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Dynamic flapping in a moving fluid

Oliver Bond

UNIVERSITY OF OXFORD

ANIMAL DYNAMICS

Contents

1 Introduction

Aquatic propulsion has been studied in considerable detail over the last 80 years, in a bid to replicate the highly evolved ways in which fish and aquatic mammals propel themselves efficiently through water for our own purposes. Fish have evolved over hundreds of millions of years into very efficient swimmers, both for steady swimming and for dodging predators. Fish can swim at remarkably high speeds. For example, some species of mackerel can reach speeds of 70 miles per hour, while the top speed of submarines is only 50 miles per hour. Animal Dynamics have developed a robot fish, based on a tuna, which consists of several rigid parts allowed to rotate around hinges. A motor inside the robot spins and causes it to imitate a flapping motion just like a real fish.

How do fishes swim?

In the study of fish swimming, there are two main theories:

- **Resistive theory:** The thrust on a fish comes from the drag forces acting on it. This idea was proposed by Taylor [1] who approximated the drag on a very long and thin fish by using measurements of drag forces on a cylinder angled inside a wind tunnel. This is a commonplace theory for describing the fluid flow around microscopic organisms and this is on a very different scale to that of a fish.
- **Reactive theory:** The thrust on a fish comes from the force acting perpendicular to it from the fluid as it passes a travelling wave down its spine. The fish thrusts the fluid backwards and sideways, and by Newton's third law, the fish experiences a forward component of the force from the fluid, pushing it through the water. This theory was pioneered by Lighthill [2], and most of the study of the mathematics of fish swimming is based on reactive theories.

Observing the flow

Despite the development of hydrodynamic theories of fish swimming, observing the actual fluid dynamics around a fish was problematic and it was not until the 1990s that technology had developed enough to obtain concrete experimental results. It remains very challenging to design an environment in which a fish can perform in an experiment, since fish are very difficult to control. There are several ways in which scientists have observed or simulated fluid flow around a fish:

1. **Digital particle image velocimetry (DPIV)** - this is a technique in which tiny tracer particles are added to a body of water, and their positions are tracked over a very narrow time interval using a laser; a computer then processes the images and shows velocity of the fluid at a large number of points.
2. **Computational fluid dynamics (CFD)** - this involves using programmed algorithms and software packages to simulate the physics of the fluid around objects.
3. **Biomimetic robots** - this involves designing robots based on aquatic wildlife with characteristics that can easily be designed, adjusted and measured.

Animal Dynamics's *Malolo*

Animal Dynamics have created a 3D-printed robot fish called *Malolo*. It is modelled on a tuna and is shown in Figure 1. During experiments, it swims with a specially designed wetsuit in order to avoid potential damage to the components. It consists of three rigid parts connected by hinges, around which each part can turn. Towards the rear of Malolo there is a motor fixed inside which oscillates in such a way as to provide Malolo with a "flapping" motion. The engineers at Animal Dynamics have total control over how they can make the motor oscillate, and one of their goals is to discover how to do this to best improve its swimming efficiency. Our aim is to develop a simple model for a swimming fish that is directly inspired by the design of Malolo, in order to examine how the oscillations propel Malolo forward.

Malolo was made using a 3D printer and swims with a tailor-made wetsuit.

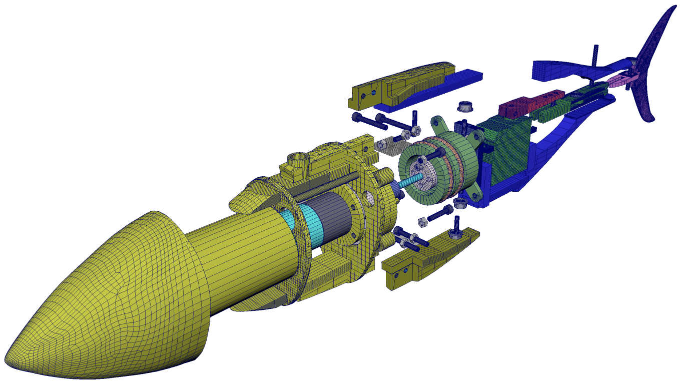


Figure 1 – An exploded CAD diagram of Malolo, the robotic tuna created by Animal Dynamics.

