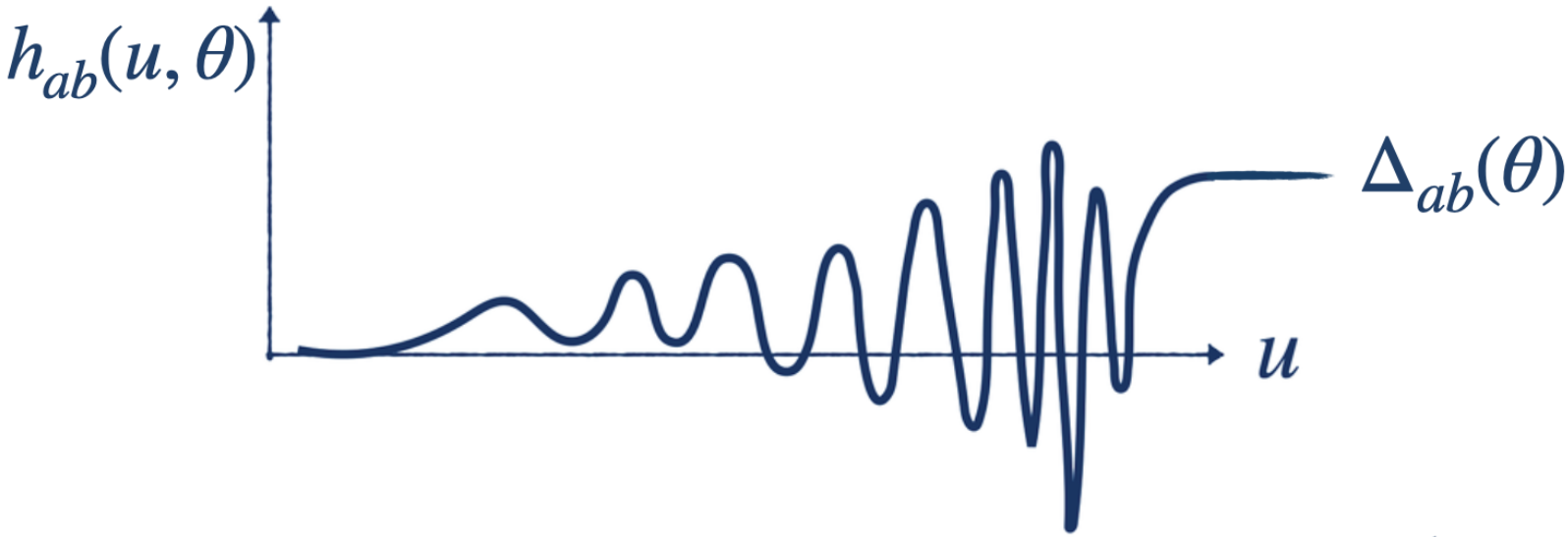
$$\Delta_{AB} = \frac{1}{2}\left(h_{AB}|_{i^+} - h_{AB}|_{i^0}\right) = \frac{1}{2}\int_{-\infty}^{\infty} N_{AB}\, du$$

where N_{AB} is the Bondi news.
Similarly, in electromagnetism in flat spacetime, let A_μ be the vector potential at $O(1/r)$ and let E_A be the angular components of the electric field at $O(1/r)$. Define the electromagnetic memory as

$$\Delta_A = -\int_{-\infty}^{\infty} E_A\, du = \left(A_A|_{i^+} - A_A|_{i^0}\right)$$

The presence of electromagnetic memory corresponds to a charged test particle receiving a net momentum kick.

Memory and Fourier Transforms



$$\Delta_{ab} \neq 0 \implies \Delta h_{ab} \neq 0 \implies \tilde{h}_{ab}(\omega,\theta) \sim \frac{1}{\omega}$$

$$\implies ||h||^2 \sim \int_0^\infty \frac{d\omega}{\omega}$$

Memory and Infrared Divergences

The "out" quantum states are determined by the "radiation field" h_{AB} at $O(1/r)$. The usual Fock space of such states is based upon a one-particle Hilbert space constructed by taking positive frequency data at $\mathscr{I}^+$ and using the Klein-Gordon (symplectic) norm

$$||h_{AB}||^2 = \int d\Omega \int_0^\infty d\omega \omega |\hat{h}_{AB}(\omega)|^2$$

But, as we have seen, for a state with memory, $\hat{h}_{AB}(\omega) \sim 1/\omega$ as $\omega \to 0$. States with memory have infinite norm in the usual Fock space. There is nothing physically or otherwise wrong with states with memory and, indeed, they unavoidably occur in scattering. But you will get divergences if you try to pretend that they are supposed to belong in the usual Fock space.

