

Execution targets: VWAP, TWAP¹

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¹Notes based on textbook “Algorithmic and High-Frequency Trading” with Sebastian Jaimungal and Jose Penalva. See also http://papers.ssrn.com/sol3/papers.cfm?abstract_id=2542314

POV and VWAP

- ▶ Executions are delegated to brokers
- ▶ How can investor's measure the performance of brokers?
- ▶ Obtaining what the market has borne is sometimes enough
- ▶ A popular benchmark is VWAP, weighted average price,

$$\text{VWAP} = \frac{\int_0^T S_u \mu_u du}{\int_0^T \mu_u du}, \quad (1)$$

where S_t is the midprice and μ_t is the volume traded by the entire market.

Targeting VWAP via POV or POCV

- ▶ Algorithms that track VWAP and those that target a percentage of the volume traded (POV) are closely interlinked because the investor also requires to minimise market impact.
- ▶ Reducing price impact forces the algorithm to slice the parent order over a time window and doing it in a volume-based fashion will help to target VWAP.
- ▶ Our formulation is to set up a performance criteria where the investor seeks to execute a large order over a trading horizon T and the speed of trading targets POV or targets a percentage of cumulative volume (POCV).

Volume patterns

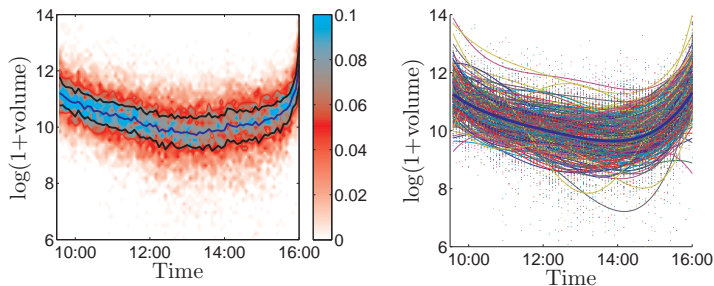


Figure : ORCL traded volume for orders sent to NASDAQ in all of 2013 using 5 minute buckets. (left) a heat-map of the data and 25%, 50% and 75% quantiles (right) functional data regression (using Legendre polynomials) on the daily curves and the estimated mean curve

Targeting VWAP via POV

The Model

- ▶ The investor searches for an optimal speed to liquidate \mathfrak{N} shares over a trading horizon T .
- ▶ Inventory

$$dQ_t^\nu = -\nu_t dt, \quad Q_0^\nu = \mathfrak{N}. \quad (2)$$

where ν_t is the speed of liquidation.

- ▶ The rest of the market trades at speed μ^+ for buy side and μ^- for sell side.
- ▶ And the idea is to target, at every instant in time, a percentage $\rho \in [0, 1]$ of the overall traded volume.
- ▶ The overall traded volume, at every instant in time, is

$$\mu_t^+ + \mu_t^- + \nu_t.$$

The Model, price impact

- ▶ Execution price

$$\hat{S}_t^\nu = S_t^\nu - k \nu_t, \quad (3)$$

$k > 0$ is the temporary impact parameter.

- ▶ The midprice satisfies

$$dS_t^\nu = b (\mu_t^+ - (\nu_t + \mu_t^-)) dt + dM_t, \quad S_0^\nu = S, \quad (4)$$

where $b \geq 0$ is the permanent impact parameter, and $M = \{M_t\}_{0 \leq t \leq T}$ is a martingale (independent of all other processes).

- ▶ Cash

$$dX_t^\nu = \hat{S}_t^\nu \nu_t dt, \quad X_0^\nu = X_0. \quad (5)$$

The Model (cont)

Performance criteria is

$$H^\nu(t, x, S, \boldsymbol{\mu}, q) = \mathbb{E}_{t, x, S, \boldsymbol{\mu}, q} \left[X_{\tau^\nu}^\nu + Q_{\tau^\nu}^\nu (S_{\tau^\nu}^\nu - \alpha Q_{\tau^\nu}^\nu) - \tilde{\varphi} \int_t^{\tau^\nu} (\nu_u - \chi_u^\nu)^2 du \right], \quad (6)$$

where $\boldsymbol{\mu} = \{\mu^+, \mu^-\}$,

$$\chi_t^\nu := \tilde{\rho} \times (\mu_t^+ + \mu_t^- + \nu_t), \quad (7)$$

$\tau^\nu = T \wedge \inf\{t : Q_t^\nu = 0\}$, and her value function

$$H(t, x, S, \boldsymbol{\mu}, q) = \sup_{\nu \in \mathcal{A}} H^\nu(t, x, S, \boldsymbol{\mu}, q), \quad (8)$$

where \mathcal{A} is the set of admissible strategies consisting of \mathcal{F} -predictable processes such that $\int_0^T |\nu_u| du < +\infty$, \mathbb{P} -a.s.