2 The model

Initial setup

In Figure 2, we show a simplified model of a fish which is comprised of two rods connected by a hinge, at which there are reaction forces (shown in green) which are equal and opposite (and whose directions are arbitrary and not important for our analysis). Each rod in the fish has a given mass and length, and each rod is assumed to be uniform so that their centres of mass lie at their midpoints. We define the angle α to be the anticlockwise angle that the first rod makes with the oncoming stream of water (which we align with the x-axis), and β is defined in the same way for the second rod, with angle $\delta = \beta - \alpha$ being the relative angle between the rods. The coordinates in Figure 2 are fixed, as in a laboratory, since the oncoming stream of water is always in the same direction.

Our model captures the essence of Malolo's structure.

Although the geometrical model we have proposed does not look much like a fish, starting with a simple model is necessary in order that we can make progress with understanding the swimming motion. Once results are obtained, the model can then be tested against reality and the model refined if necessary.

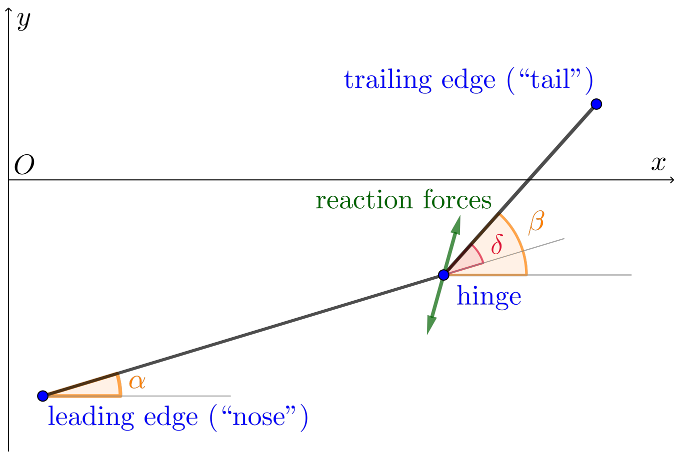


Figure 2 – A diagram of a simplified model of a fish in water (as seen from above).

Modelling assumptions

To simplify the physics of the fish, we make the following assumptions:

- all motion (fluid and fish) is restricted to two dimensions;
- the fish is represented by a pair of connected rods;
- each rod is uniform (has constant density);
- each rod is thin, so that the thickness can be ignored;
- each rod is inextensible so that the tension force is constant;
- the rods intersect at a single point, with a frictionless hinge which allows the rods to rotate about a normal axis going through this point;
- gravity can be neglected.

We couple the mechanics of the fish with the mechanics of the surrounding fluid.

Fish mechanics

To describe the mechanics of the fish, we apply *Newton's second law* which says that the net force acting on an object is equal to its mass multiplied by its acceleration. Neglecting tethering forces that arise from prescribing the motion of the head, the net force acting on the fish is due to the hydrodynamic force arising from the pressure jump across the fish. This force generates the acceleration of the centre of mass of the fish, and therefore its tendency to move as a whole. We also consider the overall rotational motion of the fish which is driven by a hydrodynamic torque. We use this fact to obtain an equation describing the conservation of angular momentum around the fish, and therefore its tendency to rotate.

Knowing both of the angles α and β at any given time, the length of each rod, and how far along the fish a specific point lies allows us to determine where any point on the fish is at any given moment by using simple trigonometry. We anticipate that the angle δ can be controlled by the fish; in the mechanical design, δ is controlled by a motor applying a torque at the hinge.

Fluid mechanics

We make the following assumptions about the water surrounding the fish.

- The water is **inviscid** (so it has no viscosity).
- The water is **incompressible** (so it cannot be compacted into a smaller volume).
- The water is **irrotational** (so the fluid experiences no local rotation).

On the scale of a tuna, the viscous effects of the water are negligible and so it can be treated as inviscid.

We impose several conditions to be able to solve the equations to do with the fluid mechanics. Firstly, we switch to a reference frame moving with the fish in which it appears the fish is cruising in a stream of constant speed with small fluid perturbations around the fish. Second, we enforce a **kinematic boundary condition** on the fish, in which the velocities of the fluid and the fish are the same in the normal direction. Lastly, we require that the fluid velocity at the tip of the tail of the fish is finite. This is called a **Kutta condition**. We make the following assumptions about the water surrounding the fish.

Modelling the vortices

As well as modelling what happens to the water around the fish, it is important to model what happens to the wake of the fish (the trail left behind it). The wake may contain what are called **vortices**, which are swirling areas of fluid. These can arise when one part of a fluid moves very quickly compared to another nearby part of fluid. Even though the water is assumed to be inviscid, we assume that there is a **vortex sheet** coming from the tail of the fish. A vortex sheet is a curve within a fluid on either side of which the fluid moves with different tangential speeds.

