

Optimal execution with Order Flow information¹

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¹Notes based on textbook “Algorithmic and High-Frequency Trading” with Sebastian Jaimungal and Jose Penalva. See also http://papers.ssrn.com/sol3/papers.cfm?abstract_id=2557457 .

Optimal Execution

- ▶ How do we trade in or out of a large position when Order-Flow affects prices?
- ▶ Optimal Execution with Temporary Impact and Inventory Control

Permanent Impact

We assume a linear relationship between net order-flow and changes in the midprice, thus for every trading day we perform the regression

$$\Delta S_n = b \mu_n + \varepsilon_n \quad (1)$$

where $\Delta S_n = S_{n\tau} - S_{(n-1)\tau}$ is the change in the midprice, μ_n is net order-flow defined as the difference between the volume of buy and sell MOs during the time interval $[(n-1)\tau, n\tau]$, and ε_n is the error term (assumed normal). In the empirical analysis we choose $\tau = 5$ min.

Temporary Impact

- ▶ Assume that temporary price impact is linear in the rate of trading so the difference between the execution price that the investor receives and the midprice is $k \nu$,
- ▶ To do this we take a snapshot of the LOB each second,
- ▶ Determine the price per share for various volumes (by walking through the LOB),
- ▶ Compute the difference between the price per share and the best quote at that time,
- ▶ Perform a linear regression.

Parameters

Table : Permanent and temporary price impact parameters for Nasdaq stocks, average volume of MOs, average midprice, σ volatility (hourly) of arithmetic price changes, mean arrival (hourly) of MOs λ^\pm , and average volume of MOs $\mathbb{E}[\eta^\pm]$. Data are from Nasdaq 2013.

	FARO		SMH		NTAP	
	mean	stdev	mean	stdev	mean	stdev
ADV	23,914	14,954	233,609	148,580	1,209,628	642,376
midprice	40.55	6.71	37.90	2.44	38.33	3.20
σ	0.151	0.077	0.067	0.039	0.078	0.045
b	1.41E-04	9.61E-05	5.45E-06	4.20E-06	5.93E-06	2.31E-06
k	1.86E-04	2.56E-04	8.49E-07	8.22E-07	3.09E-06	1.75E-06
b/k	1.02	0.83	7.43	6.24	2.04	0.77
λ^+	16.81	9.45	47.29	28.13	300.52	144.48
$\mathbb{E}[\eta^+]$	103.56	21.16	377.05	118.05	308.45	53.09
λ^-	17.62	10.69	46.37	27.62	293.83	136.13
$\mathbb{E}[\eta^-]$	104.00	21.79	381.70	126.74	312.81	49.86

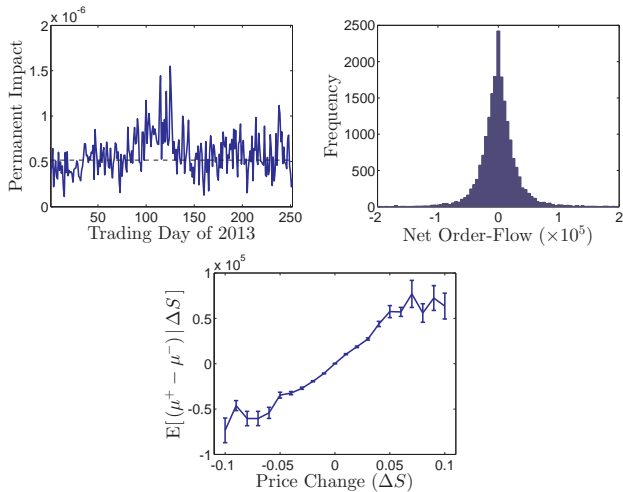


Figure : Order-Flow and effect on the drift of midprice of INTC. The first picture shows

