Suggested title of dissertation:

Semigroup theory and linear evolution equations

Dissertation supervisor:

Dr David Seifert

Description of proposal

Differential equations arising in physics and elsewhere often describe the evolution in time of quantities which also depend on other (typically spatial) variables. Examples of such evolution equations include the heat equation and the wave equation. A rigorous, functional analytic approach to studying linear autonomous evolution equations begins by considering the associated abstract Cauchy problem,

$$\begin{align*}
\dot{u}(t) &= Au(t), \quad t \geq 0, \\
u(0) &= x \in X,
\end{align*}$$

where $A$ is a linear operator (typically unbounded) acting on a suitably chosen Banach space $X$. For instance, in the case of the heat equation on a domain $\Omega$ we might choose $X = L^2(\Omega)$ and let $A$ be a suitable realisation of the Laplace operator. The abstract Cauchy problem is said to be well posed if there exists a family $(T(t))_{t \geq 0}$ of bounded linear operators on $X$ (a so-called $C_0$-semigroup) such that the unique solution (in an appropriate sense) is given by $u(t) = T(t)x$, $t \geq 0$. Hence one may study solutions of an evolution equation by investigating the associated operators $T(t)$, $t \geq 0$. There is a precise sense in which $T(t) = \exp(tA)$, $t \geq 0$, but except in special cases the operators $T(t)$ are not known explicitly, and one of the main challenges is to deduce useful information about the semigroup from what is known about $A$, in particular its spectral properties.

The starting point for this project would be an exposition of the basic theory of unbounded operators and $C_0$-semigroups. Depending on the candidate’s interests the dissertation may then focus on any of a number or possible topics, looking either at abstract operator theoretical aspects or at an application of the general theory to a particular type of evolution equation.
Possible avenues of investigation

- Abstract semigroup theory, for instance holomorphic semigroups, stability of semigroups, growth and spectral bounds, the Gearhart-Prüss theorem, hyperbolicity, perturbation theory
- Applications to particular evolution equations, such as the wave equation, delay equations or problems arising in population dynamics or control theory
- Extensions of the standard theory, for instance to non-homogeneous problems, non-autonomous equations or integrated semigroups

Pre-requisite knowledge:

- Essential: A solid understanding of the basic theory of bounded linear operators on normed vector spaces, including some spectral theory (as covered for instance in B4.1 Functional Analysis I (https://courses.maths.ox.ac.uk/node/36340))
- Recommended: Familiarity with certain slightly more advanced results in functional analysis, such as the uniform boundedness principle and the closed graph theorem (as covered for instance in B4.2 Functional Analysis II (https://courses.maths.ox.ac.uk/node/36360)); also some prior exposure to simple linear evolution equations such as the heat equation and the wave equation (as covered for instance in the Prelims course on Fourier Series and PDEs (https://courses.maths.ox.ac.uk/node/37583), so not necessarily treated rigorously), and ideally, although this can be acquired during the project, some familiarity with Sobolev spaces (which may be touched on in B4.1 or B4.2 and otherwise will appear in the course C4.3 Functional Analytic Methods for PDEs (https://courses.maths.ox.ac.uk/node/36827))
- Useful: Familiarity with the rigorous theory of PDEs (as covered for instance in C4.3)
Useful reading


Further references


• E.B. Davies. Linear Operators and their Spectra. CUP, 2004

• J.A. Goldstein. Semigroups of Linear Operators & Applications (2nd ed.). Dover, 2017