Memory Representations

States with memory are perfectly legitimate states. In the electromagnetic case, one can produce a Hilbert space representation consisting of states with memory Δ_A by starting with the usual Fock representation and replacing $\mathbf{A}_A$ by $\mathbf{A}_A + A_A\mathbf{I}$, where A_A is a classical field with memory Δ_A. (Note that A_A can be chosen to be of arbitrarily low frequency.) The corresponding Fock space representations $\mathcal{F}_\Delta$ with different Δ are unitarily inequivalent.

Similarly, in the gravitational case, one can obtain Hilbert space representations with memory by starting with the usual Fock representation and replacing $\mathbf{h}_{AB}$ by $\mathbf{h}_{AB} + H_{AB}\mathbf{I}$, where H_{AB} is a classical field with memory Δ_{AB}.

Asymptotic Symmetries, Memory and Gauge Charges: EM Case

The asymptotic conditions on the vector potential are preserved under $A_\mu \to A_\mu + \nabla_\mu \lambda$ where $\lambda = \lambda(x^A)$. Such "large gauge transformations" are actually symmetries—they are not degeneracies of the symplectic form. They have associated charges

$$\mathcal{Q}_u(\lambda) = \frac{1}{4\pi} \int_{S(u)} d\Omega \; \lambda(x^A) F^{(2)}_{ur}(u, x^A)$$

Memory corresponds to A_A at asymptotically late and early retarded times differing by a large gauge transformation. Have

$$\frac{1}{4\pi} \int_{S^2} d\Omega \; \Delta^{\text{out}}_A(x^A) \mathscr{D}^A \lambda = \mathcal{Q}_{i+}(\lambda) - \mathcal{Q}_{i0}(\lambda) + \mathcal{J}^{\text{out}}(\lambda)$$

where $\mathcal{J}^{\text{out}}$ is the integral of λ times the flux of charge through null infinity (which can be nonvanishing only if there are massless charged fields). $\mathcal{Q}_{i+}$ is determined by the outgoing massive charged particles.

Asymptotic Symmetries, Memory and Gauge Charges: Gravitational Case

The asymptotic conditions on the metric are preserved under BMS transformations; in particular, under supertranslations $u \to u + f(x^A)$. These are symmetries of the theory and have associated charges

$$\mathcal{Q}_u^{\rm GR}(f) = -\frac{1}{8\pi}\int_{S(u)} d\Omega\ f(x^A)\left[C^{(3)}_{urur}(u,x^A) - \frac{1}{4}N^{AB}h_{AB}\right]$$

Memory corresponds to having h_{AB} at asymptotically late and early retarded times differ by a supertranslation. Have

$$-\frac{1}{8\pi}\int_{S^2} d\Omega\ \Delta_{AB}^{\rm GR,out}\mathscr{D}^A\mathscr{D}^B f = \mathcal{Q}_{i^+}^{\rm GR}(f) - \mathcal{Q}_{i^0}^{\rm GR}(f) + \mathcal{J}^{\rm GR,out}(f)$$

where $\mathcal{J}^{\rm GR,out}(f)$ is the integral of the Bondi flux times f. $\mathcal{Q}_{i^+}^{\rm GR}$ is determined by the outgoing massive bodies.

Charges at Spatial Infinity

We have already seen that

$$\mathcal{Q}^{\text{out}}_{i^0} = \mathcal{Q}_{i^+}(\lambda) + \mathcal{J}^{\text{out}}(\lambda) - \frac{1}{4\pi}\int_{S^2} d\Omega\ \Delta^{\text{out}}_A(x^A)\mathscr{D}^A\lambda$$

The memory term generates large gauge transformations of the electromagnetic field at $\mathscr{I}^+$, whereas $\mathcal{Q}_{i^+}$ and/or $\mathcal{J}^{\text{out}}$ generate large gauge transformations of the charged field at i^+ and/or $\mathscr{I}^+$, so $\mathcal{Q}^{\text{out}}_{i^0}$ is the generator of large gauge transformations on all fields. Similar results hold for $\mathcal{Q}^{\text{in}}_{i^0}$. Similarly, in the gravitational case, the charges at spatial infinity generate supertranslations on all fields.