The value function satisfies

$$0 = (\partial_t + \mathcal{L}^M + \mathcal{L}^\mu) H + \sup_{\nu} \left\{ (S - k\nu) \nu \partial_x H - \nu \partial_q H + b((\mu^+ - \mu^-) - \nu) \partial_S H - \varphi(\nu - \rho \mu)^2 \right\} \quad (9)$$

for $q > 0$, with terminal condition

$$H(t, x, S, \mu, q) = x + q(S - \alpha q), \quad (10)$$

where $\mu = \mu^+ + \mu^-$, \mathcal{L}^μ represents the infinitesimal generator of μ , \mathcal{L}^M the infinitesimal generator of M , and we have introduced the re-scaled parameters

$$\varphi = \tilde{\varphi}(1 - \tilde{\rho})^2 \quad \text{and} \quad \rho = \tilde{\rho}/(1 - \tilde{\rho}).$$

Solving the DPE

The DPE (9) admits the solution

$$H(t, x, S, \mu, q) = x + qS + h_0(t, \mu) + h_1(t, \mu)q + h_2(t)q^2, \quad q > 0 \quad (11)$$

where

$$h_2(t) = - \left(\frac{T-t}{k+\varphi} + \frac{1}{\alpha - \frac{1}{2}b} \right)^{-1} - \frac{1}{2}b, \quad (12a)$$

$$\begin{aligned} h_1(t, \mu) = & \frac{\varphi \rho}{(T-t) + \zeta} \int_t^T \mathbb{E}_{t, \mu} [\mu_s^+ + \mu_s^-] ds \\ & + \frac{b}{(T-t) + \zeta} \int_t^T ((T-s) + \zeta) \mathbb{E}_{t, \mu} [\mu_s^+ - \mu_s^-] ds, \end{aligned} \quad (12b)$$

$$h_0(t, \mu) = \int_t^T \mathbb{E}_{t, \mu} \left[\frac{(h_1(t, \mu_s) - 2\varphi\rho(\mu_s^+ + \mu_s^-))^2}{4(k+\varphi)} - \varphi\rho^2(\mu_s^+ + \mu_s^-)^2 \right] ds, \quad (12c)$$

and the constant

$$\zeta = \frac{k+\varphi}{\alpha - \frac{1}{2}b}.$$

Optimal liquidation speed

$$\nu_t^* = \frac{1}{(T-t) + \zeta} Q_t^{\nu^*} \quad (13a)$$

$$+ \frac{\varphi}{k + \varphi} \rho \left\{ (\mu_t^+ + \mu_t^-) - \frac{1}{(T-t) + \zeta} \int_t^T \mathbb{E}[(\mu_s^+ + \mu_s^-) \mid \mathcal{F}_t^\mu] ds \right\} \quad (13b)$$

$$- \frac{b}{k + \varphi} \int_t^T \frac{(T-s) + \zeta}{(T-t) + \zeta} \mathbb{E}[(\mu_s^+ - \mu_s^-) \mid \mathcal{F}_t^\mu] ds, \quad \forall t < \tau^{\nu^*} \quad (13c)$$

where \mathcal{F}_t^μ denotes the natural filtration generated by μ , and $\nu_t^* = 0$ for $t \geq \tau$, is the admissible optimal control we seek.

Optimal liquidation speed

- ▶ The first component is TWAP-like.
- ▶ The second component consists of targeting instantaneous total order-flow plus a weighted average of future expected total order-flow. Although the POV target is $\rho \mu_t$, the strategy targets a lower amount since $\frac{\varphi \rho}{k + \varphi} \leq \rho$, where equality is achieved if the costs of missing the target are $\varphi \rightarrow \infty$ and k remains finite, or there is no temporary impact $k \downarrow 0$.
- ▶ The last component acts to correct her trading based on her expectations of the net order-flow from that point in time until the end of the trading horizon. When there is a current surplus of buy trades she slows down her trading rate to allow the midprice to appreciate before liquidating the rest of her order.

