A model for interest rates in repressed markets

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We consider an extension of the Vasicek short-rate model aimed to capture central bank actions to repress fixed income markets by keeping rates low for extended period of time. We also incorporate this model into a Libor Market Model.
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Chapter 1

Introduction

1.1 Economic preview

Target federal funds rate has been near zero\(^1\) from 16th of December 2008 and at the time of writing it is still not clear when monetary policy in the United States will become aggressive again\(^2\). Bernanke (2010) stated two examples of central bank commitments to easy monetary policy conditional on certain event occurring in the future\(^3\). Such central bank actions reduce uncertainty in the future path of short-term interest rates, which can be used in mathematical modelling of interest rates and pricing of fixed income products.

1.2 Modelling

In terms of the yield curve volatility, easy monetary policy results in low volatility of its short end during the period when this policy is maintained. In the present paper we study how calendar dependency of the volatility structure can be modelled in the Heath-Jarrow-Morton (HJM) framework. We use affine short rate model to predict forward rates in a financial scenario when loose monetary policy is tightened after certain period of time.

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\(^1\) Target range 0-25 basis points.

\(^2\) Although on 12th December 2012, the Federal Open Market Committee (FOMC) released the notes of its October 2012 meeting where it voted to keep the target federal funds rate low until unemployment falls to 6.5 per cent (conditional on inflation expectations). The minutes are available in Federal Open Market Committee press release (2012).

\(^3\) The Bank of Canada in April 2009 committed to low policy rate until the second quarter of 2010 (conditional on the inflation outlook), and the Bank of Japan in March 2001 committed to maintain its policy rate at zero until consumer prices stabilise or increase on a year-on-year basis.
1.3 Strategy

We construct a Libor market model (LMM) with low Markov dimensionality of the short rate by starting from an affine model with plausible dynamics of forward rates, calculating their instantaneous volatilities and then using this volatility structure to calculate volatilities of discrete Libor rates for calibration and pricing purposes.

Structure of the paper

Section 2 introduces the short-rate affine term-structure model we will use throughout the rest of the paper. In Section 3 we calculate no-arbitrage bond prices, determine basic probabilistic properties of short and forward rates and show how instantaneous forward rate volatility depends on model parameters. Section 4 is devoted to study principal components of the model and also how can it be used in modelling business cycles, long yields and central bank actions. In section 5 we build a Libor Market Model consistent with the affine model introduced above. We introduce a special probability space on which the approximate LMM with Libor rates with Gaussian dynamics is correctly defined. Section 6 justifies approximations we use when calibrating the LMM to at-the-money USD volatilities in 2012. We conclude on the model in section 7. Libor drifts are derived in Appendix A.1 and Appendix A.2 has proofs of all propositions used in the paper.
Chapter 2

The model

2.1 Short-rate model

Dynamic mean-reverting Vašíček (DMRV) model is an extension of the Vašíček model, where mean reversion level is dynamic and follows an Ornstein-Uhlenbeck process:

\[ dr_t = \kappa(\theta_t - r_t)dt + \sigma_r dW^r_t, \quad (2.1) \]
\[ d\theta_t = \alpha(\beta - \theta_t)dt + \sigma_{\theta} dW^\theta_t, \quad (2.2) \]
\[ d\langle r, \theta \rangle_t = \rho dt, \quad (2.3) \]

where \( d\langle r, \theta \rangle_t \) is the quadratic cross-variation process for processes \( r_t \) and \( \theta_t \).

Here \( r_t \) is the short rate, \( \theta_t \) is its dynamic mean-reversion level, \( \kappa \) and \( \alpha \) are the constant mean reversion speeds, \( \beta \) is the reversion level of the short rate mean reversion level; \( \sigma^\theta \) is the mean reversion level's constant volatility and \( \sigma_r \) is the time-dependent deterministic volatility of the short rate.

2.2 Model interpretation

Initial short rate \( r_0 \) is assumed to be low to represent low level of short-term USD interest rates at the time of writing. Short rate normal volatility \( \sigma^r_0 \) is also low initially to represent central bank's commitment to maintain easy monetary policy for an extended period of time. Eventual exit and tightening of short-term interest rates is represented by a combination of higher level of mean reversion than initial short rate:

\[ \beta > r_0. \]
and monotonically increasing \( \sigma_t^r \) as a function of \( t \). Typical instantaneous and root-mean squared (RMS) forward normal volatilities for constant \( \sigma_t^r \) and for monotonically increasing \( \sigma_t^r \) are plotted on Figure 2.1 alongside market implied caplet normal volatilities.

### 2.3 Short rate volatility as logistic function

We consider short rate volatility \( \sigma_t^r \) in the following form:

\[
\sigma_t^r = \sigma_0^r + \frac{\sigma_{\text{max}}^r - \sigma_0^r}{1 + e^{k(T^* - t)}}, \quad k > 0, \tag{2.4}
\]

which monotonically increases from \( \sigma_0^r \) at \( t = 0 \) to \( \sigma_{\text{max}}^r \) at \( t \to \infty \). \( T^* \) is the inflection point, and interpreted as the moment when policy rate will no longer be repressed and be set freely (that is to say to be raised, we will assume \( \sigma_{\text{max}}^r > \sigma_0^r \) throughout the rest of the paper). Typical graph of function (2.4) is plotted on Figure 2.2.
Figure 2.2: Short rate volatility as logistic function. $\sigma_r^0 = 20$ bps, $\sigma_{r, \text{max}} = 120$ bps, $T^* = 2$, $k = 2, 9$. At $k \to \infty$, logistic function converges to the Heaviside function.
Chapter 3
No-arbitrage bond pricing

3.1 Derivation

DMRV model is a particular case of a more general Gaussian model with three factors, well described by James and Webber (2000) in Chapter 7. As such, it is an **affine model**, that is a model in which zero-coupon $T$-bond prices are given by

$$P(t, T) = e^{-A(t,T)-B(t,T)r_t-C(t,T)\theta_t}. \quad (3.1)$$

By Itô’s formula, stochastic differential equation for bond price (3.1) is

$$dP(t, T) = \frac{\partial P}{\partial t} - B(t, T) P(t, T) \, dr_t - C(t, T) P(t, T) \, d\theta_t + \frac{1}{2} B^2(t, T) P(t, T) \, d\langle r \rangle_t$$

$$+ \frac{1}{2} C^2(t, T) P(t, T) \, d\langle \theta \rangle_t + B(t, T) C(t, T) P(t, T) \, d\langle r, \theta \rangle_t,$$

where $\langle x \rangle_t$ is the quadratic variation of a process $x_t$ and $\langle x, y \rangle_t$ is the quadratic cross-variation process of $x_t$ and $y_t$.

We know that bond price is driftless in the risk-neutral measure:

$$0 = -A_t - B_t r_t - C_t \theta_t - B\kappa (\theta_t - r_t) - C\alpha (\beta - \theta_t)$$

$$+ \frac{1}{2} B^2(\sigma_r^2) + \frac{1}{2} C^2(\sigma_{\theta}^2) + BC \rho \sigma_r^2 \sigma_{\theta}^2,$$ \quad (3.2)

where for brevity $A = A(t, T), A_t = \frac{\partial A}{\partial t}(t, T)$ with equivalent expressions for $B$ and $C$.

Equation (3.2) must be true for all values of $r_t$ and $\theta_t$, therefore we obtain the following system of ordinary differential equations for $A, B$ and $C$:

$$A_t = \frac{1}{2} B^2(\sigma_r^2) - C\alpha \beta + \frac{1}{2} C^2(\sigma_{\theta}^2) + BC \rho \sigma_r^2 \sigma_{\theta}^2,$$ \quad (3.3)

$$B_t = B\kappa,$$ \quad (3.4)

$$C_t = -B\kappa + C\alpha.$$ \quad (3.5)
Boundary conditions are:

\[ A(T, T) = B(T, T) = C(T, T) = 0, \]

because bond prices pull to par at maturity: \( P(T, T) = 1 \).

ODEs for \( B(t, T) \) and \( C(t, T) \) can be solved analytically:

\[ B(t, T) = \frac{1}{\kappa} \left( 1 - e^{-\kappa(T-t)} \right), \quad (3.6) \]

and

\[ C(t, T) = \frac{1}{\alpha} \left( 1 - \frac{\kappa}{\kappa - \alpha} e^{-\alpha(T-t)} \right) + \frac{1}{\kappa - \alpha} e^{-\kappa(T-t)}. \quad (3.7) \]

We give expression for \( A(t, T) \) for the case when \( \sigma^r_t \) is constant in Appendix A.

Solution for \( A(t, T) \) when \( \sigma^r_t \) is a logistic function (2.4) involves confluent hypergeometric functions for which fast, accurate and robust numerical procedure is currently not known to the author. However, it is easy to solve ODE for \( A(t, T) \) numerically, for example with Matlab’s routine \texttt{ode45}.

**Proposition 3.1.1.** In DMRV model (2.1)—(2.3), strong solutions to short rate and mean reversion levels are

\[ \theta_t = \theta_0 e^{-\alpha t} + \beta (1 - e^{-\alpha t}) + \sigma^\theta \int_0^t e^{\alpha(s-t)} dW_s^\theta, \quad (3.8) \]

and

\[ r_t = r_0 e^{-\kappa t} + \frac{\kappa (\theta_0 - \beta)}{\kappa - \alpha} (e^{-\alpha t} - e^{-\kappa t}) + \beta (1 - e^{-\kappa t}) + \frac{\kappa \sigma^\theta}{\kappa - \alpha} \int_0^t (e^{\alpha(s-t)} - e^{\kappa(s-t)}) dW_s^\theta + \sigma^r \int_0^t e^{\kappa(s-t)} \sigma_s^r dW_r. \quad (3.9) \]

**Proof.** In Appendix A.2. \( \Box \)

**Corollary 3.1.2.** Short rate \( r_t \) in DMRV model is a Gaussian normal variable with mean

\[ \mathbb{E} r_t = r_0 e^{-\kappa t} + \frac{\kappa (\theta_0 - \beta)}{\kappa - \alpha} (e^{-\alpha t} - e^{-\kappa t}) + \beta (1 - e^{-\kappa t}), \quad (3.10) \]

and variance

\[ \text{var} (r_t) = \left( \frac{\kappa \sigma^\theta}{\kappa - \alpha} \right)^2 \left[ \frac{1}{2\alpha} (1 - e^{-2\alpha t}) + \frac{1}{2\kappa} (1 - e^{-2\kappa t}) - \frac{2}{\kappa + \alpha} (1 - e^{-(\kappa + \alpha) t}) \right] 
+ \int_0^t e^{2\kappa(s-t)} (\sigma_s^r)^2 ds + \frac{\rho \kappa \sigma^\theta}{\kappa - \alpha} \int_0^t (e^{(\kappa + \alpha)(s-t)} - e^{2\kappa(s-t)}) \sigma_s^r ds. \quad (3.11) \]
If, in particular, short rate volatility is constant \( \sigma^r = \sigma^r \),

\[
\text{var} (r_t) = \left( \frac{\kappa \sigma^\theta}{\kappa - \alpha} \right)^2 \left[ \frac{1}{2\alpha} \left( 1 - e^{-2\alpha t} \right) + \frac{1}{2\kappa} \left( 1 - e^{-2\kappa t} \right) - \frac{2}{\kappa + \alpha} \left( 1 - e^{-(\kappa + \alpha)t} \right) \right] + \frac{(\sigma^r)^2}{2\kappa} \left( 1 - e^{-2\kappa t} \right) + \frac{\rho \kappa \sigma^\theta \sigma^r}{\kappa - \alpha} \left[ \frac{1}{\kappa + \alpha} \left( 1 - e^{-(\kappa + \alpha)t} \right) - \frac{1}{2\kappa} \left( 1 - e^{-2\kappa t} \right) \right].
\]

(3.12)

