STRICTLY HYPERBOLIC CONSERVATION LAWS

$$\begin{cases} \partial_t U + \partial_x F(U) = 0, & -\infty < x < \infty,\ 0 < t < \infty \\ U(x,0) = U_0(x), & -\infty < x < \infty \end{cases}$$

$$DF(U):\ \lambda_1(U) < \cdots < \lambda_n(U), \quad R(U) = [R_1(U), \dots, R_n(U)]$$

$$TV_{(-\infty,\infty)} U_0(\cdot) << 1 \implies \exists \text{ unique global BV solution}$$

$$TV_{(-\infty,\infty)} U(\cdot,t) \le c\, TV_{(-\infty,\infty)} U_0(\cdot)$$

GLIMM, LIU, BRESSAN, BIANCHINI

INHOMOGENEOUS HYPERBOLIC BALANCE LAWS

$$\begin{cases} \partial_t U + \partial_x F(U,x,t) + G(U,x,t) = 0, & -\infty < x < \infty,\ 0 < t < \infty \\ U(x,0) = U_0(x) & -\infty < x < \infty \end{cases}$$

$TV_{(-\infty,\infty)} U_0(\cdot) \ll 1 \Rightarrow \exists$ *local* BV solution

$$TV_{(-\infty,\infty)} U(\cdot,t) \le c\, e^{\alpha t}\, TV_{(-\infty,\infty)} U_0(\cdot)$$

HYPERBOLIC BALANCE LAWS WITH DAMPING

$$\begin{cases} \partial_t U + \partial_x F(U) + G(U) = 0, & -\infty < x < \infty,\ 0 < t < \infty \\ U(x,0) = U_0(x), & -\infty < x < \infty \end{cases}$$

$$A = R^{-1}(0)\, DG(0)\, R(0), \qquad A_{ii} > \sum_{j \neq i} |A_{ji}|, \quad i = 1, \dots, n$$

$TV_{(-\infty,\infty)} U_0(\cdot) \ll 1 \Rightarrow \exists$ unique global BV solution

$$TV_{(-\infty,\infty)} U(\cdot, t) \leq c\, e^{-\mu t}\, TV_{(-\infty,\infty)} U_0(\cdot)$$

DAFERMOS–HSIAO, AMADORI–GUERRA, CHRISTOFOROU

NONHOMOGENEOUS BALANCE LAWS

$$\begin{cases} \partial_t U + \partial_x F(U, x, t) + G(U, x, t) = 0, & -\infty < x < \infty,\ 0 < t < \infty \\ U(x,0) = U_0(x), & -\infty < x < \infty \end{cases}$$

$$A(x,t) = R^{-1}(0,x,t)\, DG(0,x,t)\, R(0,x,t), \quad A_{ii}(x,t) > \sum_{j \neq i} |A_{ji}(x,t)|, \quad i = 1, \dots, n$$

$$\int_{-\infty}^{\infty} \{ |F_x(0,x,t)| + |G_x(0,x,t)| \} \, dx < \delta, \qquad 0 \le t < \infty$$

$TV_{(-\infty,\infty)} U_0(\cdot) << 1 \implies \exists$ global BV solution

$$TV_{(-\infty,\infty)} U(\cdot, t) \le c e^{-\mu t} TV_{(-\infty,\infty)} U_0(\cdot) + c\delta$$

DIAGONAL DOMINANCE FAILS

Friction: $\begin{cases} u_t - v_x = 0 \\ v_t - f(u)_x + v = 0 \end{cases}$ $\quad f'(u) > 0$

Relaxation: $\begin{cases} u_t - v_x = 0 \\ v_t - f(u)_x + \mu[v - g(u)] = 0 \end{cases}$ $\quad |g'(u)| < \sqrt{f'(u)}$, $\quad \mu > 0$

Internal variable $\begin{cases} u_t - v_x = 0 \\ v_t - f(u)_x + p_x = 0 \\ p_t + \gamma p - u = 0 \end{cases}$ $\quad f'(u) > 0$, $\quad \gamma > 1$