We suppose that there are two vortex sheets: the **bound vortex sheet** which runs along the length of the fish, and a **free vortex sheet** which propagates in the water from the tail up until a certain point behind the fish, as shown in Figure 3.

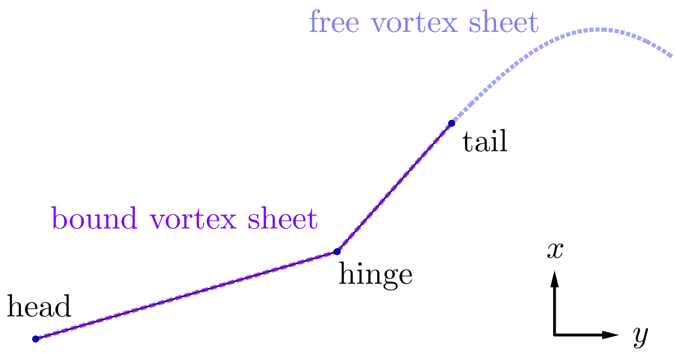


Figure 3 – A diagram showing the vortex sheets in our model.

To solve for the vorticity in these sheets, we need to impose several conditions. One of them comes from Kelvin's circulation theorem, which tells us that the total amount of **circulation** in a part of fluid does not change through time as it travels with the fluid. As a consequence, when the vortex sheet detaches from the trailing edge, it traces a curved path behind the fish and its vorticity no longer depends on how far away it has travelled from the fish. By looking at motion along the fish, we also obtain an equation describing how the bound vortex sheet strength changes over time in terms of the change in pressure from one side of the fish to the other.

Simplifying the analysis

We suppose that the two rods have the same mass and length. We follow Alben [3] and make the following simplifications.

- **Linearisation:** we assume that all angles in the system are made to be small enough so that $\sin\theta \approx \theta$ and that $\cos\theta \approx 1$. Physically, this means that the fish undergoes small oscillations.
- **Nondimensionalisation:** we scale all the variables in the system so that we can determine the key parameter groups that control the swimming behaviour.
- **Time-periodic solutions:** we assume that all the variables oscillate at a constant frequency, but may have amplitude which changes depending on this frequency.

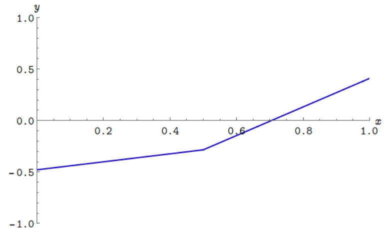
Summary of unknowns and equations

In our model, the unknowns are the vertical position of the head, the angle that the first rod makes with the horizontal, the total circulation around the fish, the vortex sheet strength, and the pressure jump across the fish. The motion of the fish and the fluid are then captured in a system of five equations, representing a global force balance in the normal direction, a global torque balance, Kelvin's circulation theorem, a vortex sheet evolution equation and the kinematic boundary condition.

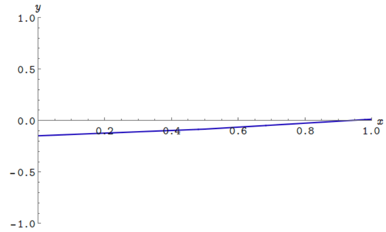
3 Results

Our model demonstrates flapping motion, just like that of a real fish.

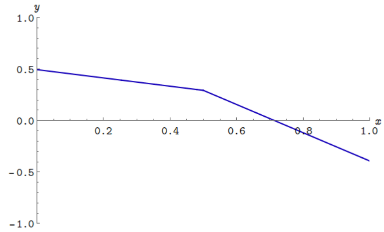
We solved the model to obtain an explicit solution for the fish motion, which we present in figure 4 for various points in the flapping cycle. This motion agrees with that commonly seen in experiments. We find that there is a point on the fish, called a *hydrodynamic pivot* that remains fixed in space throughout its motion. Our solution predicts that this point is near the tail of the fish.