**Proof.** Immediate consequence of the fact that integrands in stochastic integrals in (3.9) are deterministic functions and Itô’s isometry:

\[
\text{var} (r_t) = E \left( (r_t - E r_t)^2 \right) = E \langle r \rangle_t.
\]

\[\square\]

**Corollary 3.1.3.** Long rate \( r_\infty \) in DMRV model with constant short rate volatility is a Gaussian normal variable with mean

\[
E r_\infty = \beta,
\]

and variance

\[
\text{var} (r_\infty) = \left( \frac{\kappa \sigma^\theta}{\kappa - \alpha} \right)^2 \left[ \frac{1}{2\alpha} + \frac{1}{2\kappa} - \frac{2}{\kappa + \alpha} \right] + \frac{(\sigma^r)^2}{2\kappa} + \frac{\rho \kappa \sigma^\theta \sigma^r}{\kappa - \alpha} \left[ \frac{1}{\kappa + \alpha} - \frac{1}{2\kappa} \right].
\]

(3.14)

**Proof.** Setting \( t \to \infty \) in (3.12).

\[\square\]

### 3.2 Forward rate volatility as a function of model parameters

Bond prices can be expressed via forward rates:

\[
P(t, T) = \exp \left( - \int_t^T f(t, s) \, ds \right),
\]

where \( f(t, T) \) is the instantaneous forward rate set at \( t \) for dealing at \( T \). Forward rates are then:

\[
f(t, T) = - \frac{\partial \ln P(t, T)}{\partial T}.
\]
Instantaneous forwards in DMRV model are:

\[ f(t, T) = a^T_t + b^T_t r_t + c^T_t \theta_t, \]  

(3.15)

where

\[ a^T_t = \frac{\partial A}{\partial T}(t, T), \]

\[ b^T_t = \frac{\partial B}{\partial T}(t, T), \]

and

\[ c^T_t = \frac{\partial C}{\partial T}(t, T). \]

Equations for \( b^T_t \) and \( c^T_t \) have closed-form solutions:

\[ b^T_t = e^{-\kappa(T-t)}, \]  

(3.16)

\[ c^T_t = \frac{\kappa}{\kappa - \alpha} \left( e^{-\alpha(T-t)} - e^{-\kappa(T-t)} \right). \]  

(3.17)

Equation (3.15) is linear in \( r_t \) and \( \theta_t \) and \( C^\infty \)-differentiable in \( t \), therefore we can apply Itô’s formula:

\[
df(t, T) = \mathcal{O}(dt) + b^T_t dr_t + c^T_t d\theta_t \\
= \mathcal{O}(dt) + b^T_t \sigma^r_t dW^r_t + c^T_t \sigma^\theta_t dW^\theta_t,
\]

(3.18)

where \( \mathcal{O}(dt) \) is the drift term.
Let us denote
\[
\sigma^T_t = \sqrt{(b^T_t \sigma_t)^2 + (c^T_t \sigma^\theta_t)^2 + 2b^T_t c^T_t \rho \sigma_t \sigma^\theta_t},
\]
then another way of writing equation (3.19) is
\[
\sigma^T_t = \sigma^T_t \Delta t,
\]
so instantaneous total volatility \( \sigma^T_t \) of \( f(t, T) \) can be seen as the “derivative” of quadratic covariation w.r.t. time.

Apart from having plausible dynamics of short rate (which is automatic in DMRV model because it is built in), we would like to preserve humped shape of caplet volatility curve. As a first approximation to caplet volatilities we take instantaneous volatilities of forwards (3.20).

Figures 3.1 and 3.2 show graphs of \( \frac{\partial B}{\partial T} \) and \( \frac{\partial C}{\partial T} \) as functions of time to maturity \( T - t \) with fixed maturity \( T = 10 \) and moving calendar time \( t \).

Naturally, \( \frac{\partial B}{\partial T} \) is just an exponential decay and thus no value of reversion speed \( \kappa \) can help us to achieve humped-shaped \( \sigma^T_t \). Behaviour of \( \frac{\partial C}{\partial T} \) is richer: it starts at zero, reaches its maximum and then exponentially decays with speed dictated by
Figure 3.3: Volatility of the instantaneous forwards $\sigma^T_t$ as a function of time to maturity. Short rate volatility is constant. Volatility of the reversion level is 120 basis points per year, correlation $\rho$ is 0.4.

$\alpha$, the reversion speed of the reversion level (we assume $\kappa > \alpha$). With this complex behaviour, it is plausible that one can find a combination of values $\kappa$ and $\alpha$, that lead to humped-shaped instantaneous volatility curve.

Figures 3.3 and 3.5 show specimen instantaneous forwards’s volatility curves for various combinations of parameters. A small, but visible dip in $\sigma^T_t$ near origin is caused by a particular combination of parameters in (3.20). Attempts to remove it by increasing short rate volatility $\sigma^r$ greatly change the shape of the curve, so is not available option if one wants to calibrate to at-the-money market volatilities (of caps and swaptions). Increasing $\sigma^\theta$, the volatility of the mean reversion level does remove the dip (Figure 3.4), but it also changes the shape of the instantaneous volatility curve $\sigma^T_t$. There are three major ways to combat this dip:

- reduce short rate volatility $\sigma^r$ to a low level, for example 20 basis points per year,
- use root-mean squared averages of instantaneous volatilities $\hat{\sigma}_T$ given by equation (3.21), this will “smooth out” the dip,
- use calendar time-dependant (deterministic) instantaneous volatility function $\sigma^T_t$, such as the logistic function (2.4).
Recall the definition of root-mean square average of volatility $\hat{\sigma}_T^2$: 

$$\hat{\sigma}_T^2 = \frac{1}{T} \int_0^T (\sigma_s^T)^2 \, ds. \quad (3.21)$$

First approach is not very attractive because we are effectively “wasting” one state variable. However, since market observable quantities are cap and swaption volatilities correspond to RMS volatilities (3.21), second approach is more natural. In fact, because the dip only affects short-tenor forward rates near origin, we can combine all three approaches together: use low short rate volatility near $t = 0$, use logistic function to bridge it with higher short rate volatility after initial “repressed” period and calculate RMS average of such obtained $\sigma_T^T$. As we shall see later in section 6, this approach would give very good fit to at-the-money market volatilities (from caps and swaptions).

Figure 3.5 displays RMS volatility curves with various parameters.
Figure 3.5: RMS volatility of the instantaneous forwards $\hat{\sigma}_T$ as a function of maturity $T$. Short rate volatility given by logistic function (2.4). Volatility of the reversion level is 121 basis points per year, correlation $\rho$ is 0.4. Blue solid line corresponds to the blue line on Figure 3.3. Red dashed line corresponds to short rate volatility being "repressed" for 2 years at 20 basis points per year. The effect of this repression on forward rates with long maturity $T$ is muted. The dotted line represents RMS volatility with constant short rate volatility of 20 basis points per year. Logistic function acts as a bridge between the two curves.

3.3 Correlation

Instantaneous volatilities of forward rates $\sigma^T_t$ in DMRV model have a rich dependency on correlation coefficient $\rho$. On Figures 3.6 and 3.7 we have graphs for $\sigma^T_t$ as a function of residual time to maturity $T - t$ (and also as a function of calendar time $t$ on Figure 3.7). Note that function $\sigma^T_t$ is time homogeneous when short rate volatility $\sigma^r_t = \text{const}$, that is map $T \mapsto \sigma^{T+t}_t$ does not depend on $t$. It is not so when $\sigma^r_t$ has calendar time dependency and we showed this as a series of charts with varying $t$.

We shall see later in Section 5.3 how this correlation structure affects the correlation surface between Libor rates in LMM-DMRV model and how it brings calendar time dependency into it.
Figure 3.6: Effect of correlation coefficient $\rho$ on instantaneous volatility $\sigma_t^T$. Short rate volatility is constant at 80 basis points and volatility of the reversion level is 121 basis points per year.
Figure 3.7: Dynamics of the $\sigma^T_t$ as a function of calendar time $t$ and residual time to maturity $T - t$ when short rate volatility is a logistic function (2.4). All parameters are identical to that of Figure 3.6. Different charts show snapshots taken at calendar time points $t_{start} \in \{0, 1, 1\frac{1}{2}, 1\frac{3}{4}, 2$ and 4 years}, and the functions are $\sigma^{T+t_{start}}_{t_{start}}$ with $T - t_{start}$ on the horizontal axis.
Chapter 4
Principal components

To establish more knowledge of the dynamics of the yield curve in the DMRV model, we shall analyse its principal components. Principal component analysis, or PCA is a traditional tool in fixed income modelling to study the driving factors behind yield curve moves, typically parallel shifts, tilts and changes of curvature.\(^1\) In other words, since short-rate models are specified in terms of dynamics of the short-rate, which is not directly observable one needs to transform it into movements of a more observable quantity, for example bond yields.

Recall that yield \(y(t, T)\) at time \(t\) of a zero-coupon bond maturing at \(T\) is
\[
y(t, T) = -\frac{\ln P(t, T)}{T - t},
\]
that is to say the yield is the discount rate which reprices the bond today (or at a given point in the future before bond’s maturity).

For PCA, we need to fix a set of bond maturities, say 6 months, 1 year, 2 years and so on:
\[
\mathcal{T} = \{T_i : i = 1, 2, \ldots, n\}.
\]

We then discretise calendar time \(t\), simulate short rates \(r_t\) and reversion levels \(\theta_t\), calculate bond yields, calculate covariance matrix of changes in yields and then run principal component analysis on this matrix (for covariance matrices this is equivalent to finding its eigenvalues and eigenvectors).

Figures 4.1 and 4.2 show simulated short rates with weekly time steps for 30 years (and with daily steps in Figure 4.2).

\(^1\) These movements can be interpreted as frequencies in Fourier series expansion, with parallel shift being lowest frequency and curvature highest. Even higher order frequencies can also be studied, but often ignored as noise. This is arguable for fourth and fifth principal components, but is certainly true for higher order terms.
Figure 4.1: Simulation of short rates. Model parameters: short-rate volatility is a logistic function with $\sigma^0_r = 20$ bps, $\sigma^\text{max}_r = 120$ bps, $k = 4$ and $T^* = 1.8$ years, mean-reversion level volatility is constant $\sigma^\theta = 133$ bps, initial short rate $r_0 = 1.25\%$ and initial reversion level $\theta_0 = 3.5\%$, reversion level of the mean reversion $\beta = 9.5\%$, reversion speeds: for short rate $\kappa = 0.3$, for reversion level $\alpha = 0.0561$. These parameters are typical when calibrating to USD market data of late 2012. Both graphs on the left- and right-hand side are simulated using Brownian motion drivers with the same seed.

We take 10 maturities for our PCA:

$$\mathcal{T} = \{6\text{m}, 1\text{y}, 2\text{y}, 3\text{y}, 5\text{y}, 7\text{y}, 10\text{y}, 15\text{y}, 20\text{y}, 30\text{y}\},$$

and discretise calendar time in weekly time steps:

$$0 = t_0 < t_1 < \ldots < t_N = 30\text{y}.$$

On each time step $t_j$ we calculate bond yields for bonds with constant maturity:

$$y(t_j, t_j + T_i) = A^T_{t_j} + B^T_{t_j} r_{t_j} + C^T_{t_j} \theta_{t_j}, \quad \forall T_i \in \mathcal{T},$$

and calculate changes in yields:

$$Y = \{ y(t_j, t_j + T_i) - y(t_{j-1}, t_{j-1} + T_i) | T_i \in \mathcal{T}, 1 \leq j \leq N \}, \quad (4.2)$$

where $Y$ is the matrix with bond maturities $T_i$ as rows and calendar time steps $t_j$ as columns.
Figure 4.2: The same simulation as in Figure 4.1, zoomed in to 5 years calendar time. One can clearly see the effect of the logistic function on short-rate volatility in about 2 years’s time.