REDISTRIBUTION OF DAMPING

$$\begin{cases} u_t - v_x = 0 \\ v_t - f(u)_x + v = 0 \end{cases}$$

$$\varphi_x = u, \quad \varphi_t = v, \quad w = v + \tfrac{1}{2}\varphi$$

$$\begin{cases} u_t - w_x + \tfrac{1}{2}u = 0 \\ w_t - f(u)_x + \tfrac{1}{2}w = \tfrac{1}{4}\varphi \end{cases}$$

Need: $\displaystyle\int_{-\infty}^{\infty} |\varphi_x(x,t)|\,dx = \int_{-\infty}^{\infty} |u(x,t)|\,dx < \delta$

$$\begin{cases} u_t - v_x = 0 \\ v_t - f(u)_x + p_x = 0 \\ p_t + \gamma p - u = 0 \end{cases} \qquad f'(0) = 1, \ \gamma > 1$$

$\varphi_x = u$, $\varphi_t = v$, $\psi_x = v$, $\psi_t = f(u) - p$, $w = v + \frac{1}{2}\varphi$, $z = p - \psi$

$$\begin{cases} u_t - w_x + \frac{1}{2}u = 0 \\ w_t - f(u)_x + z_x + \frac{1}{2}w = \frac{1}{4}\varphi \\ z_t + (\gamma - 1)z + h(u) = -(\gamma-1)\psi \end{cases} \qquad h(u) = f(u) - u = O(u^2)$$

Need: $\int_{-\infty}^{\infty} [|\varphi_x(x,t)| + |\psi_x(x,t)|]\,dx = \int_{-\infty}^{\infty} [|u(x,t)| + |v(x,t)|]\,dx < \delta$

REDISTRIBUTION OF DAMPING IN GENERAL SYSTEMS OF BALANCE LAWS

$$\partial_t U + \partial_x F(U) + G(U) = 0$$

$$A = R^{-1}(0)\, DG(0)\, R(0)\,, \quad A_{ii} > 0\,, \quad i = 1, \dots, n$$

$$M_{ii} = 0\,, \quad i = 1, \dots, n\,, \quad M_{ij} = \frac{A_{ij}}{\lambda_j(0) - \lambda_i(0)}\,, \quad i \neq j$$

$$N = R(0)\, M\, R^{-1}(0)$$

$$P(x,t) = \int_{-\infty}^{x} N\, U(y,t)\, dy\,, \quad Q(x,t) = \int_{-\infty}^{x} N\, G(U(y,t))\, dy$$

$$\partial_t U + \partial_x F(U) + G(U) = 0$$

Change of variable: $W(x,t) = U(x,t) - P(x,t)$

$$\partial_t W + \partial_x \hat{F}(W, P(x,t)) + \hat{G}(W, P(x,t), Q(x,t)) = 0$$

$$\hat{F}(W,P) = F(W+P) - F(P)$$

$$\hat{G}(W,P,Q) = G(W+P) - NF(W+P) + DF(P)N[W+P] - Q$$

$$R^{-1}(0,0)\, D_W \hat{G}(0,0,0)\, R(0,0) = \mathrm{diag}\{A_{11}, \dots, A_{nn}\}$$

Need: $\int_{-\infty}^{\infty} |U(x,t)|\, dx < \delta$

L^1 BOUND VIA SPECIAL ENTROPY

$$\partial_t U + \partial_x F(U) + G(U) = 0$$

$$\partial_t \eta(U) + \partial_x q(U) + h(U) \leq 0$$

$$Dq(U) = D\eta(U)\, DF(U), \quad h(U) = D\eta(U)\, G(U)$$

$$\eta(U) \text{ convex}, \qquad h(U) \geq 0$$

$$\alpha^{-1}|U| \leq \eta(U) \leq \alpha |U|$$

$$\int_{-\infty}^{\infty} |U(x,t)|\, dx \leq \alpha^2 \int_{-\infty}^{\infty} |U_0(x)|\, dx$$

L^1 BOUND VIA COMMON ENTROPY

Assume: $\begin{cases} (\eta, q) \text{ entropy-entropy flux pair} \\ D^2\eta(0) \text{ positive definite} \\ D^2\eta(0)\, DG(0) \text{ positive definite} \end{cases}$