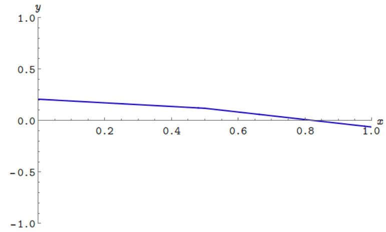
(a) $t = 0$



(b) $t = T/4$



(c) $t = T/2$



(d) $t = 3T/4$

Figure 4 – Graphs showing the motion of the fish in the case of (dimensionless) angular frequency equal to 6π, at (a) its initial position, (b) a quarter of the way through its flapping cycle, (c) half of the way through its flapping cycle, and (d) three-quarters of the way through its flapping cycle, where T denotes the flapping period (how long it takes for the fish to complete a flapping cycle).

We also find that the vortex sheet strength and the pressure jump get infinitely big half-way along the fish and thus do not smoothly change around the hinge. This is to be expected, since the kinematic condition is discontinuous at the hinge. Our results highlighted that we need to find a way to "smooth out" the hinge so that the vortex sheet strength does not blow up there.

4 Discussion, conclusions, & recommendations

We have built a mathematical model of a swimming fish in which we assumed that the fish comprised of two rigid oscillating rods. We worked in a frame in which the position of the oscillating fish remains fixed in a steady stream of fluid. Our model involved five coupled equations for the vertical position of the fish head, the angle the first rod makes with the horizontal, the total circulation around the fish, the vortex sheet strength, and the pressure jump across the fish. We solved our model explicitly to obtain time-periodic solutions. We found that our model reproduced some experimentally observed swimming behaviour, particularly the flapping of the spine and the hydrodynamic pivot (although this has been observed to be significantly closer to the nose than the tail in practice). Despite being very simple, our model has many aspects that are of interest to Animal Dynamics, who can prescribe data such as the flapping profile of the motor, and the masses and lengths of the oscillating parts. This information can be fed into our mathematical model and used to simulate the motion of the fish and fluid.

A natural future step would be to use three rods with a motor attached. It might also be possible to model the fish as an elastic beam which is stiff along the body parts but very flexible at the hinges. Further, we could consider a mathematical model which involves several rods with a flexible tail attached to the end (a bit like a flipper). Researchers at Animal Dynamics have also suggested extending the model to an arbitrary number of rods, each with the scope to have more than one motor applying a turning effect. A key goal in the future is to develop a performance metric, for example, a modified version of *Froude efficiency*, which depends on the mean swimming speed of the fish, the power input into the swimming motion, and the thrust generated. A modified version of this can be used to formulate an optimisation problem, which we could then use to decide how heavy and how long each part of the robot should be in order to make swimming as efficient as possible at a range of speeds.

Our model can be readily adapted to include more rods and motors to resemble future iterations of Malolo and ascertain how efficiently it swims.

5 Potential Impact

Our model can be used by Animal Dynamics to predict the swimming motion of their robot fish, Malolo. Our model provides a framework from which to expand and answer questions such as:

1. *How should the motor oscillate in order to cause Malolo to swim at different speeds?*
2. *How does the hydrodynamic thrust depend on the motor speed?*
3. *How efficient is the current 3-rod Malolo at swimming?*
4. *Do the rods in Malolo oscillate out of phase, and if so, how do the input parameters affect this oscillation?*

Dr Jay Willis, Senior Systems Design Lead at Animal Dynamics, says, *"The key issue is that, while there have been many theories about fish swimming, there have been few, if any, demonstrations of a physical pragmatic result. This mathematical project is aimed at improving the most advanced theories by confronting them with physical results. We suspect that swimming animals (and plants!) have developed some highly desirable capabilities while travelling huge distances quickly at sea. The predominant ones are efficiency, speed, and quietness. We are inspired by these natural systems to build our own super-efficient and quiet marine vehicles – if we can make a step change in efficiency when compared with a propeller we can change the world for the better.*

We need to broadly understand the domain of the model, so find local optima that we can design the robot fish toward. This includes things like size of body in relation to tail, position of hinge in body, shape of body, number of body segments, flexibility of tail. We need to look at flexibility of sprung joints, torque, efficiency, speed of travel, and speed of motor, and so on. We can calibrate and validate the mathematical model with physical changes to the robot fish (configured as a scientific instrument). Without a mathematical model we would be testing blind, or hoping that we understand the optima in fish design that has been developed through evolution – but it is not always clear what the constrains and targets of evolution have been. So we need a theoretical basis for our tests and a model to refine so that we can scale to large expensive designs (10m plus in linear dimension) with which our experimentation of basic design will be limited.

The impact already has been to provide a focus to identify these requirements as tangible questions and to gauge the difficulty and cost of developing answers. It has been valuable to get a new perspective on the problem and a new appreciation of the challenges and the available resources in the literature."

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