Of course, (4.2) is just a discretisation of the stochastic differential equation:

\[ dy_t = \left( \frac{\partial a_t^{T-t}}{\partial t} + \frac{\partial b_t^{T-t}}{\partial t} r_t + \frac{\partial c_t^{T-t}}{\partial t} \theta_t \right) \, dt + \tilde{b}_t^{T-t} \, dr_t + \tilde{c}_t^{T-t} \, d\theta_t, \]

where for brevity

\[ \tilde{a}_t = \frac{A_t^T}{T-t}, \]
\[ \tilde{b}_t = \frac{B_t^T}{T-t}, \]
\[ \tilde{c}_t = \frac{C_t^T}{T-t}. \]

General form of the equation above may be complicated, but since we are only interested in covariances between changes in yields, we shall just calculate quadratic cross-variations between yields as stochastic processes:

\[ d\langle y^i, y^k \rangle_t = \tilde{b}_t^{T-t} \, d\langle r \rangle_t + \tilde{c}_t^{T-t} d\langle \theta \rangle_t, \]

where \( \tilde{b}_t^{T-t} \) and \( \tilde{c}_t^{T-t} \) were shortened to \( \tilde{b}^T \) and \( \tilde{c}^T \), since they only depend on residual time to maturity, and not on calendar time.

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\(^2\) Which is an application of the Itô’s formula to equation (4.1). We can use Itô’s formula because expression for yields \( y(t, T) \) is linear in \( r_t \) and \( \theta_t \) and is continuously differentiable in \( t \).
The matrix we are interested in, $Y$, consists of integrated cross-variances:

$$Y_{ik} = \frac{1}{T} \int_0^T d\langle y^i, y^k \rangle_t. \tag{4.4}$$

For example, for standard Vašíček model, this matrix is

$$Y_{ik} = \bar{b}^T \bar{b}^T (\sigma^r)^2,$$

or in matrix form,

$$Y = \bar{b} \bar{b}^T (\sigma^r)^2,$$

where $\bar{b}$ is the column-vector with values $\bar{b}^{ri}$.

For this matrix, the single non-zero eigenvalue is

$$\lambda = \|\bar{b}\|^2 (\sigma^r)^2,$$

where

$$\|\bar{b}\|^2 = \bar{b}^T \bar{b}.$$

Corresponding eigenvector is:

$$\frac{\bar{b}}{\|\bar{b}\|},$$

which has exponentially decaying shape.

To find the eigenvalues of (4.4) for DMRV, we need to arrive to instantaneous cross-variances $d\langle y^i, y^k \rangle_t$ differently. First, let us orthogonalise Brownian motions in the formula for $dy_t$:

$$dy_t = \mathcal{O}(dt) + \begin{pmatrix} \bar{b}^T \sigma_t^r \\ \bar{c}^T \sigma_t^\theta \end{pmatrix}^T \begin{pmatrix} W_t^r \\ W_t^\theta \end{pmatrix} = \mathcal{O}(dt) + \begin{pmatrix} \bar{b}^T \sigma_t^r \\ \bar{c}^T \sigma_t^\theta \end{pmatrix}^T \begin{pmatrix} 1 \\ \rho \sqrt{1 - \rho^2} \end{pmatrix} \begin{pmatrix} W_t^1 \\ W_t^2 \end{pmatrix},$$

where Brownian motions $W^1$ and $W^2$ are independent:

$$d\langle W^1, W^2 \rangle_t = 0.$$

Define $\bar{c}$ as a column-vector with $\bar{c}^{ri}$, then instantaneous quadratic cross-variances for yields is thus (in formal vector form)

$$d\langle y, y^T \rangle_t = (\sigma_t^r \bar{b} + \rho \sigma_t^\theta \bar{c}) \sqrt{1 - \rho^2 \sigma_t^\theta} \left( \begin{array}{c} \bar{b}^T + \rho \sigma_t^\theta \bar{c}^T \\ \sqrt{1 - \rho^2 \sigma_t^\theta} \bar{c}^T \end{array} \right) dt,$$
where instantaneous covariation matrix is square and has the same dimensions as
the number of maturities we are interested in. Because of its particular form, the
eigenvalues are easy to calculate analytically (although expressions involved are
rather cumbersome):

$$\lambda_{1,2} = \frac{1}{2} \left( \tilde{b}^\top \tilde{b} + \tilde{c}^\top \tilde{c} ± \sqrt{(\tilde{b}^\top \tilde{b})^2 + (\tilde{c}^\top \tilde{c})^2} - 2\tilde{b}^\top \tilde{b} \tilde{c}^\top \tilde{c} + 4(\tilde{b}^\top \tilde{c})^2 \right) dt,$$

(4.5)

where for simplicity

$$\tilde{b} = \sigma^r \bar{b} + \rho \sigma^b \bar{c},$$

and

$$\tilde{c} = \sqrt{1 - \rho^2} \sigma^a \bar{c}.$$  

Interpreting these factors in general form is difficult, but we can illustrate them
via a numerical experiment.

On Figure 4.3 we have eigenvalues and corresponding eigenvectors for DMRV
model with the same parameters as in simulation on Figure 4.1, which are typi-
cal when calibrating LMM-DMRV model\(^\text{3}\) to at-the-money volatilities of USD cap
and swaption markets of late 2012. Second principal component (represented by
green line in figure 4.3) distributes shocks to the curvature of the yield curve and
in this particular example controls approximately 14 per cent of the total variation
of changes in yields. In other words, DMRV can be thought of as a “1.5” factor
model, rather than full two factor model, such as additive 2-factor Hull-White or
2-factor Vašiček G2++ in terminology of Brigo and Mercurio (2006). Let this not
upset the reader because DMRV explains and is good at incorporating business
cycles and central bank actions as we shall see in the next subsection. Also, in
order to get a hump-shaped cap volatility curve, we do not have to impose a rather
counter-intuitive negative correlation between factors. As we shall see, correlation
\(\rho\) in DMRV stays between 0.2 and 0.5 to give good fit to USD cap vols.

Balduzzi et al. (1996) and Chen (1996) study three-factor models, similar to
DMRV but richer to include stochastic variance as a third driving Brownian mo-
tion.\(^\text{4}\) Unfortunately, neither model admits closed-form solutions for bond prices (in
functions that are fast to compute) and also pose restrictions on model parameters
(such as correlation between short rate and its dynamic reversion level must be
zero).

\(^3\) That is the LMM model with forward rate volatilities given by DMRV model, equation (3.20). We study this model in section 5.

\(^4\) Balduzzi et al use Gaussian family of affine models, while Chen’s model is from the CIR (Cox-Ingersoll-Ross) family.
4.1 Business cycles

Models with dynamic mean reversion level can be used to model business cycles. James and Webber (2000) have a good overview of various models available for this purpose. In particular, in Section 11.3.2 they consider generalised Vašíček model:

\[
\begin{align*}
    dr_t &= \kappa (\theta_t - r_t) dt + \sigma r dW^r_t, \\
    d\theta_t &= \alpha (pr_t + (1-p)\beta - \theta_t) dt + \sigma \theta dW^\theta_t,
\end{align*}
\]

they also show how this can lead to chaotic behaviour in short rate even in deterministic case ($\sigma^r = 0$ and $\sigma^\theta = 0$).

Shiryaev (1998) in §4a.4 (p. 340) derives a similar model (although non-linear) considering short rate as the conditional mathematical expectation of an unobservable “economy internal state” process $X_t$, consisting of a Markov chain with two values: “expansion” and “contraction”, and a noise term. See also Diebold and Rudebusch (1994) for a survey of economic papers on business cycle modelling.

Beaglehole and Tenney (1991) and Dai and Singleton (2000) have a very good description of dynamic mean-reversion models, including non-linear ones.

Modern developments in this field include papers by Diebold and Li (2006) for extension of 3-factor Nelson-Siegel model controlling parallel moves, changes in
slope and curvature of bond yield curve, and Christensen et al. (2008) for non-arbitrage version of it.

Andersen and Piterbarg (2010b) in Chapter 12.1.4, p. 497, suggest that when central bank actions become predictable, this can be incorporated in a DMRV model by setting short rate volatility to 0. Note that in this case, bond yield covariance matrix (4.4) is simple and equation (4.5) has a single eigenvalue: \[ \lambda_0 = \left\| \bar{c} \right\|^2 \left( \sigma^2 \right) dt, \]
where \[ \left\| \bar{c} \right\|^2 = \bar{c}^T \bar{c}. \]
Corresponding eigenvector is just \( \frac{\bar{c}}{\left\| \bar{c} \right\|} \), it has a humped shape, peaked around 10-15 years (naturally, dependent on model parameters) and its specimen is shown on Figure 4.4. Figure 4.5 shows simulation of short rates with no short rate volatility, one can clearly see long periods of similar values in \( r_t \) with no short-term “noise”.

Lastly in this section, let us look at the dynamics of long-term yields:

---

5 We consider DMRV with constant short-rate volatility here: \( \sigma_t^r = 0 \) for all \( t > 0 \).
Figure 4.5: Simulation of short rates with no short rate volatility. Volatility of the mean reversion level is constant at 133 basis points. Reversion speeds: for short rate $\kappa = 0.3$, for reversion level $\alpha = 0.0561$. Reversion level of the mean reversion $\beta = 9.5\%$.

**Proposition 4.1.1.** The long yield $y_\infty$ in DMRV model, that is the limit in (4.1) with $T \to \infty$ is

$$y_\infty = \beta - \frac{(\sigma^r_{\text{max}})^2}{2\kappa^2} - \frac{(\sigma^\theta)^2}{2\alpha^2} - \frac{\rho \sigma^r_{\text{max}} \sigma^\theta}{\kappa \alpha}.$$  \hspace{1cm} (4.6)

**Proof.** In Appendix A.2. \hfill $\Box$

The fact that long yield is a constant is of course no surprise. Dybvig et al. (1996) have proven it in a very general setting.
Chapter 5
Libor Market Model with DMRV forwards

5.1 Basic setup

Our main model is LMM-DMRV model, that is a hybrid of Libor market model, but with forward rate volatilities given by short-rate model DMRV, formula (3.20).

We take a filtered probability space \((Ω, ℱ, ℱ, P)\), where \(Ω\) is the sample space, \(ℱ\) the \(σ\)-algebra of measurable sets in \(Ω\) interpreted as “events”, \(P\) the probability measure and \(ℱ\) the filtration, or the set of non-decreasing sub-\(σ\)-fields:

\[ ℱ = (ℱ_t)_{t≥0}, \quad ℱ_s ⊆ ℱ_t ⊆ ℱ, \]

for \(0 ≤ s ≤ t < ∞\). We assume the usual conditions for \(ℱ\).

Measure \(P\) is called the objective measure, but it will not be used directly anywhere in this paper (apart from its existence and full mass: \(P(Ω) = 1\), we do not require anything from it).

Key modelling objects in the Libor market model are tenor dates\(^1\),

\[ 0 = T_0 < T_1 < \cdots < T_N < ∞, \]

and Libor rates, discrete forward rates set in arrears at \(T_i\) and paid in advance at \(T_{i+1}\):

\[ L_i(t) = \frac{1}{τ_i} \left( \frac{P(t, T_i)}{P(t, T_{i+1})} - 1 \right), \quad (5.1) \]

here \(τ_i\) is the coverage, or day-count fraction between \(T_i\) and \(T_{i+1}\) (with floating leg convention, typically “Act/360”).

\(^1\) Libor fixing dates, reset times, . . .
In a general LMM setup, Libor rates $L_1(t), L_2(t), \ldots, L_N(t)$ are individually modelled as stochastic processes:

$$dL_i(t) = \mathcal{O}(dt) + \sigma_i^i(t, L_i) dW^i_t,$$

(5.2)

here $\mathcal{O}(dt)$ are drifts (we will calculate them later), $\sigma_i(t, L_i)$ is the vector of Libor volatilities and $W^i_t$ is the vector of driving P-Brownian motions adapted to the filtration $\mathcal{F}$. Initial conditions for equations (5.2) are to reproduce the initial (deterministic) yield curve:

$$L_i(0) = L^*_i.$$

We will be working with two types of local volatility functions:

- shifted log-normal, $\sigma_i(t, L_i) = (1 + \tau_i L_i) \sigma_f(t, T_i)$,
- Gaussian, $\sigma_i(t, L_i) = \sigma_f(t, T_i)$,

where $\sigma_f(t, T)$ is the vector of volatilities of instantaneous forward rates $f(t, T)$.