Targeting VWAP

Investor must liquidate all shares. Do this by letting $\alpha \rightarrow \infty$

$$\begin{aligned} \lim_{\alpha \rightarrow \infty} \nu_t^* &= \frac{1}{T-t} Q_t^{\nu^*} \\ &+ \frac{\varphi}{k+\varphi} \left(\rho \mu_t - \frac{1}{T-t} \int_t^T \mathbb{E}[\mu_s^+ + \mu_s^- | \mathcal{F}_t^\mu] ds \right) \\ &- \frac{b}{k+\varphi} \frac{\int_t^T (T-s) \mathbb{E}[\mu_s^+ - \mu_s^- | \mathcal{F}_t^\mu] ds}{T-t}, \end{aligned}$$

and follow this by taking the limit $\varphi \rightarrow \infty$, resulting in

$$\lim_{\varphi \rightarrow \infty} \lim_{\alpha \rightarrow \infty} \nu_t^* = \frac{1}{T-t} Q_t^{\nu^*} + \rho \left((\mu_t^+ + \mu_t^-) \right. \quad (15)$$

$$\left. - \frac{1}{T-t} \int_t^T \mathbb{E}[(\mu_s^+ + \mu_s^-) | \mathcal{F}_t^\mu] ds \right). \quad (16)$$

Targeting VWAP via POCV

Accumulated Volume

- Accumulated volume V of orders, excluding the agent's, is

$$V_t = \int_0^t (\mu_u^+ + \mu_u^-) du.$$

- The investor's performance criteria is now modified to

$$H^\nu(t, x, S, \mu, V, q) = \mathbb{E}_{t, x, S, \mu, V, q} \left[X_{\tau^\nu}^\nu + Q_{\tau^\nu}^\nu (S_{\tau^\nu}^\nu - \alpha Q_{\tau^\nu}^\nu) - \tilde{\varphi} \int_t^{\tau^\nu} \left((\mathfrak{N} - Q_u^\nu) - \tilde{\rho} (V_u + (\mathfrak{N} - Q_u^\nu)) \right)^2 du \right]$$

and $\tau^\nu = T \wedge \inf\{t : Q_t^\nu = 0\}$.

- The investor's value function is

$$H(t, x, S, \mu, V, q) = \sup_{\nu \in \mathcal{A}} H^\nu(t, x, S, \mu, V, q). \quad (17)$$

DPE

The value function should satisfy the DPE

$$0 = (\partial_t + \mathcal{L}^M + \mathcal{L}^{\mu, V}) H - \varphi ((\mathfrak{N} - q) - \rho V)^2 \\ + \sup_{\nu} \{ b((\mu^+ - \mu^-) - \nu) \partial_S H + (S - k \nu) \nu \partial_x H - \nu \partial_q H \} , \quad (18)$$

for $q > 0$, subject to $H(T, x, S, y, q) = x + q(S - \alpha q)$ and

$$\varphi = \tilde{\varphi}(1 - \tilde{\rho})^2 \quad \text{and} \quad \rho = \tilde{\rho}/(1 - \tilde{\rho}).$$

Here, $\mathcal{L}^{\mu, V}$ denotes the infinitesimal generator of the joint process $(\mu_t, V_t)_{0 \leq t \leq T}$.

The DPE (18) admits the solution

$$H(t, x, S, \mu, V, q) = x + q S + h_0(t, \mu, V) + h_1(t, \mu, V) q + h_2(t) q^2, \quad q > 0 \quad (19)$$

where

$$h_2(t) = -\sqrt{k\varphi} \frac{\gamma e^{\xi(T-t)} + e^{\xi(T-t)}}{\gamma e^{\xi(T-t)} - e^{-\xi(T-t)}} - \frac{1}{2} b, \quad (20a)$$

$$\begin{aligned} h_1(t, \mu, V) = & 2\varphi \int_t^T \ell(u, t) (\mathfrak{N} - \rho \mathbb{E}_{t, \mu, V} [V_u]) du \\ & + b \int_t^T \ell(u, t) \mathbb{E}_{t, \mu, V} [(\mu_u^+ - \mu_u^-)] du, \end{aligned} \quad (20b)$$

$$h_0(t, \mu, V) = \int_t^T \mathbb{E}_{t, \mu} \left[\frac{1}{4k} (h_1(u, \mu_u, V_u))^2 - \varphi (\mathfrak{N} - \rho V_u)^2 \right] du, \quad (20c)$$

the function

$$\ell(u, t) := \frac{\gamma e^{\xi(T-u)} - e^{-\xi(T-u)}}{\gamma e^{\xi(T-t)} - e^{-\xi(T-t)}}, \quad (20d)$$

and the constants

$$\xi = \sqrt{\frac{\varphi}{k}}, \quad \text{and} \quad \gamma = \frac{\alpha - \frac{1}{2}b + \sqrt{k\varphi}}{\alpha - \frac{1}{2}b - \sqrt{k\varphi}}.$$