Then: $A_{ii} > 0 \,, \quad i = 1, \dots, n$

Assume: $\displaystyle\int_{-\infty}^{\infty} (1+|x|)^{2\sigma} |v_0(x)|^2 dx = \delta^2 \,, \qquad \sigma > 1$

Then: $\displaystyle\int_{-\infty}^{\infty} |v(x,t)|\, dx \leq c\delta$

$$R^T(0)\, D^2\eta(0)\, R(0) = I$$

$$A = R^{-1}(0)\, DG(0)\, R(0)$$

$$A = R^T(0)\, D^2\eta(0)\, DG(0)\, R(0)$$

$$\therefore \quad A_{ii} > 0, \quad i = 1, \dots, n$$

Normalize: $\quad \eta(0) = 0, \quad g(0) = 0, \quad D\eta(0) = 0$

$$\alpha^{-1}|U|^2 \leq \eta(U) \leq \alpha|U|^2$$

$$|q(U)| \leq \lambda\,\eta(U)$$

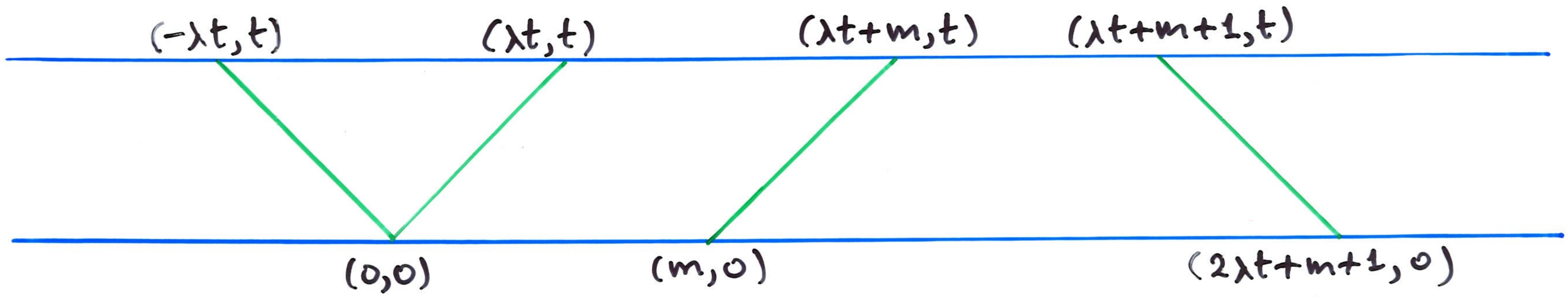
$$D\eta(U)G(U) \geq \beta^{-1}|U|^2$$

$$\int_{-\infty}^{\infty} \eta(U(x,t))\,dx + \int_0^t \int_{-\infty}^{\infty} h(U(x,\tau))\,dx\,d\tau \leq \int_{-\infty}^{\infty} \eta(U_0(x))\,dx$$

$$\int_0^t \int_{-\infty}^{\infty} \eta(U(x,\tau))\,dx\,d\tau \leq \alpha^2\beta\delta^2$$

$$t\int_{-\infty}^{\infty} |U(x,t)|^2\,dx \leq \alpha^3\beta\delta^2$$

$$\int_{-\lambda t}^{\lambda t} |U(x,t)|\,dx \leq (2\lambda\alpha\beta)^{1/2}\alpha\delta$$



$$\int_{\lambda t+m}^{\lambda t+m+1} |v(x,t)|^2\,dx \le \alpha^2 \int_m^\infty |v_0(x)|^2\,dx \le \alpha^2 (1+m)^{-2\sigma} \int_m^\infty (1+x)^{2\sigma} |v_0(x)|^2\,dx$$

$$\int_{\lambda t+m}^{\lambda t+m+1} |v(x,t)|\,dx \le (1+m)^{-\sigma} \alpha\delta$$

$$\int_{\lambda t}^{\infty} |v(x,t)|\,dx \le \zeta(\sigma)\,\alpha\delta\,, \qquad \int_{-\infty}^{-\lambda t} |v(x,t)|\,dx \le \zeta(\sigma)\,\alpha\delta$$

$$\int_{-\infty}^{\infty} |v(x,t)|\,dx \le \left[(2\lambda\alpha\beta)^{1/2} + 2\zeta(\sigma)\right]\alpha\delta$$

