On the representation of numbers in different bases

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Numbers are usually written in decimal format, otherwise known as base 10. However, it is possible to use other bases. Here it is shown how to translate a number from one base into another.

I. INTRODUCTION

In place-value notation the position of a numeral within a number indicates its value. For example, in the number 11 the first digit "1" represents the number 10, since it lies in the second ("tens") column, while the second digit "1" represents just 1, since it lies in the first ("units") column. Not all number systems follow this rule: in Roman numerals, for example, 1, 10, 100 and 1000 are each given separate, distinct symbols (I, X, C and M, respectively [1]). However, calculations with numbers are much easier using positional notation, and the system has now been universally adopted.

On the other hand, the convention that the second column represents *tens*, and the third *hundreds*, etc. is somewhat arbitrary (and in fact probably arose because we have ten fingers and thumbs). It is quite possible to use a number other than ten as the *base* of the system. Here we investigate the representation of numbers in some other bases.

II. BINARY NOTATION

Perhaps the simplest base to use is the number two. Base two is also known as binary notation (in the same way that base 10 is known as decimal notation) [2]. In this case the second column represents the "twos", the third column the "fours" ($4 = 2 \times 2 = 2^2$), the fourth column the "eights" ($8 = 2 \times 2 \times 2 = 2^3$), etc. Now each column can contain only the digits 0 or 1, since the number 2 is represented by a "1" in the twos column. The binary representation of the first 21 numbers is shown in Table I.

binary representation					decimal representation	
2^4	2^3	2^2	2^1	2^0	10^1	10^0
16	8	4	2	1	10	1
				1		1
			1	0		2
			1	1		3
		1	0	0		4
		1	0	1		5
		1	1	0		6
		1	1	1		7
	1	0	0	0		8
	1	0	0	1		9
	1	0	1	0	1	0
	1	0	1	1	1	1
	1	1	0	0	1	2
	1	1	0	1	1	3
	1	1	1	0	1	4
	1	1	1	1	1	5
1	0	0	0	0	1	6
1	0	0	0	1	1	7
1	0	0	1	0	1	8
1	0	0	1	1	1	9
1	0	1	0	0	2	0
1	0	1	0	1	2	1

TABLE I: Binary representation of the first 21 numbers.

We see from this table, that the decimal number 13 for example (which represents $1 \times 10 + 3 \times 1$) is represented in binary by the number 1101 (which represents $1 \times 8 + 1 \times 4 + 1 \times 1$).

III. TERNARY NOTATION

The next simplest base to use is the number three. Base three is also known as ternary notation. In this case the second column represents the "threes", the third column the "nines" ($9 = 3 \times 3 = 3^2$), the fourth column the "twenty-sevens" ($27 = 3 \times 3 \times 3 = 3^3$), etc. Now each column can contain only the digits 0, 1, or 2, since the number 3 is represented by a "1" in the threes column. The ternary representation of the first 21 numbers is shown in Table II [3].

We see from this table, that the decimal number 14 for example (which represents $1 \times 10 + 4 \times 1$) is represented in ternary by the number 112 (which represents $1 \times 9 + 1 \times 3 + 2 \times 1$).

IV. HEXADECIMAL NOTATION

It is also possible to use bases which are greater than ten. One base which is in common use in computing is base 16, which is also known as hexadecimal notation [4].

ternary representation				decimal representation	
3^3	3^2	3^1	3^0	10^1	10^0
27	9	3	1	10	1
			1		1
			2		2
		1	0		3
		1	1		4
		1	2		5
		2	0		6
		2	1		7
		2	2		8
	1	0	0		9
	1	0	1	1	0
	1	0	2	1	1
	1	1	0	1	2
	1	1	1	1	3
	1	1	2	1	4
	1	2	0	1	5
	1	2	1	1	6
	1	2	2	1	7
	2	0	0	1	8
	2	0	1	1	9
	2	0	2	2	0
	2	1	0	2	1

TABLE II: Ternary representation of the first 21 numbers.

When we use a base greater than 10 we need some more symbols, since the numbers 10 to 15 must now be represented by a single digit in the "units" column. It is common to adopt the symbols A to F for these numbers.

In base 16 the second column represents the "sixteens", the third column the "two-hundred-and-fifty-sixes" ($256 = 16 \times 16 = 16^2$), etc. The hexadecimal representation of the first 21 numbers is shown in Table III.

We see from this table, that the decimal number 18 for example (which represents $1 \times 10 + 8 \times 1$) is represented in hexadecimal by the number 12 (which represents $1 \times 16 + 2 \times 1$).

V. CONCLUSION

Positional notation in which the value of a digit depends on its position within a number is an incredibly useful concept. However, the use of ten as the base of such a numbering system, though common, is somewhat arbitrary. We have demonstrated in this paper how to translate between numbers written in different bases.

hexadecimal representation			decimal representation	
16^2	16^1	16^0	10^1	10^0
256	16	1	10	1
		1		1
		2		2
		3		3
		4		4
		5		5
		6		6
		7		7
		8		8
		9		9
		A	1	0
		B	1	1
		C	1	2
		D	1	3
		E	1	4
		F	1	5
	1	0	1	6
	1	1	1	7
	1	2	1	8
	1	3	1	9
	1	4	2	0
	1	5	2	1

TABLE III: Hexadecimal representation of the first 21 numbers.

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