Under suitable asymptotic flatness assumptions, the large gauge charges at $\mathscr{I}^+$ and $\mathscr{I}^-$ match at i^0 under antipodal identification:

$$\mathcal{Q}^{\text{out}}_{i^0}(\lambda) = \mathcal{Q}^{\text{in}}_{i^0}(\lambda\circ\Upsilon)\,, \quad \mathcal{Q}^{\text{GR,out}}_{i^0}(f) = \mathcal{Q}^{\text{GR,in}}_{i^0}(f\circ\Upsilon)$$

where Υ is the antipodal map on S^2.

Charges at Spatial Infinity: Superselection

I will focus attention now on the electromagnetic case, returning to the gravitational case briefly near the end of the talk.

Local experiments performed in the bulk in the electromagnetic case should all be describable in terms of gauge invariant local observables (e.g., $F_{ab}(x)$, $J^a(x)$, smeared with test functions of compact support). However, "large gauge transformations" are indistinguishable from ordinary gauge transformations in a local region, so for any local gauge invariant observable, O, we have

$$[O, \mathcal{Q}_{i0}(\lambda)] = 0$$

Consequently, the eigensubspaces of large gauge charges define superselection sectors for the gauge invariant local observables, i.e., all states effectively decohere into eigenstates of large gauge charges. This is a familiar result for ordinary charge ($\lambda = 1$) but it also holds for large gauge charges of arbitrary λ.

Consequence of Superselection: Incoming "Bare Electrons" do Not Correspond to Bulk States!

Consider a single incoming "bare electron" state, i.e., an incoming state of a single electron in a normalized wave packet state (*not* a plane wave) with no incoming electromagnetic radiation. Then the large gauge charges at spatial infinity of the incoming state are then given by $\mathcal{Q}_{i0} = \mathcal{Q}_{i-}$. But $\mathcal{Q}_{i-}$ has continuous spectrum and its (improper) eigenstates are plane wave states $|p\rangle$. Consequently, with regard to local, gauge invariant bulk observables, there would be a complete decoherence into the "superselection sectors" of the large gauge charges, i.e., momentum eigenstates. This, in turn, implies a translation invariance of the state in the bulk. But there is no translation invariant one-electron state. This means that bare electron incoming wave packet states do not describe physically possible states in the bulk!

Incoming States that Do Correspond to Physical Bulk States: Faddeev-Kulish States

Since

$$\mathcal{Q}_{i0}(\lambda) = \mathcal{Q}_{i-}(\lambda) + \frac{1}{4\pi}\int \Delta_A^{\text{in}} \mathscr{D}^A \lambda d\Omega$$

we can get eigenstates of $\mathcal{Q}_{i0}$ by entangling each momentum mode of incoming electrons/positrons with incoming photon states of opposite memory. More precisely, the (improper) eigenstates of the charges $\mathcal{Q}_{i-}$ of n incoming massive charged particles are the plane wave states $|p_1 \dots p_n\rangle_{\text{in}}$. Pair these states with incoming photon states belonging to the representation with memory Δ_A^{in} such that for all $\lambda(x^A)$

$$-\frac{1}{4\pi}\int \Delta_A^{\text{in}} \mathscr{D}^A \lambda d\Omega = Q_{i-}[\lambda]$$

Then take the direct integral over the p_i's and the direct sum over n to get the full "in" Hilbert space. This is the Faddeev-Kulish construction and is described as *dressing the electrons*.

Understanding Faddeev-Kulish States: An Electron Released from a Trap

Suppose one releases an electron from a trap at some finite time and allows the electron wave packet to propagate freely to infinity. Prior to the release, there is no electromagnetic radiation present and the large gauge charges at spatial infinity vanish. However, some electromagnetic radiation will be emitted during the release. The longer the electron propagates, the more entangled it will become with soft radiation arising from its release. In the asymptotic limit to infinity, this will result in a complete decoherence of the electron due to entanglement with soft photons. The asymptotic state will be a Faddeev-Kulish state. However, as I will now illustrate, this decoherence will occur only logarithmically in time and should be completely negligible in the finite time required to do a real experiment. Thus, there is a major difference between the behavior of a free electron at "late times" relevant to a practical experiment as compared with the idealization of "infinitely late times" as used in scattering theory.

How Long Does it Take for Decoherence to Occur?

Suppose that at time $t = 0$ we separate an electron wave packet into two "beams" with relative velocity v. After letting the beams drift for time T (so they are a distance vT apart) we turn them around over a timespan $\tau \ll T$ and recombine them at $t = 2T$. How much decoherence has occurred?