OTHER SYSTEMS

$$
\begin{cases}
u_t(x,t) - v_x(x,t) = 0 \\[2ex]
v_t(x,t) - f(u(x,t))_x = \displaystyle\int_0^t a'(t-\tau)\, g(u(x,\tau))_x \, d\tau
\end{cases}
$$

HEAT FLOW WITH MEMORY

$$\theta_t(x,t) = \int_0^t a(t-\tau)\, f(\theta_x(x,\tau))_x \, d\tau$$

$$\theta_x = u, \quad \theta_t = v$$

$$\begin{cases} u_t(x,t) - v_x(x,t) = 0 \\ v_t(x,t) - f(u(x,t))_x = \int_0^t a'(t-\tau) f(u(x,\tau))_x \, d\tau \end{cases}$$

$$a(t) > 0, \quad a'(t) < 0, \quad a''(t) > 0, \quad [\log |a'(t)|]'' > 0$$

$$a(t) \in L^1(0,\infty), \quad t\,a(t) \in L^1(0,\infty)$$

$$\begin{cases} u_t(x,t) - v_x(x,t) = 0 \\ v_t(x,t) - f(u(x,t))_x = \int_0^t a'(t-\tau) f(u(x,\tau))_x \, d\tau \end{cases} \qquad -\infty < x < \infty, \; 0 < t < \infty$$

$$u(x,0) = u_0(x) = \theta_{0x}(x), \quad v(x,0) = 0, \qquad -\infty < x < \infty$$

Assume: $\int_{-\infty}^{\infty} \theta_0^2(x)\,dx \le \delta^2$, $\int_{-\infty}^{\infty} (1+|x|)^{2\sigma} u_0^2(x)\,dx \le \delta^2$

$$TV_{(-\infty,\infty)} u_0(\cdot) \le \delta$$

Then: $\exists$ global BV solution (u,v)

$$TV_{(-\infty,\infty)} u(\cdot,t) + TV_{(-\infty,\infty)} v(\cdot,t) \le c\,\delta$$

VISCOELASTICITY

$$\begin{cases} u_t(x,t) - v_x(x,t) = 0 \\ v_t(x,t) - f(u(x,t))_x = \int_0^t a'(t-\tau)\, g(u(x,\tau))_x \, d\tau \end{cases}$$

$$f'(0) = g'(0) = 1$$

$$a(t) > 0, \quad a'(t) < 0, \quad a''(t) > 0$$

$$a(t) \in L^1(0,\infty), \quad a(0) < 1$$

REDUCTION TO INTERNAL VARIABLE

$$\begin{cases} u_t(x,t) - v_x(x,t) = 0 \\ v_t(x,t) - f(u(x,t))_x = -\int_0^t e^{-\gamma(t-\tau)} u_x(x,\tau)\,d\tau, \quad \gamma > 1 \end{cases}$$

$$p(x,t) = \int_0^t e^{-\gamma(t-\tau)} u(x,\tau)\,d\tau$$

$$\begin{cases} u_t(x,t) - v_x(x,t) = 0 \\ v_t(x,t) - f(u(x,t))_x + p_x(x,t) = 0 \\ p_t(x,t) + \gamma p(x,t) - u(x,t) = 0 \end{cases}$$

$$\begin{cases} u_t(x,t) - v_x(x,t) = 0 \\ v_t(x,t) - f(u(x,t))_x + p_x(x,t) = 0, & -\infty < x < \infty,\ 0 < t < \infty \\ p_t(x,t) = u(x,t) - \gamma p(x,t) \end{cases}$$

$$u(x,0) = u_0(x),\ v(x,0) = v_0(x),\ p(x,0) = p_0(x),\quad -\infty < x < \infty$$

