EPSRC OxPDE CDT
Research Case Study
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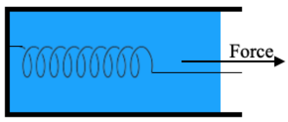
‘Existence and Uniqueness Results for Problems in Strain-limiting Viscoelasticity.’

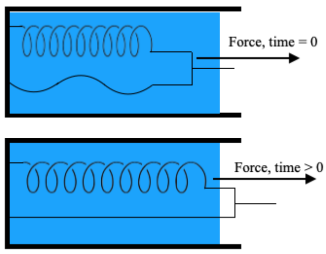
My research focuses on the need for the accurate mathematical modelling of the behaviour of certain materials over time, and under the influence of external forces. It considers materials with and without fractures and/or material discontinuities. Experimental data suggests that biological tissues (especially those made of collagen fibrils), and engineered materials which contain flexible micro-structures may exhibit a strain-limiting response, making this work relevant to real-world problems. Also, it adds to the body of knowledge surrounding the behaviour of viscoelastic solids, as it allows us to systematically study the fracture of materials in the small strain range -without the presence of a mathematical singularity.

Viscoelastic solids are ones which show characteristics of **both elastic solids and viscous fluids**. Many solids are in fact viscoelastic, e.g. putty, ligaments and tendons in the body, and things like polymer foams in seat cushions. Often, ***the elastic properties arise after a deformation of the material*** in question, while ***viscosity properties result from the rate of a deformation*** occurring.

To define a solid mathematically, we use a *constitutive relation – which* encodes specific properties of a material e.g. here viscoelasticity, and gives a relationship between the *stress* T (a measure of the internal forces in a solid), the *strain* ε (a measure of the deformation) and the *strain rate* ε_t (a measure of the rate of deformation). The subscript t refers to a derivative in time; i.e. where the term rate comes from.



An elastic spring moving through a viscous fluid in the above set up gives an intuitive idea of what viscoelasticity means.



For a strain-limiting viscoelastic material we can consider an elastic spring and an inextensible string in parallel moving through a viscous fluid. At time t = 0 the string is slack. Then, as the force works over time, eventually the string becomes taught. Then no matter how large the force (i.e., stress) becomes, the strain now remains constant. The string does not allow any more movement in the direction of the force. Thus the strain is limited.

We are dealing with solids, so to simplify the model we demand that the *small displacement gradient assumption* holds, which means that the value of the strain must be close to 0. So, if we encounter large strains in our problem, we are contradicting the basic assumption on which the model is constructed, meaning it is likely to be unreliable/ invalid. The classical Kelvin-Voigt model of a viscoelastic solid gives the constitutive relation as $T = E\varepsilon + \eta\varepsilon_t$, where E and η are positive real numbers. We call this a *linear model (*linear relationship between the stress and sum of the strain & strain rate). However, this linear model has a problem in that, as the stress T becomes very large, the strain ε also becomes large, which is important when we are considering the fracture of solids.

Imagine a material with a crack where we wish to assign every point a stress and a strain that accurately models the reaction of the material when forces act upon it – we do this via a set of balance equations for the *mass*, *linear momentum* & *angular momentum.* When combined with the *constitutive relation*, this forms a set of partial differential equations (PDEs) which we can solve to find stress & strain functions that accurately model the motion of the material over time. As stated, there is a problem with the model if the ***strain*** becomes large, and as we are working with a linear model, this happens if the

Image of spring adapted from https://www.shutterstock.com *stress* becomes large. Near the tip of a crack, the stress becomes large (as expected), and the validity of the model breaks down in the linear case (basic assumption is not valid).

This suggests that the *linear model* may not be the best choice when investigating the fracture of viscoelastic solids. Instead, we introduce a constitutive relation that ensures the strain is always bounded, i.e. we cannot contradict the basic modelling assumptions. This is where the term *strain-limiting* comes from and explains our interest in strain-limiting viscoelastic solids.
In essence, my work studies the usual equations of motion coupled with a constitutive relation of the form $F(T) = \varepsilon + \eta\varepsilon_t$ (where F is a bounded function), forming a set of PDEs which needs to be solved. The main difficulty in solving this problem is due to the strain-limiting property.

My research to date has focused on a body without a crack. We have been able to show that there exists a solution to the problem for this simplified situation. My next goal is to reintroduce the idea of material with a fracture and adapt the mathematical analysis.

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