These two sub-models are similar and with a relatively high level of shift in the former, both produce distributions for Libor rates which are close to Gaussian. Our calibration experiment is done on the Gaussian model and therefore we need to theoretically justify it first. Let us begin with a link between short-rate models and Libor market model.

### 5.2 LMM and HJM Volatility

Recall a useful formula (14.11) on p. 598 in Andersen and Piterbarg (2010b), linking Libor market model and instantaneous forward volatilities (Heath-Jarrow-Morton framework):

$$\sigma_i(t, L_i) = \frac{1}{\tau_i} (1 + \tau_i L_i(t)) \int_{T_i}^{T_{i+1}} \sigma_f(t, u) du,$$

(5.3)

where $\sigma_f(t, T)$ is the vector of volatilities of instantaneous forward rates $f(t, T)$.

Recall formula (3.15) for instantaneous forward rates in DMRV model:

$$f(t, T) = a_t^T + b_t^T r_t + c_t^T \theta_t,$$

where $x_t^T$ is a symbolic shorthand for $x_t^T = \frac{\partial X}{\partial T}(t, T)$. 

25
By Itô’s formula, the dynamics of $f(t, T)$ is:

$$
\begin{align*}
\frac{df(t, T)}{dtdt} &= \Omega(dt) + b_T^t \sigma^*_r dW_r^t + c_T^t \sigma^*\theta dW_\theta^t,
\end{align*}
$$

(5.4)

where $\Omega(dt)$ is the drift term which we don’t need now.

We now can proceed to finalise the set up of LMM-DMRV model. Because we know explicitly how forward rates evolve, and we know that they are driven by two factors, our version of LMM is going to have two factors. This naturally limits possible shapes of correlation matrices we can have in our model, but instead we gain knowledge about financial side (rate repression and eventual exit).

Vector $\sigma_f(t, T)$ is

$$
\sigma_f(t, T) = (b_T^t \sigma^*_r, c_T^t \sigma^*\theta)^\top.
$$

The first approximation we make is to treat function $\sigma_f(t, T)$ piecewise-constant between two fixing dates, in which case equation (5.3) becomes

$$
\sigma_i(t, L_i) = (1 + \tau_i L_i(t)) \sigma_f(t, T_i).
$$

This is the shifted log-normal formulation of local volatility. Andersen and Piterbarg (2010b) prove in their Lemma 14.2.5 that this formulation of LMM has strong solutions $L_i(t)$ and that forward measures $Q_T^i$ are theoretically sound (that is they are equivalent martingale measures, or that bond prices are positive almost surely).

For calibration we will use the Gaussian formulation of $\sigma_i(t, L_i)$, that is we are going to approximate discrete forward rate with instantaneous forward rate of the same expiry,

$$
\sigma_i(t, L_i) = \sigma_f(t, T_i).
$$

(5.5)

Assuming forward rates below 10%2 and 3-month3 floating frequency, shifted log-normal distribution is close to Gaussian, with advantage of the Gaussian model in calibration to swaptions. However, because in a model with constant local volatility, rates are unbounded from below, we need to say a few words about the probability space where this model operates.

---

2 After all, we are modelling very low interest rates.
3 Default frequency of floating rate payments in USD markets is quarterly and hence default tenor for Libor rates is 3 month.
A note on the sample space

We need to make sure that bond prices in LMM-DMRV model with Gaussian volatility are positive so that we can use discount bonds as numéraire assets. Unfortunately, this is not the case in general, so a restriction of the sample space is required: with Gaussian model we shall be working in the sub-space $$(\Omega_0, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}, \tilde{P})$$, where

$$\Omega_0 = \left\{ \omega \in \Omega : L_i(t, \omega) \geq -\frac{1}{\tau_i} + \varepsilon_0, \forall t, \forall i \right\},$$

(5.6)

where $\varepsilon_0$ is a small positive constant, $L_i(t, \omega)$ is the Libor rate, and $\tau_i$ is its coverage, or day-count fraction. The other parameters are restrictions of their respective counterparts to $\Omega_0$:

$$\tilde{\mathcal{F}} = \{ A \in \mathcal{F} : A \subseteq \Omega_0 \},$$

$$\tilde{\mathcal{F}} = (\tilde{\mathcal{F}}_t)_{t \geq 0} \tilde{\mathcal{F}}_t = \mathcal{F}_t \cap \tilde{\mathcal{F}},$$

$$\tilde{P}(A) = P(A|\Omega_0).$$

Proposition 5.2.1. Space $$(\Omega_0, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}, \tilde{P})$$ is a filtered probability space satisfying the usual conditions.

Proof. In Appendix A.2.$\square$

Proposition 5.2.2. Assume coordinate-wise Lipschitz continuity and linear growth for Gaussian volatility function

$$\sigma^i = \sigma_f(t, T_i),$$

for all $i$. Then the system (5.2) has unique, non-explosive and $t$-continuous solutions. Moreover, when solutions $L_i(t)$ are considered on restricted probability space $$(\Omega_0, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}, \tilde{P})$$, then bond prices are strictly positive and solutions exist in all forward measures $Q^{T_i}$.

Proof. In Appendix A.2.$\square$

5.3 Correlation

The vector $W = (W^1, W^2)$ of driving Brownian motions in the LMM-DMRV model is two-dimensional, with correlations given as part of the DMRV model setup:

$$\langle W^1, W^2 \rangle_t = \rho t.$$  

(5.7)
Consider the Gaussian version of LMM-DMRV model. First, let us calculate instantaneous covariances between two Libor rates, \( L_i(t) \) and \( L_j(t) \). This can be done easily with the usual rules on quadratic cross-variation between Itô integrals from equations (5.2), (5.5) and (5.7):

\[
\begin{align*}
\langle L_i(t), L_j(t) \rangle &= b_i^t b_j^t (\sigma_I^r)^2 dt \\
&+ (b_i^t c_i^t + c_i^t b_i^t) \rho \sigma_I^r \sigma_\theta^r dt \\
&+ c_i^t c_j^t (\sigma_\theta^r)^2 dt \\
&\overset{\text{def}}{=} \Lambda_{ij}(t) dt,
\end{align*}
\]  

(5.8)

where

\[
b_i^t = \frac{\partial B_i^T}{\partial T} = e^{-\kappa(T_i - t)},
\]

and

\[
c_i^t = \frac{\partial C_i^T}{\partial T} = \frac{\kappa}{\kappa - \alpha} \left( e^{-\alpha(T_i - t)} - e^{-\kappa(T_i - t)} \right).
\]

\[4\] This restriction for USD markets turns out to be rather loose: in this market typically \( \tau = \frac{1}{4} \), so the additional requirement on Libor rates: \( L_i > -400\% \) is not very restrictive.
Instantaneous correlation coefficient between the two Libor rates is therefore:

\[
\rho_{ij}(t) = \frac{\Lambda_{ij}(t)}{\sqrt{\Lambda_{ii}(t)\Lambda_{jj}(t)}}. \tag{5.9}
\]

Figures 5.1 and 5.2 show charts of \( \rho_{ij}(t) \) with typical model parameters for various \( t \). Note how long-term Libor rates initially are almost perfectly correlated, but become decorrelated with time because of strong calendar time dependency of \( \sigma_t^r \).
Figure 5.2: Dynamics of correlation matrix $\rho_{ij}(t)$ with $t = 0, 1, 1\frac{1}{2}, 2$ and 3 years. We use rolling maturity so that time to maturity remains fixed on all charts. Model parameters are the same as on Figure 5.1.
Chapter 6
Calibration

Let us first describe calibration instruments and their LMM-DMRV model implied prices and volatilities.

6.1 Caps

In LMM, model price of a $T$-cap struck at $K$ is

$$\sum_{i=1}^{n-1} P(0, T_i+1) \tau_i E_{i+1}(L_i(T_i) - K)^+, \quad (6.1)$$

where $\tau_i$ is the day-count fraction (or coverage) of the $i$-th caplet\(^2\) and $L_i$ is the Libor rate set at $T_i$ and paid at $T_{i+1}$ (summation starts from 1 because there is almost no variation in $L_0$: $T_0$ is typically only 2 business days from today), $n$ is such that $T_n = T$ and $E_{i+1}$ is the mathematical expectation operator under $T_{i+1}$-forward measure (that is the measure with $T_{i+1}$-bond as numéraire).

Andersen and Piterbarg (2010b) show in Proposition 14.4.1 that $L_i$ is driftless under $Q_{T_{i+1}}$, or $T_{i+1}$-forward measure, and has the dynamics:

$$dL_i(t) = \| \sigma_i(t, L_i) \| dY_{i+1}^t, \quad (6.2)$$

where $Y_{i+1}^t$ is a Brownian motion in measure $Q_{T_{i+1}}$.

Because of Gaussian dynamics of Libor rates, we can’t use Black’s formula for expectation in (6.1). However, Bachelier (1900) provides us with such a formula:

$$E(F + \sigma W_T - K)^+ = (F - K) \Phi(d) + \sigma \sqrt{T} \phi(d), \quad (6.3)$$

\(^1\) See for example Proposition 14.4.1 in Andersen and Piterbarg (2010b), p. 615.

\(^2\) That is $\tau_i$ is the day-count fraction between $T_i$ and $T_{i+1}$ with floating payment day-count convention, “Act/360” in USD markets.
where \( F \) is forward, \( d \) is option’s moneyness:

\[
d = \frac{F - K}{\sigma \sqrt{T}},
\]

and \( \varphi \) and \( \Phi \) are respectively standard normal probability density and cumulative distribution functions:

\[
\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}},
\]

and

\[
\Phi(x) = \int_{-\infty}^{x} \varphi(u)du.
\]

**Proposition 6.1.1.** In LMM-DMRV model, \( T_i \)-caplet price struck at \( K \) is given by the formula:

\[
E_{i+1}(L_i(T_i) - K)^+ = (L_i(0) - K)\Phi(d) + \hat{\sigma}_i \sqrt{T_i} \varphi(d), \tag{6.4}
\]

where

\[
d = \frac{L_i(0) - K}{\hat{\sigma}_i \sqrt{T_i}},
\]

and

\[
\hat{\sigma}_i^2 = \frac{1}{T_i} \int_0^{T_i} (\sigma^T_s)^2 \, ds.
\]

**Proof.** In Appendix A.2.

RMS volatilities in LMM-DMRV with calendar-time dependent instantaneous forward volatilities \( \sigma^T_t \) can be computed using a numerical scheme, for example Gaussian quadrature. When this dependency is a logistic function (2.4), closed-form solutions are available but they involve confluent hypergeometric functions and therefore have limited practical use.

For our calibration we are going to minimise the following functional

\[
C_{\text{cap}} \overset{\text{def}}{=} \sum_{i=2}^{N} (\hat{\sigma}_i - \hat{\sigma}_i^{MKT})^2 \to \min, \tag{6.5}
\]

where \( N \) is such that \( T_N \) is the maturity of the longest market quoted cap and \( \hat{\sigma}_i^{MKT} \) is the market implied Bachelier volatility\(^3\) of \( i \)-th caplet. Caplet volatilities are obtained by the usual bootstrap procedure described in Section 6.4.3 by Brigo and Mercurio (2006).

\(^3\) Also known as basis point vol, Black (normal) vol or Gaussian vol.
6.2 Swaptions

Let us begin with a quick introduction of swap rate and its decomposition into a linear combination of Libor rates.