Optimal control for POCV

Moreover, the trading speed, for $q > 0$, is given by

$$\nu_t^* = \xi \frac{1 + \gamma}{\gamma e^{\xi(T-t)} - e^{-\xi(T-t)}} Q_t^{\nu^*} \quad (21a)$$

$$- \xi^2 \int_t^T \ell(u, t) \left\{ \left(\mathfrak{N} - Q_t^{\nu^*} \right) - \rho \mathbb{E} \left[V_u \mid \mathcal{F}_t^{\mu, \nu} \right] \right\} du \quad (21b)$$

$$- \frac{b}{2k} \int_t^T \ell(u, t) \mathbb{E} \left[(\mu_u^+ - \mu_u^-) \mid \mathcal{F}_t^{\mu, \nu} \right] du, \quad \forall t < \tau^{\nu^*}. \quad (21c)$$

And the optimal inventory path

$$\begin{aligned} Q_t^{\nu^*} = & \ell(t, 0) \mathfrak{N} \\ & + \int_0^t \int_u^T \ell(u, s) \left\{ \frac{\varphi}{k} \left(\mathfrak{N} - \rho \mathbb{E} \left[V_s \mid \mathcal{F}_u^{\mu, \nu} \right] \right) + \frac{b}{2k} \mathbb{E} \left[(\mu_s^+ - \mu_s^-) \mid \mathcal{F}_u^{\mu, \nu} \right] \right\} ds du. \end{aligned} \quad (22)$$

Interpreting liquidation speed

- ▶ (21a) may be interpreted as an Almgren-Chriss like strategy which approaches TWAP near maturity.
- ▶ The second term (21b) corrects for the weighted average of the difference between what the investor has liquidated so far $(\mathfrak{N} - Q_t^{\nu*})$ and her future targeted volume $\rho \mathbb{E} \left[V_u \mid \mathcal{F}_t^{\mu, \nu} \right]$. This term will be positive most of the time because: i) total volume is a submartingale and so $\mathbb{E} \left[V_u \mid \mathcal{F}_t^{\mu, \nu} \right] \geq V_t$, and ii) since at time t the investor targets $(\mathfrak{N} - Q_t^{\nu*})$ to ρV_t , it is likely that

$$\left(\mathfrak{N} - Q_t^{\nu*} \right) - \rho \mathbb{E} \left[V_u \mid \mathcal{F}_t^{\mu, \nu} \right] \sim \rho V_t - \rho \mathbb{E} \left[V_u \mid \mathcal{F}_t^{\mu, \nu} \right] \leq 0.$$

- ▶ The third term (21c) contributes a weighted average of future net order-flow and accounts for the permanent impact which trades have on the midprice. If future net order-flow is expected to be buy heavy, then investor slows down
- ▶ Finally, as the trading horizon ends, t approaching T , the second and third terms become negligible and the investor ignores order-flow entirely and instead focuses on completing her trades.

Targeting VWAP

Let $\alpha \rightarrow \infty$, so that the investor ensures that she completely liquidates her position by the terminal time, $\gamma \rightarrow 1$, therefore $\ell(u, t) \rightarrow \frac{\sinh(\xi(T-u))}{\sinh(\xi(T-t))}$ and so we have

$$\begin{aligned} \lim_{\alpha \rightarrow \infty} \nu_t^* &= \xi \frac{Q_t^{\nu^*}}{\sinh(\xi(T-t))} \\ &\quad - \xi^2 \int_t^T \frac{\sinh(\xi(T-u))}{\sinh(\xi(T-t))} \left\{ (\mathfrak{N} - Q_t^{\nu^*}) - \rho \mathbb{E} [V_u \mid \mathcal{F}_t^{\mu, V}] \right\} du \\ &\quad - \frac{b}{2k} \int_t^T \frac{\sinh(\xi(T-u))}{\sinh(\xi(T-t))} \mathbb{E} [(\mu_u^+ - \mu_u^-) \mid \mathcal{F}_t^{\mu, V}] du, \end{aligned}$$