From the Larmor formula, the energy radiated during the "turn-around" is $\sim (ev/\tau)^2\tau = (ev)^2/\tau$. This energy will mainly go into photons of frequency $1/\tau$, and the number of "entangling hard photons" will be $\langle N\rangle \sim (ev)^2$. We assume $ev \ll 1$ (and, in particular $v \ll 1$) so that this decoherence is negligible.

However, from the Lienard-Wiechert formula, one sees that the boosted Coulomb fields give an $\ell = 1$ contribution to A_A at order $1/r$ of order ev lasting for the time interval $2T$. The upshot is that the total number of entangling "soft photons" is

$$\langle N\rangle \sim e^2v^2\ln(T/\tau)$$

This is negligible for any laboratory experiment but is infinite in the idealization of scattering from infinity!

Why Does Scattering Theory Work at All?

At asymptotic infinity, the incoming (and outgoing) Faddeev-Kulish states have an infinite degree of entanglement between the charged particles and radiation. It would be utterly impossible to produce an incoming Faddeev-Kulish state from infinity. But "bare electrons" incoming from infinity do not describe physical states in the bulk at all!

However, the entanglement and decoherence that occur in scattering from infinity are not important for actual scattering experiments occurring over realistic time spans. For the calculation of, e.g., a cross-section, one can replace a finite time incoming electron wave packet by a plane wave. Since plane waves are eigenstates of $\mathcal{Q}_{i0}$, one can then pretend that it is incoming from infinity, i.e., scattering of plane wave "bare electron" states from infinity is well defined. The electromagnetic radiation resulting from the scattering will have memory, but if one is not interested in this, one can employ the usual procedures to obtain "inclusive cross-sections."

Massless Charged Fields

If the charged field is massless, then $\mathcal{Q}_{i^-} = 0$ but $\mathcal{J} \neq 0$. The (improper) eigenstates of $\mathcal{J}$ are the momentum eigenstates $|k_1 \ldots k_n\rangle_{\text{in}}$. So, again one could try to pair (or "dress") these states with incoming photon states belonging to the representation with memory Δ_A such that

$$\frac{1}{4\pi} \int \Delta_A \mathscr{D}^A \lambda d\Omega = \mathcal{J}(\lambda)$$

However, for the states $|k_1 \ldots k_n\rangle_{\text{in}}$, $\mathcal{J}$ has angular delta-function singularities. This implies that Δ_A—and, therefore, E_A—has sufficiently singular angular behavior ("collinear divergences") that the stress-energy tensor is singular. All of the states in the Faddeev-Kulish Hilbert space are unphysical! Quantum scattering theory from infinity for massless charged field probably does not make sense, even though classical scattering theory is not problematic.

Gravitational Case

In gravity, one could attempt to use a construction similar to the Faddeev-Kulish construction to "dress" incoming "hard gravitons." In fact, in gravity the collinear divergences are not as severe, and the required Faddeev-Kulish dressing is not as singular. However, the dressing consists of "soft gravitons" and although one can make the energy content of the dressing as small as one wishes, one cannot make it zero. These additional soft gravitons contribute to Bondi fluxes, which spoils the eigenstate property. One must therefore also "dress the dressing," etc. It can be directly proven that the only eigenstate of $\mathcal{Q}_{i0}^{\mathrm{GR}}(f)$ is the vacuum, i.e., apart from the vacuum state, there are no Faddeev-Kulish states at all—nor any other eigenstates of supertranslation charges $\mathcal{Q}_{i0}^{\mathrm{GR}}(f)$.
However, in full quantum gravity, it is not clear how to obtain gauge invariant local observables that would be subject to superselection. Thus, it is not obvious what criteria should be imposed on scattering states so as to correspond to physically relevant bulk states.

The Idealization of Scattering from Infinity

In classical field theory, one can take asymptotic limits of the massless/massive fields in the theory to null/timelike infinity. The behavior of these fields at infinity provides a good description of the behavior of these fields at the large but finite times relevant to experiments and observations.

However, this is not the case for quantum field theory in 4 dimensions with a massless field. Entanglement with the "soft modes" of the massless field grows (logarithmically) with time and becomes infinite in the asymptotic limit. The behavior of states at asymptotic infinity is very different from the behavior of states at the large but finite times relevant to experiments.

This fact is not of major consequence for the use of quantum field theory to do various "practical calculations," such as obtaining inclusive cross-sections at given 4-momenta. However, it may be of considerable significance for attempts to elevate scattering from infinity to a fundamental status in the formulation of the theory.