Coupon rate for a swap starting at \( T_\alpha \) with final payment at \( T_\beta \) is defined as

\[
S_{\alpha,\beta}(t) = \frac{P(t, T_\alpha) - P(t, T_\beta)}{A_{\alpha,\beta}(t)},
\]

where \( A_{\alpha,\beta}(t) \) is the annuity\(^4\)

\[
A_{\alpha,\beta}(t) = \sum_{j=\alpha+1}^{\beta} \delta_j P(t, T_j).
\]

By definition of Libor rate, formula (5.1):

\[
P(t, T_\alpha) - P(t, T_\alpha + \tau_\alpha) = \tau_\alpha L_\alpha(t) P(t, T_\alpha + \tau_\alpha),
\]

that is to say:

\[
P(t, T_\alpha) - P(t, T_\beta) = \sum_{i=\alpha}^{\beta-1} \tau_i L_i(t) P(t, T_{i+1}).^6^7
\]

Now swap rates allow simple decomposition as weighted sum of Libor rates:

\[
S_{\alpha,\beta}(t) = \sum_{i=\alpha}^{\beta-1} w_i(t) L_i(t),
\]

where

\[
w_i(t) = \frac{\tau_i P(t, T_i)}{A_{\alpha,\beta}(t)}.
\]

Although \( w_i(t) \) is stochastic, its variation \( \langle w_i \rangle_t \) is assumed to be small. For calibration purposes we will make this approximation:

\[
S_{\alpha,\beta}(t) \approx \sum_{i=\alpha}^{\beta-1} w_i(0) L_i(t).
\]

\(^4\) Level, swap PV01, swap duration, \ldots

\(^5\) By round deltas \( \delta_j \), we shall denote year fractions under the swap’s fixed leg convention: day-count rule, business day convention and payment frequency. \( \tau_i \) will refer to floating leg’s year fractions. This can cause some nuisance, but is necessary to get right if one wants to implement accurate calibration.

\(^6\) We allow for slight abuse of notation here: \( T_i \) denote payment dates of the floating leg (with indexing of the floating leg), while \( T_j \) are the fixed leg payment dates, so \( T_1 \) when \( i = 1 \) is not the same as \( T_1 \) when \( j = 1 \). We shall be clear on which date schedule is used where lest confusion arises.

\(^7\) Also with slight abuse of terminology, \( \beta \) shall indicate the payment index of the last payment of the swap, potentially different on the fixed and floating sides.
This function is linear in $L_i(t)$ so we can apply Itô’s formula to both sides of the equation above to get approximately:

$$dS_{\alpha,\beta}(t) \approx \sum_{i=\alpha}^{\beta-1} w_i(0)dL_i(t),$$

with approximate quadratic variation

$$d\langle S_{\alpha,\beta} \rangle_t \approx \sum_{i,k=\alpha}^{\beta-1} w_i(0)w_k(0)d\langle L_i(t), L_k(t) \rangle.$$  

We know (Lemma 4.2.4 by Andersen and Piterbarg (2010a)) that there exists a $P$-equivalent martingale measure$^8$ in which $A_{\alpha,\beta}$ is a numéraire asset and therefore $S_{\alpha,\beta}$ is a martingale (as a combination of traded assets discounted by numéraire). Denote this measure $Q^A$ and let $E^A$ be the corresponding mathematical expectation operator.

A European payer swaption has payoff

$$(S_{\alpha,\beta} - K)^+,\)$$

so it’s an option to pay fixed rate $K$ and receive floating (Libor).

By the change of numéraire technique, no-arbitrage price of such option is

$$A_{\alpha,\beta}(0)E^A(S_{\alpha,\beta} - K)^+. \quad (6.8)$$

To value this integral we need to know the distribution of $S_{\alpha,\beta}(T_\alpha)$ (we already know it has mean zero, thanks to martingale property). This is a complex and potentially non-elementary distribution, but we know that approximate process $\tilde{S}_{\alpha,\beta}(t)$ has Gaussian marginals:

$$d\tilde{S}_{\alpha,\beta} = \sum_{i,k=\alpha}^{\beta-1} w_i(0)dL_i(t),$$

this is due to Gaussian distribution of increments $L_i(t+s) - L_i(t)$ and deterministic volatility coefficient of $d\tilde{S}_{\alpha,\beta}$ (note that neither of this is true for the original $dS_{\alpha,\beta}$).

The last approximation we are going to make in order to be able to quickly calibrate to the swaption volatility surface is that $\tilde{S}_{\alpha,\beta}$ is also a martingale under $Q^A$. In this case, the $Q^A$-distribution of $\tilde{S}_{\alpha,\beta}(T_\alpha)$ is Gaussian with mean 0 and variance $\langle \tilde{S}_{\alpha,\beta} \rangle_{T_\alpha}$.  

$^8$ Here as always, $P$ is the objective probability measure.
We can now use Bachelier’s formula to approximate integral (6.8):

\[
A_{\alpha,\beta}(0)E_{Q^A}(S_{\alpha,\beta}(T_\alpha) - K)^{+} \\
\approx A_{\alpha,\beta}(0) \left( [S_{\alpha,\beta}(0) - K]\Phi(d) - \hat{\sigma}_{\alpha,\beta}\sqrt{T_\alpha}\phi(d) \right),
\]

(6.9)

where

\[
\hat{\sigma}_{\alpha,\beta} = \langle S_{\alpha,\beta} \rangle_{T_\alpha},
\]

and

\[
d = \frac{S_{\alpha,\beta}(0) - K}{\hat{\sigma}_{\alpha,\beta}\sqrt{T_\alpha}}.
\]

Of course, for analytical tractability we are going to use approximate variance

\[
\langle \tilde{S}_{\alpha,\beta} \rangle_{T_\alpha},
\]

rather than exact \( \langle S_{\alpha,\beta} \rangle_{T_\alpha} \).\(^9\)

Volatility for purposes of Bachelier formula is then

\[
\tilde{\sigma}_{\alpha,\beta} = \sqrt{\frac{1}{T_\alpha} \langle \tilde{S}_{\alpha,\beta} \rangle_{T_\alpha}},
\]

(6.10)

with

\[
\langle \tilde{S}_{\alpha,\beta} \rangle_{T_\alpha} = \sum_{i,k=\alpha}^{\beta-1} w_i(0)w_k(0) \int_0^{T_\alpha} \Lambda_{ik}(s)ds,
\]

(6.11)

where \( \Lambda_{ik}(s) \) is the instantaneous covariance defined in equation (5.8).

This is the volatility we are going to use for minimisation of the calibration functional \( C_{\text{swpn}}(\mathcal{F}) \):

\[
C_{\text{swpn}}(\mathcal{F}) \overset{\text{def}}{=} \sum_{i \in \mathcal{F}} (\tilde{\sigma}_i - \sigma_i^{\text{MKT}})^2 \rightarrow \min,
\]

(6.12)

where \( \mathcal{F} \) is the set of calibration swaptions, and \( \sigma_i^{\text{MKT}} \) their market implied basis point vols.\(^{10}\)

Another way of computing the calibration functional is to take model implied Black (log-normal) volatilities for swaptions and compare them with market implied Black volatilities. One could argue that this would introduce unnecessary numerical error from the conversion (from model’s natural quantity, Gaussian vol, into log-normal vol); we do not pursue this approach in our experiment.

\(^9\) It would be desirable to know \( \langle S_{\alpha,\beta} \rangle_{T_\alpha} \) exactly, of course, because in this case the only approximation we will be making is about marginal distributions of \( S_{\alpha,\beta} \) and not about its variance. Options at or near the money depend on variance more than on the shape of the probability distribution.

\(^{10}\) These are the volatilities that if used in Bachelier formula, would give current market swaption prices.
6.3 Calibration results

We calibrate LMM-DMRV model to at-the-money US Dollar caps and swaptions on 7-July-2012, 13-Sep-2012, 7-Mar-2013 and 8-Mar-2013.

Our strategy for calibration problem is to use multidimensional Levenberg-Marquardt method to minimise cap calibration functional $C_{\text{cap}}$ from a cold start\(^{11}\), then also using Levenberg-Marquardt method proceed to minimisation of $C_{\text{swptn}}(\mathcal{S})$ from the point obtained on the cap calibration step.

More formally, we proceed as follows

1. With multidimensional Levenberg-Marquardt method, find

   $$(\kappa_c, \alpha_c, \beta_c, \sigma_0^c, \sigma_{\text{max}}^c, \rho_c, T^*_{\text{c}}) = \text{arg min } C_{\text{cap}}(\kappa, \alpha, \beta, \sigma_0^r, \sigma_{\text{max}}^r, \sigma^\theta, \rho T^*, k),$$

2. fix a set of calibration swaptions $\mathcal{S}$, we used the following set of at-the-money swaptions: \(^{12}\)

   $$\mathcal{S} = \begin{array}{cccccc}
   \text{expiry} & 1y & 2y & 3y & 4y & 5y & 10y \\
   1y & x & x & x & x & x & x \\
   2y & x & x & x & x & x & \\
   3y & x & x & x & x & \\
   4y & x & x & x & x & \\
   5y & x & x & x & x & \\
   7y & x & x & x & x & \\
   10y & x & x & x & x & 
   \end{array}$$

3. using this point as initial guess, find

   $$(\kappa_s, \alpha_s, \beta_s, \sigma_0^s, \sigma_{\text{max}}^s, \rho_s, T^*_{\text{s}}) = \text{arg min } C_{\text{swptn}}(\mathcal{S}, \kappa, \alpha, \beta, \sigma_0^r, \sigma_{\text{max}}^r, \sigma^\theta, \rho T^*, k).$$

Note that we do not calibrate $k$, parameter of the logistic function responsible for the “speed of loosening”, that is how fast short rate volatility rises from $\sigma_0^r$ to $\sigma_{\text{max}}^r$ around $T^*$. We prefer to keep this parameter fixed, since it is a purely model parameter with no direct impact on the dynamics of Libor rates. The risk to this parameter however, is useful and should be calculated, because it shows sensitivity of swaption value (or any other fixed income product we wish to price with thus calibrated LMM-DMRV) to how fast central bank policy will be changed.

When doing the minimisations, we also enforce a number of relationships and constraints:

\(^{11}\) That is without a good initial guess.

\(^{12}\) There is nothing particularly special about this set of instruments except that it is reasonably global, that is representative of the entire swaption volatility surface up to 10 years of expiry, and covers most of liquid points.
• $\kappa > \alpha$,
• $\sigma_r^0 < \sigma_{\text{max}}^r$,
• $\sigma_r^0 < \sigma_\theta$,
• $0.2 \leq \rho \leq 0.5$.

This is done with model’s explanation power in mind. Our goal is not goodness of fit (standard LMM with piecewise constant local volatility with parametric correlation will probably calibrate a lot better), but having control over “financial side” of things.

Note that we do not use constrained optimisation, but rather use functional relationships:

$$x > y \iff x = y + z^2,$$

and

$$a \leq x \leq b \iff x = a + (b - a) \cos^2 \varphi.$$

As with any other LMM, computational bottleneck of the calibration routine is the calculation of covariance integrals

$$\int_0^{T_n} \Lambda_{ij}(s) \, ds,$$

and for LMM-DMRV this is made worse by having to calculate them numerically (recall again that as it stands, there is no fast and robust way of calculating confluent hypergeometric functions that are necessary to integrate the logistic function).

Obviously, parts of these integrals that are common for different swaptions can be cached in memory, but unfortunately on each iteration we will have to empty the caches because the model parameters would change.

Non-global algorithm, such as the “cascade” described by Brigo and Mercurio (2006) in section 7.4, is obviously not suitable for LMM-DMRV because it has too few parameters and each parameter has non-local impact.

By and large, computation time for LMM-DMRV calibration is acceptable and from a cold start on a modern laptop computer, it takes about 2 minutes to calibrate to the set of swaptions described above.\(^{13}\)

\(^{13}\) Needless to say that calibration to the cap volatilities is almost instant, because of linear computational complexity.
6.4 Data

We used end-of-day quotes\(^\text{14}\) for at-the-money USD cap/floor and swaption volatility as published by ICAP on their Reuters screen (VCAP23 for caps and floors and VCAP21 for swaptions). USD swap rates for initial yield curve construction were obtained from the end-of-date database of one investment bank in London.