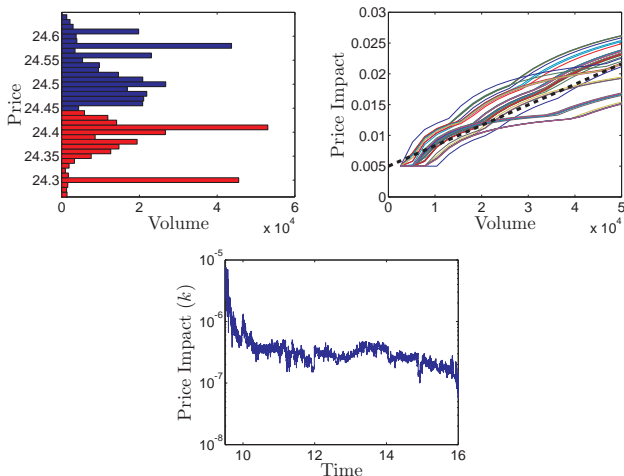


Figure : An illustration of how the temporary impact may be estimated from snapshots of the LOB using INTC on Nov 1, 2013. The first panel is at 11:00am, the second from 11:00am to 11:01am and the third contains the entire day.

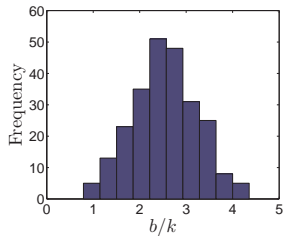
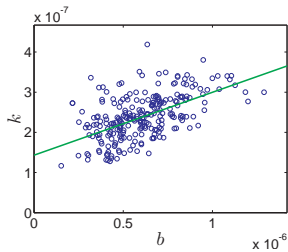


Figure : Price Impact INTC using daily observations for 2013.

The Model

- ▶ Inventory (investor is liquidating)

$$dQ_t^\nu = -\nu_t dt, \quad Q_0^\nu = \mathfrak{N}, \quad (2)$$

- ▶ Midprice

$$dS_t^\nu = b(\mu_t - \nu_t) dt + \sigma dW_t, \quad (3)$$

where $\mu_t = \mu_t^+ - \mu_t^-$ is net order-flow of other market participants.

- ▶ Execution price

$$\hat{S}_t^\nu = S_t^\nu - \left(\frac{1}{2}\Delta + k\nu_t\right), \quad (4)$$

- ▶ Cash

$$dX_t^\nu = \hat{S}_t^\nu \nu_t dt, \quad X_0^\nu = x. \quad (5)$$

Value Function

$$H^\nu() = \mathbb{E}_{t,x,S,\mu,q} \left[X_{\tau^\nu} + Q_{\tau^\nu}^\nu (S_{\tau^\nu}^\nu - \frac{1}{2}\Delta - \alpha Q_{\tau^\nu}^\nu) - \phi \int_t^{\tau^\nu} (Q_u^\nu)^2 du \right] \quad (6)$$

where $\tau^\nu = T \wedge \inf\{t : Q_t^\nu = 0\}$, and $\mu = \{\mu^+, \mu^-\}$. Her value function is

$$H(t, x, S, \mu, q) = \sup_{\nu \in \mathcal{A}} H^\nu(t, x, S, \mu, q), \quad (7)$$

The DPP suggests that $H(t, x, S, \mu, q)$ satisfies

$$0 = (\partial_t + \frac{1}{2}\sigma^2 \partial_{SS}) H + \mathcal{L}^\mu H - \phi q^2 + \sup_\nu \left\{ \left(\nu (S - \frac{1}{2}\Delta - k\nu) \partial_x + b(\mu - \nu) \partial_S - \nu \partial_q \right) H \right\} \quad (8)$$

for $q > 0$, subject to the terminal condition.

$$H(T, x, S, \mu, q) = x + q (S - \frac{1}{2}\Delta) - \alpha q^2.$$

Solving the DPE.