and

$$\begin{aligned} \lim_{\alpha \rightarrow \infty} Q_t^{\nu^*} &= \frac{\sinh(\xi(T-t))}{\sinh(\xi T)} \mathfrak{N} + \xi^2 \int_0^t \int_u^T \frac{\sinh(\xi(T-u))}{\sinh(\xi(T-s))} \left(\mathfrak{N} - \rho \mathbb{E} [V_s \mid \mathcal{F}_u^{\mu, V}] \right) ds du \\ &\quad + \frac{b}{2k} \int_0^t \int_u^T \frac{\sinh(\xi(T-u))}{\sinh(\xi(T-s))} \mathbb{E} [(\mu_s^+ - \mu_s^-) \mid \mathcal{F}_u^{\mu, V}] ds du. \end{aligned}$$

Strategy Performance

Assume that other market participants trades follow

$$d\mu_t^+ = -\kappa^+ \mu_t^+ dt + \eta_{1+N_t^+}^+ dN_t^+, \quad (23a)$$

$$d\mu_t^- = -\kappa^- \mu_t^- dt + \eta_{1+N_t^-}^- dN_t^-, \quad (23b)$$

where

- ▶ $\kappa^\pm \geq 0$ are the mean-reversion rates,
- ▶ N_t^+ and N_t^- are independent homogeneous Poisson processes with intensities λ^+ and λ^- , respectively,
- ▶ $\{\eta_1^\pm, \eta_2^\pm, \dots\}$ are non-negative i.i.d. random variables with distribution function F , with finite first moment, independent from all processes.
- ▶ In addition, we require $\kappa^\pm > \lambda^\pm \mathbb{E}[\eta_1^\pm]$ to ensure that μ^\pm remain bounded \mathbb{P} -a.s..

VWAP as target

Compute VWAP as

$$\text{VWAP} = \frac{\int_0^T S_u^\nu (\mu_u^+ + \mu_u^- + \nu_u) du}{\int_0^T (\mu_u^+ + \mu_u^- + \nu_u) du}, \quad (24a)$$

while the execution price is computed as

$$\text{Exec. Price} = \frac{X_T^\nu}{\mathfrak{N}} = \frac{\int_0^T \hat{S}_u^\nu \nu_u du}{\mathfrak{N}}. \quad (24b)$$

POV-VWAP performance

Table : The statistics of the execution price, VWAP, and relative error (computed as $(Exec.Price - VWAP)/VWAP$ for each simulation) and reported in basis points (i.e., $\times 10^4$).

		FARO		SMH		NTAP	
		VWAP	Rel.Error	VWAP	Rel.Error	VWAP	Rel.Error
quantile	mean	\$ 40.54	8.9	\$ 37.90	2.98	\$ 38.30	0.19
	stdev	\$0.11	16.9	\$0.04	6.10	\$0.06	0.87
	5%	\$40.35	-4.2	\$37.83	-1.03	\$38.20	-0.78
	25%	\$40.46	-0.4	\$37.87	-0.16	\$38.26	-0.20
	50%	\$40.54	2.5	\$37.90	0.53	\$38.30	0.03
	75%	\$40.61	12.0	\$37.92	3.60	\$38.34	0.36
	95%	\$40.72	42.6	\$37.96	15.29	\$38.40	1.66
		27.3%		18.4%		0.6%	

POCV-VWAP performance

Table : The statistics of the execution price, VWAP, and relative error (computed as $(Exec.Price - VWAP)/VWAP$ for each simulation) and reported in basis points (i.e., $\times 10^4$). **For POCV we set $\varphi = 10^5 \times k$**

		FARO		SMH		NTAP	
		VWAP	Rel.Error	VWAP	Rel.Error	VWAP	Rel.Error
quantile	mean	\$40.54	5.0	\$37.89	2.75	\$38.30	1.88
	stdev	\$0.11	14.2	\$0.04	6.02	\$0.06	2.35
	5%	\$40.35	-13.6	\$37.83	-5.43	\$38.20	-1.65
	25%	\$40.46	-3.0	\$37.87	-0.78	\$38.26	0.28
	50%	\$40.54	1.9	\$37.89	1.79	\$38.30	1.72
	75%	\$40.61	11.0	\$37.92	5.51	\$38.34	3.27
	95%	\$40.73	31.9	\$37.96	13.76	\$38.40	6.00
		0. 83%		0.013%		0.55%	