Calibrated model parameters are presented below in Table 6.1. A few things stand out: firstly, calibration is quite robust (even unaided by special methods such as Tikhonov regularisation) and parameters are quite stable over time. Secondly, repression time horizon shrinks over time, which only adds support to the model’s ability to explain things, not just reproduce market data. Thirdly, we included a very short period of one day between 7th and 8th of March 2013 when USD market anticipated an important labour statistic announcement: non-farm payroll report on the 8th of March 2013. The report came particularly optimistic and somewhat reduced US unemployment\(^\text{15}\), which brought, according to the model, market expectation of a possible “exit” date from short rate repression almost 3 months closer (from about 2 years and 2 month to 1 year and 11 months).

<table>
<thead>
<tr>
<th>Market date</th>
<th>$\kappa$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\sigma_0^r$</th>
<th>$\sigma_{\max}^r$</th>
<th>$\sigma^\theta$</th>
<th>$\rho$</th>
<th>$T^*$</th>
<th>$k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6-Jul-2012</td>
<td>0.2369</td>
<td>0.0505</td>
<td>9.50</td>
<td>0.20</td>
<td>0.83</td>
<td>1.38</td>
<td>0.4719</td>
<td>2.3187</td>
<td>4</td>
</tr>
<tr>
<td>13-Sep-2012</td>
<td>0.1573</td>
<td>0.0789</td>
<td>9.50</td>
<td>0.20</td>
<td>0.80</td>
<td>1.75</td>
<td>0.4523</td>
<td>2.2401</td>
<td>4</td>
</tr>
<tr>
<td>7-Mar-2013</td>
<td>0.2414</td>
<td>0.0610</td>
<td>9.50</td>
<td>0.20</td>
<td>1.04</td>
<td>1.43</td>
<td>0.4004</td>
<td>2.1649</td>
<td>4</td>
</tr>
<tr>
<td>8-Mar-2013</td>
<td>0.2429</td>
<td>0.0632</td>
<td>9.50</td>
<td>0.20</td>
<td>1.02</td>
<td>1.45</td>
<td>0.4141</td>
<td>1.9068</td>
<td>4</td>
</tr>
</tbody>
</table>

Table 6.1: Calibrated model parameters. All columns except for $\beta$, $\sigma_0^r$, $\sigma_{\max}^r$ and $\sigma^\theta$ are in units, unscaled. Aforementioned columns are in per cent.

---

\(^{14}\) Snapshots were taken around 21 p.m. London time each day, to include as much of New York trading hours as possible.

\(^{15}\) Recall that Federal Open Market Committee tied monetary policy to unemployment and inflation.
Table 6.2: Calibration errors for swaptions, in basis points. Volatility surface from 13-Sep-2012.

<table>
<thead>
<tr>
<th>Expiry</th>
<th>1Y</th>
<th>2Y</th>
<th>3Y</th>
<th>5Y</th>
<th>7Y</th>
<th>10Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>1Y</td>
<td>4.0987</td>
<td>10.6167</td>
<td>8.7321</td>
<td>2.7828</td>
<td>−3.4937</td>
<td>−11.6133</td>
</tr>
<tr>
<td>2Y</td>
<td>4.7916</td>
<td>6.3448</td>
<td>2.4611</td>
<td>−3.9022</td>
<td>−5.9246</td>
<td>−10.5157</td>
</tr>
<tr>
<td>3Y</td>
<td>5.2758</td>
<td>0.3281</td>
<td>−0.3879</td>
<td>−3.2267</td>
<td>−4.8194</td>
<td>−5.3212</td>
</tr>
<tr>
<td>5Y</td>
<td>−3.5097</td>
<td>−4.6805</td>
<td>−1.6920</td>
<td>−2.0600</td>
<td>0.0360</td>
<td>−0.5856</td>
</tr>
<tr>
<td>7Y</td>
<td>0.9954</td>
<td>1.0473</td>
<td>0.3212</td>
<td>3.2225</td>
<td>3.5115</td>
<td>2.7505</td>
</tr>
<tr>
<td>10Y</td>
<td>5.0787</td>
<td>8.2367</td>
<td>8.7169</td>
<td>7.4360</td>
<td>7.4113</td>
<td>5.9300</td>
</tr>
</tbody>
</table>

Table 6.3: Market implied basis point volatilities of swaptions. Volatility surface from 13-Sep-2012.

<table>
<thead>
<tr>
<th>Expiry \ Tenor</th>
<th>1Y</th>
<th>2Y</th>
<th>3Y</th>
<th>5Y</th>
<th>7Y</th>
<th>10Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>1Y</td>
<td>30.2669</td>
<td>31.6864</td>
<td>40.6199</td>
<td>57.2382</td>
<td>70.5528</td>
<td>84.2018</td>
</tr>
<tr>
<td>2Y</td>
<td>40.0084</td>
<td>45.4308</td>
<td>55.1823</td>
<td>70.3975</td>
<td>78.0412</td>
<td>86.6865</td>
</tr>
<tr>
<td>3Y</td>
<td>57.4644</td>
<td>67.0851</td>
<td>71.8554</td>
<td>80.7340</td>
<td>85.8169</td>
<td>88.1939</td>
</tr>
<tr>
<td>5Y</td>
<td>86.5049</td>
<td>90.1151</td>
<td>89.1965</td>
<td>92.0986</td>
<td>90.9380</td>
<td>90.6733</td>
</tr>
<tr>
<td>7Y</td>
<td>90.3680</td>
<td>91.5012</td>
<td>93.1470</td>
<td>91.0878</td>
<td>90.3534</td>
<td>88.8343</td>
</tr>
<tr>
<td>10Y</td>
<td>90.3671</td>
<td>87.6183</td>
<td>87.1328</td>
<td>87.6913</td>
<td>86.1965</td>
<td>84.3071</td>
</tr>
</tbody>
</table>

Figure 6.1: Market implied swaption volatility surface (left-hand side) and calibration errors in basis points (right-hand side). Market as of 13-Sep-2012. The data are the same as in Tables 6.2 and 6.3.
Chapter 7

Conclusion

Main motivation for introducing affine term-structure model into LMM is to be able to use explicit dynamics of forward rates that can explain future changes in interest rates. Monetary policy makers in the US have committed to start raising short-term interest rates sooner or later and a model that is built for this purpose is a useful tool for price discovery and risk-management. We saw in Section 6.4 an example of how this model can interpret economic facts, and also give reasonable quality fit to market data (stressing again that goodness of fit is not our goal, but having a model that can explain future dynamics of the yield curve is).
Appendix A

A.1 Libor drifts in LMM-DMRV

Proposition A.1.1. In LMM-DMRV model, drift of $L_i(t)$ under $T_i$-forward measure is:

$$dL_i(t) = (1 + \tau_i L_i(t))\sigma_f(t, T_i) \begin{pmatrix} \tau_i \sigma_f(t, T_i) \end{pmatrix} dt + dW_i^t,$$

(A.1)

or in shifted log-normal formulation, the drift is deterministic

$$\frac{dL_i(t)}{1 + \tau_i L_i(t)} = \sigma_f(t, T_i) \begin{pmatrix} \tau_i \sigma_f(t, T_i) \end{pmatrix} dt + dW_i^t.$$

Here $W^i$ is a $T_i$-Brownian motion.

Proof. Recall from Heath et al. (1992) that discount bond price has risk-neutral dynamics:

$$dP(t, T) = \mathcal{O}(dt) - P(t, T) \int_t^T \sigma_f(t, s)^T ds \, dW_t,$$

where we omit the drift term for now and $W$ is a Brownian motion under the risk-neutral measure.

Libor rates in LMM are discretely compounded forward rates:

$$L_i(t) = \frac{1}{\tau_i} \left( \frac{P(t, T_i)}{P(t, T_{i+1})} - 1 \right),$$

with dynamics by Itô’s formula,

$$dL_i(t) = \mathcal{O}(dt) + \frac{1}{\tau_i P(t, T_{i+1})} dP(t, T_i) - \frac{P(t, T_i)}{\tau_i P(t, T_{i+1})^2} dP(t, T_{i+1}).$$

From this formula, simple algebra gives us

$$dL_i(t) = \mathcal{O}(dt) + \frac{1}{\tau_i} (1 + \tau_i L_i(t)) \int_{T_i}^{T_{i+1}} \sigma_f(t, s)^T ds \, dW_t.$$
We agreed to use piecewise-constant volatilities between two fixing dates, so we can integrate the above expression into:

\[ dL_i(t) = \gamma dt + (1 + \tau_i L_i(t)) \sigma_f(t, T_i)^\top dW_t. \]

\( L_i(t) \)’s natural numéraire is \( P(t, T_{i+1}) \), hence in \( T_{i+1} \)-forward measure it is a martingale as a linear combination of tradable assets, and therefore the drift is zero:

\[ dL_i(t) = (1 + \tau_i L_i(t)) \sigma_f(t, T_i)^\top dW_t^{i+1}. \quad (A.2) \]

To establish \( T_i \)-dynamics of \( L_i(t) \), we need to use the Girsanov theorem and change of numéraire technique.

Let \( Z_t \) be the process

\[ Z_t = \frac{P(t, T_i)}{P(0, T_i)} \frac{P(t, T_{i+1})}{P(0, T_{i+1})}, \]

then by Theorem 1 in Geman et al. (1995), there exists measure \( Q^i \), the \( T_i \)-forward measure and Radon-Nikodym derivative between it and \( T_{i+1} \)-forward measure is

\[ \frac{dQ^i}{dQ^{i+1}} \bigg|_{\mathcal{F}_t} = Z_t, \quad Q^{i+1}\text{-a.s.} \]

We need to find a process \( X_t \), such that

\[ Z_t = \mathcal{E}(X)_t, \quad Q^{i+1}\text{-a.s.} \]

where \( \mathcal{E}(\xi)_t \) is the stochastic exponent:

\[ \mathcal{E}(\xi)_t = \exp \left( \xi_t - \frac{1}{2} \langle \xi \rangle_t \right), \]

and then by Girsanov theorem (Theorem 5.1 in Karatzas and Shreve (1991)),

\[ W^i = W^{i+1} - \langle X, W^{i+1} \rangle_t, \]

would be a \( Q^i \)-Brownian motion.

Rearranging \( Z_t \) in terms of Libor rate \( L_i(t) \), we get

\[ Z_t = \frac{1 + \tau_i L_i(t)}{1 + \tau_i L_i(0)}, \]

and by Itô’s formula

\[ dZ_t = \frac{\tau_i}{1 + \tau_i L_i(0)} dL_i(t), \]

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or
\[
\frac{dZ_t}{Z_t} = \tau_i \sigma_f(t, T_i) \top dW^{i+1}_t.
\]

This SDE admits explicit solution, a geometric Brownian motion:
\[
Z_t = Z_0 \exp \left( -\frac{\tau_i^2}{2} \int_0^t (\sigma_s^T)^2 ds + \tau_i \int_0^t \sigma_f(s, T_i) \top dW^{i+1}_s \right),
\]
where
\[(\sigma_t^T)^2 = \sigma_f(t, T) \top \sigma_f(t, T).\]

That is to say,
\[
Z_t = \mathcal{E}(X_t),
\]
with
\[
X_t = \tau_i \int_0^t \sigma_f(s, T_i) \top dW^{i+1}_s.
\]

Now calculate quadratic cross-variation of this process with \(W^{i+1}\):
\[
d\langle X, W^{i+1} \rangle_t = \tau_i \sigma_f(t, T_i) dt.
\]

Using Girsanov theorem, rewrite equation (A.2) with \(W^i\):
\[
dL_i(t) = (1 + \tau_i L_i(t)) \sigma_f(t, T_i) \top (d\langle X, W^{i+1} \rangle_t + dW^i_t)
\]
\[
= (1 + \tau_i L_i(t)) \sigma_f(t, T_i) \top (\tau_i \sigma_f(t, T_i) dt + dW^i_t).
\]

\[\square\]

**Corollary A.1.2.** Libor rate under \(T_i\)-forward measure is
\[
L_i(t) = \left( L_i(0) + \frac{1}{\tau_i} \right) \exp \left( \frac{\tau_i^2}{2} \int_0^t (\sigma_s^T)^2 ds + \tau_i \int_0^t \sigma_f(s, T_i) \top dW^i_s \right) - \frac{1}{\tau_i}, \quad (A.3)
\]
where \(\sigma^T_t = \|\sigma_f(t, T)\|^2\) is the total volatility of instantaneous forwards in DMRV as given by equation (3.20).