The DPE (8) admits the solution

$$H(t, x, S, \boldsymbol{\mu}, q) = x + q \left(S - \frac{1}{2} \Delta \right) + h_0(t, \boldsymbol{\mu}) + q h_1(t, \boldsymbol{\mu}) + q^2 h_2(t), \quad q > 0,$$

where

$$h_2(t) = \sqrt{k\phi} \frac{1 + \zeta e^{2\gamma(T-t)}}{1 - \zeta e^{2\gamma(T-t)}} - \frac{1}{2} b, \quad (9a)$$

$$h_1(t, \boldsymbol{\mu}) = b \int_t^T \left(\frac{e^{-\gamma(T-u)} - \zeta e^{\gamma(T-u)}}{e^{-\gamma(T-t)} - \zeta e^{\gamma(T-t)}} \right) \mathbb{E}_{t, \boldsymbol{\mu}} [\mu_u] du, \quad (9b)$$

$$h_0(t, \boldsymbol{\mu}) = \frac{1}{4k} \int_t^T \mathbb{E}_{t, \boldsymbol{\mu}} [h_1^2(t, \boldsymbol{\mu}_u)] du, \quad (9c)$$

with the constants γ and ζ :

$$\gamma = \sqrt{\frac{\phi}{k}}, \quad \text{and} \quad \zeta = \frac{\alpha - \frac{1}{2}b + \sqrt{k\phi}}{\alpha - \frac{1}{2}b - \sqrt{k\phi}}.$$

Proof

To solve (8) we make the ansatz

$$H(t, x, S, \mu, q) = x + q \left(S - \frac{1}{2} \Delta \right) + h(t, \mu, q),$$

and upon substitution of the ansatz in the DPE $h(t, \mu, q)$ satisfies

$$\partial_t h + \mathcal{L}^\mu h + b \mu q - \phi q^2 + \sup_{\nu} \left\{ -k \nu^2 - (b q + \partial_q h) \nu \right\} = 0, \quad q > 0,$$

subject to $h(T, \mu, q) = -\alpha q^2$, so

$$\nu^* = -\frac{1}{2k} (b q + \partial_q h). \quad (10)$$

Upon substitution back into the DPE we find that h satisfies

$$(\partial_t + \mathcal{L}^\mu) h + b \mu q - \phi q^2 + \frac{1}{4k} (b q + \partial_q h)^2 = 0, \quad q > 0. \quad (11)$$

Proof (cont)

Due to the existence of linear and quadratic terms in q in (11), and its terminal conditions, we expect $h(t, \mu, q)$ to be a quadratic form in q , and we assume the ansatz

$$h(t, \mu, q) = h_0(t, \mu) + q h_1(t, \mu) + q^2 h_2(t, \mu).$$

Inserting this into (11) and collecting like terms in q leads to

$$(\partial_t + \mathcal{L}^\mu) h_0 + \frac{1}{4k} h_1^2 = 0, \quad (12a)$$

$$(\partial_t + \mathcal{L}^\mu) h_1 + b \mu + \frac{1}{2k} h_1 (b + 2h_2) = 0, \quad (12b)$$

$$(\partial_t + \mathcal{L}^\mu) h_2 - \phi + \frac{1}{4k} (b + 2h_2)^2 = 0, \quad (12c)$$

subject to the terminal conditions

$$h_0(T, \mu) = 0, \quad h_1(T, \mu) = 0, \quad h_2(T, \mu) = -\alpha.$$

Proof (cont)

To solve for h_2 we note that since Equation (12c) for h_2 contains no source terms in μ and its terminal condition is independent of μ , the solution must be independent of μ , i.e. h_2 is a function only of time. In this case, (12c) is an ODE of Riccati type and can be solved explicitly:

$$h_2(t, \mu) = \chi(t) - \frac{1}{2} b, \quad \text{where} \quad \chi(t) = \sqrt{k\phi} \frac{1 + \zeta e^{2\gamma(T-t)}}{1 - \zeta e^{2\gamma(T-t)}},$$

with the constants γ and ζ :

$$\gamma = \sqrt{\frac{\phi}{k}}, \quad \text{and} \quad \zeta = \frac{\alpha - \frac{1}{2}b + \sqrt{k\phi}}{\alpha - \frac{1}{2}b - \sqrt{k\phi}}.$$

Proof (cont)