Assume: $\displaystyle\int_{-\infty}^{\infty} (1+|x|)^{2\sigma} [u_0^2(x) + v_0^2(x) + p_0^2(x)]\, dx \le \delta^2$

$$TV_{(-\infty,\infty)} u_0(\cdot) + TV_{(-\infty,\infty)} v_0(\cdot) + TV_{(-\infty,\infty)} p_0(\cdot) \le \delta$$

Then: $TV_{(-\infty,\infty)} u(\cdot,t) + TV_{(-\infty,\infty)} v(\cdot,t) + TV_{(-\infty,\infty)} p(\cdot,t) \le c\delta$

ENTROPY ESTIMATE

$$\partial_t \eta(u,v,p) + \partial_x q(u,v,p) + r(u,p) \leq 0$$

$$\eta = \int^{u} f(\xi)\,d\xi + \frac{1}{2}v^2 - up + \frac{1}{2}\gamma p^2$$

$$q = -v[f(u) - p]$$

$$r = (u - \gamma p)^2$$

$$\int_{-\infty}^{\infty} [u^2(x,t) + v^2(x,t) + p^2(x,t)]\,dx \leq c\delta^2$$

$$\int_0^{\infty}\int_{-\infty}^{\infty} [u(x,t) - \gamma p(x,t)]^2\,dx\,dt \leq c\delta^2$$

$$\psi(x,t) = \int_{-\infty}^{x} v(y,t)\,dy$$

$$\omega(x,t) = \int_{-\infty}^{x} u_0(y)\,dy + \int_0^t v(x,\tau)\,d\tau$$

$$h(u) = f(u) - u = O(u^2)$$

$$\partial_t H(\psi,\omega,p) + \partial_x Q(\psi,\omega) + R(u,p,\psi) = 0$$

$$H = \omega^2 + \frac{1}{\gamma}p^2 + \frac{1}{\gamma(\gamma-1)}[\gamma\psi - p]^2$$

$$Q = -2\psi\omega$$

$$R = 2p^2 - \frac{2}{\gamma-1}[\gamma\psi - p]h(u)$$

$$\partial_t v - \partial_x f(u) + \partial_x p = 0$$

Multiply by ω and integrate:

$$\int_0^t \int_{-\infty}^{\infty} v^2(x,\tau)\, dx\, d\tau = \int_0^t \int_{-\infty}^{\infty} f(u(x,\tau))\, u(x,\tau)\, dx\, d\tau - \int_0^t \int_{-\infty}^{\infty} p(x,\tau)\, u(x,\tau)\, dx\, d\tau$$

$$+ \int_{-\infty}^{\infty} \omega(x,t)\, v(x,t)\, dx - \int_{-\infty}^{\infty} \omega(x,0)\, v_0(x)\, dx$$

Premise:

$$\int_0^\infty \int_{-\infty}^\infty [u^2(x,t) + v^2(x,t) + p^2(x,t)]\, dx\, dt \le \rho^2 \delta^2$$

$$|u(x,t)| + |v(x,t)| + |p(x,t)| \le \varepsilon$$

Have:

$$\int_0^\infty \int_{-\infty}^\infty [u(x,t) - \gamma p(x,t)]^2\, dx\, dt \le c\delta^2$$

$$\int_0^\infty \int_{-\infty}^\infty p^2(x,t)\, dx\, dt \le c\delta^2 + [(c + c\rho)\delta + c\varepsilon]\rho^2\delta^2$$

$$\int_{-\infty}^\infty w^2(x,t)\, dx \le c\delta^2 + [(c + c\rho)\delta + c\varepsilon]\rho^2\delta^2$$

$$\int_{-\infty}^\infty v^2(x,t)\, dx \le c\delta^2$$

$$\int_0^\infty \int_{-\infty}^\infty v^2(x,t)\, dx\, dt \le c\delta^2 + [(c + \rho c)\delta + c\varepsilon]\rho^2\delta^2$$

$$\int_0^\infty \int_{-\infty}^\infty [u^2(x,t) + v^2(x,t) + p^2(x,t)]\,dx\,dt \le c\delta^2 + [(c + c\rho)\delta + c\varepsilon]\rho^2\delta^2$$

$$\overset{?}{\le} \rho^2\delta^2$$

Yes if ρ is fixed sufficiently large and then δ and ε are fixed sufficiently small