**Proof.** Note that,
\[
\tau_i \frac{dL_i(t)}{1 + \tau_i L_i(t)} = \frac{d \left( \frac{1}{\tau_i} + L_i(t) \right)}{1 + \tau_i L_i(t)}.
\]

So that the process \(\xi_t = \frac{1}{\tau_i} + L_i(t)\) is a geometric Brownian motion with drift:
\[
\xi_t = \xi_0 \mu_t dt + \xi_t \sigma^\xi_t dW^\xi_t,
\]

and as such can be integrated directly:
\[
\xi_t = \xi_0 \exp \left[ \int_0^t \left( \mu_s - \frac{1}{2} \sigma_s^T \sigma_s \right) ds + \int_0^t \sigma^\xi_s dW^\xi_s \right].
\]

\[\square\]
Definition A.1.1. Spot measure $Q^B$ is the measure defined by numéraire $B(t)$:

$$B(t) = P(t, T_{\eta_t}) \prod_{i=0}^{\eta_t-1} (1 + \tau_i L_i(T_i)),$$

where $\eta_t$ is the first Libor fixing time after $t$.

Corollary A.1.3. In LMM-DMRV under the spot measure, $L_i(t)$ dynamics is

$$\frac{dL_i(t)}{1 + \tau_i L_i(t)} = \sum_{j=\eta_t}^i \tau_j L_{ij}(t) dt + \sigma_f(t, T_i) dW^B,$$

where $L_{ij}(t)$ is the instantaneous covariance between $L_i(t)$ and $L_j(t)$ as given by equation (5.8) and $W^B$ is a 2-dimensional Brownian motion under $Q^B$.

Proof. Immediate consequence of Lemma 14.2.3 by Andersen and Piterbarg (2010b). For convenience of the reader, the idea of the proof is to split the numéraire asset under $Q^B$

$$B(t) = P(t, T_{\eta_t}) \prod_{i=0}^{\eta_t-1} (1 + \tau_i L_i(T_i)),$$

into random part, the discount bond $P(t, T_{\eta_t})$ and the remainder. Then we apply Proposition A.1.1 iteratively to get shifted log-normal dynamics of $L_i(t)$ in $T_i^-, T_{i-1}^-, \ldots, T_{\eta_t}^-$. forward measures, adding an extra drift term on each step:

$$\tau_j \sigma_f(t, T_i)^T \sigma_f(t, T_j) dt,$$

for $j$ “rolling” backwards: $j = i, i-1, \ldots, \eta_t$.

Corollary A.1.4. In LMM-DMRV under the terminal measure, or $T_N^-$-forward measure, shifted log-normal dynamics of $L_i(t)$ is

$$\frac{dL_i(t)}{1 + \tau_i L_i(t)} = -\sum_{j=i+1}^{N-1} \tau_j L_{ij}(t) dt + \sigma_f(t, T_i) dW^{T_N},$$

Proof. Use the same idea as in the proof of previous corollary.

Corollary A.1.5. Libor rate under the terminal measure is

$$L_i(t) = \left( L_i(0) + \frac{1}{\tau_i} \right) \exp \left[ \int_0^t \left( \mu_i^t - \frac{\tau_i^2}{2} \Lambda_{ii}(s) \right) ds + \tau_i \int_0^t \sigma_f(s, T_i) dW_i^s \right] - \frac{1}{\tau_i},$$

where

$$\mu_i^t = \sum_{j=i+1}^{N-1} \tau_j \tau_{ij}(t).$$
The fact that in LMM-DMRV model, Libor rates have shifted log-normal distribution with deterministic drift has many advantages over the usual log-normal or shifted log-normal Libor Market Model. In a Monte Carlo implementation one can directly simulate terminal distributions (without predictor corrector or similar discretisation schemes) which increases the performance and accuracy. Assuming forward rates below 10%, the shift of 1 is fairly high and leads to a distribution that is close to a normal distribution. The implied skew is therefore similar to the skew of a normal model (e. g. Vašíček or Hull-White).

A.2 Proofs

Proposition A.2.1. In DMRV model with constant parameters, \( A(t, T) \) is given by:

\[
A(t, T) = -\frac{(\sigma^r)^2}{2\kappa^2} \tau + \frac{(\sigma^r)^2}{\kappa^2} (1 - e^{-\kappa \tau}) - \frac{(\sigma^r)^2}{4\kappa^2} (1 - e^{-2\kappa \tau})
\]

\[\left(-\alpha \beta + \frac{\rho \sigma^r \sigma^\theta}{\kappa}\right) \left(-\frac{1}{\alpha} + \frac{\kappa}{\alpha^2 (\kappa - \alpha)} (1 - e^{-\alpha \tau}) - \frac{1}{\kappa (\kappa - \alpha)} (1 - e^{-\kappa \tau})\right)\]

\[\frac{\sigma^\theta}{2} \left(-\frac{1}{\alpha^2} \tau + \frac{2\kappa^2}{\alpha^2 (\kappa - \alpha)} (1 - e^{-\alpha \tau}) - \frac{\kappa^2}{2 \alpha^2 (\kappa - \alpha)^2} (1 - e^{-2\alpha \tau}) - \frac{1}{2 \kappa (\kappa - \alpha)} (1 - e^{-2\kappa \tau})\right)\]

\[+ \frac{\rho \sigma^r \sigma^\theta}{\kappa^2 \alpha} (1 - e^{-\kappa \tau}) - \frac{\rho \sigma^r \sigma^\theta}{\alpha (\kappa^2 - \alpha^2)} (1 - e^{-(\kappa + \alpha) \tau}) - \frac{\rho \sigma^r \sigma^\theta}{\kappa^2 (\kappa - \alpha)} (1 - e^{-\kappa \tau}),\]

where \( \tau = T - t \) for brevity.

**Proof.** By direct integration of \( \frac{\partial A(t, T)}{\partial t} \). □

**Proof of Proposition 3.1.1.**

**Proof.** Existence of strong solutions and their properties (uniqueness, continuity) are given by the general theory of stochastic calculus (for example, Theorem 5.2.1 by Øksendal (1998)), as long as diffusion coefficients are well behaved (Lipschitz continuous and grow no faster than linear). Coefficients of (2.1)—(2.3) satisfy both these conditions (with finite \( k \) in the definition of \( \sigma^r_t \), that is we do not allow the step function as short rate volatility because it will fail to be Lipschitz continuous).

We apply Itô’s formula to \( e^{\alpha t} \theta_t \)

\[
d(e^{\alpha t} \theta_t) = \alpha e^{\alpha t} \theta_t dt + e^{\alpha t} d\theta_t
\]

\[= \alpha \beta e^{\alpha t} dt + e^{\alpha t} \sigma^\theta dW^\theta.
\]

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The latter equation can be integrated:

\[ e^{\alpha t} \theta_t = \theta_0 + \beta (e^{\alpha t} - 1) + \sigma^\theta \int_0^t e^{\alpha s} dW^\theta_s. \]

Dividing both sides by \( e^{\alpha t} \), we obtain (3.8).

We can now apply Itô’s formula to \( e^{\kappa t} r_t \):

\[ d(e^{\kappa t} r_t) = \kappa e^{\kappa t} r_t dt + e^{\kappa t} dW^r_t. \]

We already know \( \theta_t \), so can integrate \( d(e^{\kappa t} r_t) \) too:

\[ e^{\kappa t} r_t = r_0 + \kappa \int_0^t e^{\kappa s} \left[ \theta_0 e^{-\alpha s} + \beta (1 - e^{-\alpha s}) + \sigma^\theta \int_0^s e^{\alpha(u-s)} dW^\theta_u \right] ds + \int_0^t e^{\kappa s} \sigma^r_s dW^r_s. \]

By stochastic Fubini’s theorem (for example, Theorem 65 in Protter (2004)), we can interchange the order of integration:

\[ \int_0^t \int_0^s e^{\kappa s+\alpha(u-s)} dW^\theta_u ds = \int_0^t \int_0^t e^{\kappa s+\alpha(u-s)} ds dW^\theta_u = \frac{1}{\kappa - \alpha} \int_0^t e^{\alpha u} (e^{(\kappa-\alpha)t} - e^{(\kappa-\alpha)u}) dW^\theta_u. \]

So, dividing both sides of expression for \( e^{\kappa t} r_t \) by \( e^{\kappa t} \) we get the desired result:

\[ r_t = r_0 e^{-\kappa t} + \kappa (\theta_0 - \beta) \int_0^t e^{\alpha(u-t)} dW^\theta_u + \frac{1}{\kappa - \alpha} \int_0^t e^{(\kappa-\alpha)u} (e^{(\kappa-\alpha)t} - e^{(\kappa-\alpha)u}) dW^\theta_u. \]

Proof of Proposition 4.1.1.

**Proof.** Integrating \( \frac{\partial A}{\partial t} \) taking into account boundary condition \( A(T, T) = 0 \), we get

\[ A(t, T) = \alpha \beta \int_t^T C(s, T) ds - \frac{1}{2} \int_t^T B(s, T)^2 (\sigma^r_s)^2 ds - \frac{(\sigma^\theta)^2}{2} \int_t^T C(s, T)^2 ds - \rho \sigma^\theta \int_t^T B(s, T) C(s, T) \sigma^r_s ds. \]
Let us fix $t, T$ and find $t_1$ such that

\[ \sigma_{t_1}^r \geq \sigma_{\text{max}}^r - \frac{\varepsilon}{T}, \]

where $\varepsilon$ is a constant which we will determine later.

Then

\[ \int_t^T B(s, T)^2(\sigma_s^r)^2 ds = \left( \int_t^{t_1} + \int_{t_1}^T \right) B(s, T)^2(\sigma_s^r)^2 ds. \]

Due to monotonicity of $\sigma_t^r$, we can calculate upper and lower bounds for the latter integral:

\[ \int_{t_1}^T B(s, T)^2(\sigma_{t_1}^r)^2 ds \leq \int_{t_1}^T B(s, T)^2(\sigma_s^r)^2 ds \leq \int_{t_1}^T B(s, T)^2(\sigma_{\text{max}}^r)^2 ds. \]

In other words,

\[ \int_{t_1}^T B(s, T)^2 \left( \sigma_{\text{max}}^r - \frac{\varepsilon}{T} \right)^2 ds \leq \int_{t_1}^T B(s, T)^2(\sigma_s^r)^2 ds \leq \int_{t_1}^T B(s, T)^2(\sigma_{\text{max}}^r)^2 ds. \]

We can integrate $B(t, T)^2$:

\[ \int_t^T B(s, T)^2 ds = \frac{1}{\kappa^2} (T - t) - \frac{2}{\kappa} (1 - e^{-\kappa(T-t)}) + \frac{1}{2\kappa} (1 - e^{-2\kappa(T-t)}). \]

Now the bounds are explicit:

\[ \left( \sigma_{\text{max}}^r - \frac{\varepsilon}{T} \right)^2 \left[ \frac{T}{\kappa^2} + \mathcal{O}(1) \right] \leq \int_{t_1}^T B(s, T)^2(\sigma_s^r)^2 ds \leq \frac{(\sigma_{\text{max}}^r)^2 T}{\kappa^2} + \mathcal{O}(1), \]

where $\mathcal{O}(1)$ is a constant plus terms infinitesimal to $T$. We can take $\varepsilon = 1$.