Now we turn to solving (12b) which is a linear PIDE for h_1 where $h_2 + \frac{1}{2}b$ acts as an effective discount rate and $b\mu$ is a source term. The general solution of such an equation can be represented using the Feynman-Kac theorem. Thus we write

$$h_1(t, \mu) = b \mathbb{E}_{t, \mu} \left[\int_t^T \exp \left\{ \frac{1}{k} \int_t^u (h_2(s) + \frac{1}{2}b) ds \right\} \mu_u du \right]$$

which can be simplified to

$$h_1(t, \mu) = b \int_t^T \left(\frac{e^{-\gamma(T-u)} - \zeta e^{\gamma(T-u)}}{e^{-\gamma(T-t)} - \zeta e^{\gamma(T-t)}} \right) \mathbb{E}_{t, \mu} [\mu_u] du. \quad (13)$$

Finally, we can solve for $h_0(t, \mu)$ by again noticing it is a linear PDE with non-linear source term and a straight forward application of Feynman-Kac, and interchanging integration and expectation, we obtain (9c). □

Limiting cases

All shares must be liquidated, $\alpha \rightarrow \infty$, and the optimal trading speed simplifies to

$$\lim_{\alpha \rightarrow \infty} \nu_t^* = \gamma \frac{\cosh(\gamma(T-t))}{\sinh(\gamma(T-t))} Q_t^{\nu^*} - \frac{b}{2k} \int_t^T \frac{\sinh(\gamma(T-u))}{\sinh(\gamma(T-t))} \mathbb{E}[\mu_u | \mathcal{F}_t^\mu] du. \quad (14)$$

Another interesting limiting case is if the above limit is followed by letting $\phi \rightarrow 0$:

$$\lim_{\phi \rightarrow 0} \lim_{\alpha \rightarrow \infty} \nu_t^* = \frac{1}{(T-t)} Q_t^{\nu^*} - \frac{b}{2k} \int_t^T \frac{(T-u)}{(T-t)} \mathbb{E}[\mu_u | \mathcal{F}_t^\mu] du. \quad (15)$$

Model for Order-Flow

Order-flow μ_t^\pm satisfy the SDEs

$$d\mu_t^\pm = -\kappa \mu_t^\pm dt + \eta_{1+L_{t-}^\pm}^\pm dL_t^\pm, \quad (16)$$

where L_t^\pm are independent Poisson processes with equal intensity λ , $\{\eta_1^\pm, \eta_2^\pm, \dots\}$.

The solutions to (16), for $s > t$, are

$$\mu_s^\pm = e^{-\kappa^\pm(s-t)} \mu_t^\pm + \int_t^s e^{-\kappa^\pm(s-u)} \eta_{1+N_{u-}^\pm}^\pm dL_u^\pm,$$

so that

$$\mathbb{E}[\mu_s^\pm | \mathcal{F}_t^\mu] = e^{-\kappa^\pm(s-t)} (\mu_t^\pm - \psi^\pm) + \psi^\pm,$$

where

$$\psi^\pm = \frac{1}{\kappa^\pm} \lambda^\pm \mathbb{E}[\eta^\pm].$$

Therefore, under this particular model for order-flow, we follow the optimal trading strategy

$$\lim_{\alpha \rightarrow \infty} \nu_t^* = \gamma \frac{\cosh(\gamma(T-t))}{\sinh(\gamma(T-t))} Q_t^{\nu^*} - \frac{b}{2k} \left[\ell_1^+(t) (\mu_t^+ - \psi^+) - \ell_1^-(t) (\mu_t^- - \psi^-) + \ell_0(t) (\psi^+ - \psi^-) \right]$$

where

$$\ell_0(t) = \frac{1}{\gamma} \frac{\cosh(\gamma(T-t)) - 1}{\sinh(\gamma(T-t))},$$

and

$$\ell_1^\pm(t) = \frac{1}{2} \left(\frac{e^{\gamma(T-t)} - e^{-\kappa^\pm(T-t)}}{\kappa^\pm + \gamma} - \frac{e^{-\gamma(T-t)} - e^{-\kappa^\pm(T-t)}}{\kappa^\pm - \gamma} \right) / \sinh(\gamma(T-t)).$$