Now we can calculate the limit

\[ \lim_{T \to \infty} \frac{1}{T - t} \int_t^T B(s, T)^2(\sigma_s^r)^2 ds. \]

Calculate both bounds first:

\[ \lim_{T \to \infty} \frac{1}{T - t} \left( \sigma_{\text{max}}^r - \frac{\varepsilon}{T} \right)^2 \left[ \frac{T}{\kappa^2} + \mathcal{O}(1) \right] = \frac{(\sigma_{\text{max}}^r)^2}{\kappa^2}, \]

and

\[ \lim_{T \to \infty} \frac{1}{T - t} \left( \frac{(\sigma_{\text{max}}^r)^2 T}{\kappa^2} + \mathcal{O}(1) \right) = \frac{(\sigma_{\text{max}}^r)^2}{\kappa^2}. \]

Both bounds exist in the limit $T \to \infty$, limits are identical, therefore by the sandwich theorem\(^1\), we have

\[ \lim_{T \to \infty} \frac{1}{T - t} \int_{t_1}^T B(s, T)^2(\sigma_s^r)^2 ds = \frac{(\sigma_{\text{max}}^r)^2}{\kappa^2}. \]

\(^1\) Also known as the two policemen theorem, or two gendarme theorem.
Finally,
\[
\lim_{T \to \infty} \frac{1}{T-t} \int_t^T B(s, T)^2 (\sigma_s^r)^2 \, ds = \lim_{T \to \infty} \frac{1}{T-t} \left( \int_t^{t_1} + \int_{t_1}^T \right) B(s, T)^2 (\sigma_s^r)^2 \, ds \\
= \lim_{T \to \infty} \frac{1}{T-t} \int_{t_1}^T B(s, T)^2 (\sigma_s^r)^2 \, ds \\
= \frac{(\sigma_{\text{max}}^r)^2}{\kappa^2}.
\]

Using the same argument,
\[
\lim_{T \to \infty} \frac{1}{T-t} \int_t^T B(s, T) C(s, T) \sigma_s^r \, ds = \frac{\sigma_{\text{max}}^r}{\kappa \alpha}.
\]

We are now ready to calculate the limit in (A.6), integration of other terms is straightforward:
\[
\lim_{T \to \infty} A(t, T) = \beta - \frac{(\sigma_{\text{max}}^r)^2}{2 \kappa^2} - \frac{r^2}{2 \alpha^2} - \frac{r \sigma_{\text{max}}^r \sigma^\theta}{\kappa \alpha}.
\]

Proof of Proposition 5.2.1.

Proof. We need to check the definition. There are no restrictions on the sample space, so \( \Omega_0 \) is fine. First, show that \( \Omega_0 \in \mathcal{F} \), that is \( \Omega_0 \) is measurable set in the original probability space. By Theorem 5.2.1 on existence and uniqueness of solutions to SDEs in Øksendal (1998), a strong, unique, non-explosive and \( t \)-continuous (for fixed \( \omega \in \Omega \)) solution to (5.2) exists as long as the coefficients of drift and volatility \( \sigma^i(t, L_i) \) satisfy conditions of linear growth and Lipschitz continuity (separate for each component of the vector \( \sigma^i(t, L_i) \)). In LMM-DMRV, volatility is given as vector
\[
\sigma^i(t, L_i) = (1 + \tau_i L_i(t)) \begin{pmatrix} b^T L_i(t) \\ c^T L_i(t) \end{pmatrix},
\]
where \( b^T_i \) and \( c^T_i \) are given by equations (3.16) and (3.17). Both functions \( b^T_i \) and \( c^T_i \) are Lipschitz continuous (they are even \( C^\infty \)-differentiable). Clearly, both components of \( \sigma^i(t, L_i) \) is growing linearly in \( L_i(t) \), so the conditions of Theorem 5.2.1 are met. Solution \( L_i(t) \) is strong in that \( L_i(t) \) is defined on the same probability space \( (\Omega, \mathcal{F}, P) \). Processes \( L_i(t) \) have progressively measurable modifications as Itô integrals with cádlág sample paths (the paths are actually continuous on both sides), this is due to for example Proposition 1.13 by Karatzas and Shreve (1991).
Then the level set
\[ \{ L_i(s, \omega) \geq c \}, \]
is \( \mathcal{B}([0, t]) \times \mathcal{F} \)-measurable for all \( i \), here \( \mathcal{B}([0, t]) \) is the Borel sigma-algebra of open sets in \([0, t] \).

And because we have a finite set of Libor rates, the union set
\[ \{ L_i(s, \omega) \geq c, i = 0, 1, \ldots, N \} \]
is \( \mathcal{B}([0, T_N]) \times \mathcal{F} \)-measurable. Now because all Libor rates become deterministic after their respective fixing date, we have a finite time horizon in \( \Omega_0 \). The set of single points
\[ \{ \{ t \} : 0 \leq t \leq T_N \} \]
is a Borel set, and hence
\[ \{ L_i(t, \omega) \geq c, 0 \leq t \leq T_N, i = 0, 1, \ldots, N \} \]
is \( \mathcal{F} \)-measurable. Now we take constants \( c_i = -\frac{1}{\tau_i} + \varepsilon_0 \) (there is no loss of generality in the above arguments to take index dependent constant \( c_i \), rather than global \( c \)).

We showed that \( \Omega_0 \in \mathcal{F} \), and hence \( \Omega_0 \in \hat{\mathcal{F}} \).

For \( \hat{\mathcal{F}} \) to be a \( \sigma \)-algebra, it needs to satisfy:

- \( \emptyset \in \hat{\mathcal{F}} \), obviously because \( \emptyset \in \mathcal{F} \) and \( \emptyset \subset \Omega_0 \).

- \( A \in \hat{\mathcal{F}} \Rightarrow A^c \in \hat{\mathcal{F}} \). Let \( A \in \hat{\mathcal{F}} \), then its \( \Omega_0 \)-complement set \( \Omega_0 \setminus A \subset \Omega_0 \) and \( \Omega_0 \setminus A \in \mathcal{F} \) because of the sigma-ring closedness under relative complementation and due to \( \mathcal{F} \)-measurability of \( \Omega_0 \). But then by definition \( \Omega_0 \setminus A \in \hat{\mathcal{F}} \).

- Countable additivity, \( A_1, A_2, \ldots \in \hat{\mathcal{F}} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \hat{\mathcal{F}} \). Clearly, \( \bigcup_{n=1}^{\infty} A_n \in \mathcal{F} \) and we just need to show that this union lies in \( \Omega_0 \). Suppose this isn’t true and there exists a point \( \omega_0 \in \bigcup_{n=1}^{\infty} A_n \) such that \( \omega_0 \notin \Omega_0 \). Form a sequence of sets \( B_n = \bigcup_{i=1}^{n} A_i \), then by definition of the limit, \( \exists N \forall n > N \omega_0 \in B_n \). But \( B_n \subset \Omega_0 \) because it is a finite union of sets, and hence we have a contradiction. It means that \( \bigcup_{n=1}^{\infty} A_n \subset \Omega_0 \).

Filtration \( \hat{\mathcal{F}} \) inherits the usual conditions (right-continuity and completeness) from \( \mathcal{F} \).
Probability $\bar{P}$ is defined as Bayesian conditional probability,

$$
\bar{P}(A) \overset{\text{def}}{=} P(A|\Omega_0), \ A \in \tilde{\mathcal{F}},
$$

such that the axioms of probability measure are met: $\bar{P}(\emptyset) = 0$, $\bar{P}(\Omega_0) = 1$ and for $A_i \in \tilde{\mathcal{F}}$, $i = 1, 2, \ldots$

$$
\bar{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \frac{P(\bigcup_{i=1}^{\infty} A_i)}{P(\Omega_0)} = \frac{\sum_{i=1}^{\infty} P(A_i)}{P(\Omega_0)} = \sum_{i=1}^{\infty} \bar{P}(A_i),
$$

where the infinite union and summation commute because $P$ is a probability measure.\(^2\)

\[ \square \]

**Proof of Proposition 5.2.2.**

*Proof.* Existence of solutions on the original space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ is proved in the proof of Proposition 5.2.1, we just need to remove the local volatility function $(1 + \tau_i L_i(t))$ (neither linear growth condition nor Lipschitz continuity are violated by doing so).

Then by definition of Libor rate,

$$
P(t, T_{i+1}) = \frac{P(t, T_i)}{1 + \tau_i L_i(t)},
$$

in other words, for all $\omega \in \Omega_0$, $P(t, T_i) > 0$ as long as $P(0, 0) > 0$. Since bonds pull to par, $P(0, 0) = 1$, all bond prices with maturities on fixing dates are positive. Hence by Theorem 1 in Geman et al. (1995), we can define measures $Q^{T_i}$ associated with appropriate discount bond numéraire, and that each $L_i(t)$ can be redefined in any other measure $Q^{T_j}$. \[ \square \]

\(^2\) Probability $\bar{P}$ of course could have been defined as conditional probability on sigma-algebra $\tilde{\mathcal{F}}$,

$$
\tilde{P}(A) = P(A|\tilde{\mathcal{F}}) \overset{\text{def}}{=} E(1_A|\tilde{\mathcal{F}}),
$$

and for sets $A \in \tilde{\mathcal{F}}$, this is just

$$
\tilde{P}(A) = 1_A.
$$

But in this case the system $(\Omega_0, \tilde{\mathcal{F}}, \tilde{P})$ would fail the axioms of probability space. This "conditional stochastic calculus" is a topic of original research.
Proof of Proposition 6.1.1.

Proof. By definition of stochastic integral,

\[ L_i(T_i) = L_i(0) + \int_0^{T_i} dL_i(s). \]

Formula (2.19) on p. 138 by Karatzas and Shreve (1991) states that \( L_i(T_i) \) is a continuous, square-integrable martingale and gives us that quadratic variation of \( L_i(T_i) \) is:

\[ \langle L_i(T_i) \rangle = \int_0^{T_i} \| \sigma_i(s, L_i) \|^2 d\langle Y^{i+1} \rangle_s, \]

where the last equality comes from (6.2).

We know that \( Y^{i+1} \) is a Brownian motion under \( Q_{i+1} \), therefore by Lévy’s characterisation of Brownian motion,

\[ \langle L_i(T_i) \rangle = \int_0^{T_i} \| \sigma_i(s, L_i) \|^2 ds. \]

By definition,

\[ \sigma_i(t, L_i) = \sigma_f(t, T_i), \]

where \( \sigma_f(t, T_i) \) is the vector of instantaneous forward volatilities from DMRV model.

Define a continuous martingale

\[ X_t := \frac{L_i(t)}{\hat{\sigma}^2_t}, \]

where

\[ \hat{\sigma}^2_t = \frac{1}{t} \int_0^t \| \sigma_f(s, T_i) \|^2 ds. \]

Quadratic variation of \( X_t \) is thus:

\[ \langle X_t \rangle = \frac{\langle L_i(t) \rangle}{\hat{\sigma}^2_t} = t, \quad (A.7) \]

and by Lévy’s characterisation of Brownian motion (Theorem 3.16 in Karatzas and Shreve (1991)), \( X_t \) is a Brownian motion. Therefore, all its marginal distributions are Gaussian and in particular \( L_i(T_i) - L_i(0) \) is also Gaussian with zero and variance \( \langle L_i(T_i) \rangle \).

\[ ^3 \text{By l'Hôpital rule, } \hat{\sigma}_0 = 1. \]
Note that we can only take $\hat{\sigma}_t$ out of quadratic variation operator in (A.7), because it is deterministic.

For total volatility of forward rate from DMRV model, or $\|\sigma_f(s, T_i)\|$ we have by formula (3.20):

$$\|\sigma_f(s, T_i)\| = \sigma_s^{T_i}.$$

Now we can apply Bachelier’s formula for a call option on $L_i(T_i)$ with forward $F = E_{s+1}L_i(T_i) = L_i(0)$ and volatility:

$$\sqrt{\frac{1}{T_i} \int_0^{T_i} (\sigma_s^{T_i})^2 \, ds}.$$
Bibliography