Sims

- ▶ We assume that the trading horizon is $T = 1$ hour,
- ▶ the running inventory penalty parameter is $\phi = 10 \times k$,
- ▶ the liquidation target \mathfrak{N} shares is set to 1% of the expected traded volume over the trading window (including the investor's trades),
- ▶ run 10,000 simulations,
- ▶ the strategy trades at the same frequency as that of the arrival of market sell orders from all other agents

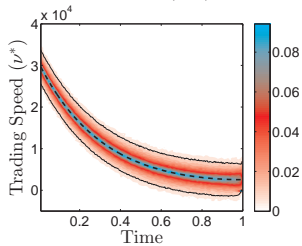
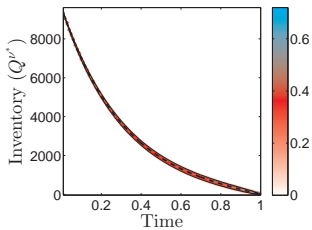
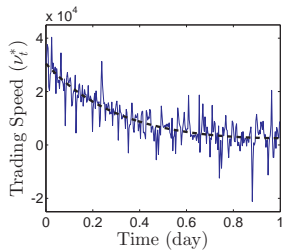
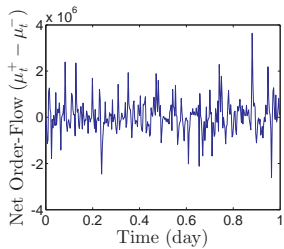
Use INTC with parameters

$$k = 2.50 \times 10^{-7}, \quad b = 6.15 \times 10^{-}, \quad \mathfrak{N} = 9,612,$$

$$\lambda^{\pm} = 328, \quad \kappa = 360, \quad \eta \sim \text{Exp}(1,453 \times \kappa),$$

$$\sigma = 0.01, \quad S_0 = 23.04,$$

and $\eta \sim \text{Exp}(\eta_0)$ denotes the exponential distribution with mean size $\mathbb{E}[\eta] = \eta_0$,



Histogram of the financial performance of the strategy relative to Almgren-Chriss. This performance is measured in basis points using

$$\frac{X_T^{\nu^*} - X_T^{AC}}{X_T^{AC}} \times 10^4 \quad (17)$$

where X_T^{AC} is the cash obtained from running the Almgren-Chriss strategy with the same level of ϕ .

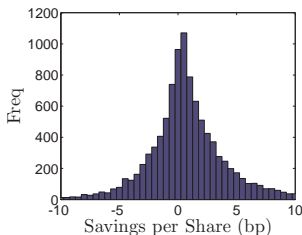


Figure : The savings per share (INTC) measured in basis points, $(X_T^{\nu^*} - X_T^{AC})/X_T^{AC} \times 10^4$, from following the optimal strategy relative to Almgren-Chriss.

Table : Simulation performance relative to AC

		ν^*				
		FARO	SMH	NTAP	ORCL	INTC
quantile	mean	529	602	0.78	0.64	0.93
	stdev	739	708	3.71	3.02	3.78
	5%	-438	5	-4.70	-3.83	-4.48
	25%	151	157	-0.98	-0.82	-0.87
	50%	396	428	0.50	0.44	0.55
	75%	835	895	2.23	1.85	2.45
	95%	1862	1957	6.98	5.58	7.51
$\%t \nu_t^* < 0$		12.3%	17.2%	2.7%	2.6%	3.4%
$\%X_T < X_T^{AC}$		10.2%	4.9%	38.9%	39.2%	38.5%

Table : Simulation performance relative to AC

		$\max(\nu^*, 0)$				
		FARO	SMH	NTAP	ORCL	INTC
quantile	mean	615	670	1.12	0.91	1.43
	stdev	648	654	3.80	3.14	3.97
	5%	-1	42	-4.10	-3.46	-3.79
	25%	179	203	-0.79	-0.67	-0.58
	50%	434	470	0.61	0.52	0.76
	75%	868	942	2.52	2.08	2.89
	95%	1887	1952	7.85	6.41	8.57
$\%t \nu_t^* < 0$		0.0%	0.0%	0.0%	0.0%	0.0%
$\%X_T < X_T^{AC}$		5.0%	0.1%	36.5%	37.0%	34.7%