Optimal estimates
in stochastic homogenization

Antoine Gloria, Stefan Neukamm*, Felix Otto*

* Max Planck Institute for Mathematics in the Sciences

Leipzig, Germany

Summary

Discrete elliptic equations with random coefficients:
Setting and qualitative homogenization result

A computable approximation of the homogenized coefficient:
two approximation errors and their expected scaling

Statement of main result

Elements of a calculus: Horizontal derivative, vertical derivative,
spectral gap, moment bounds on the corrector

Systematic error and semi group decay

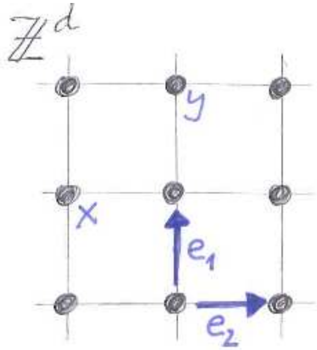
Mathematical context:
Yurinskii and random walks in random environments,
Naddaf & Spencer and Gradient Gibbs measures

Elements of the proof,
connection to classical elliptic regularity theory

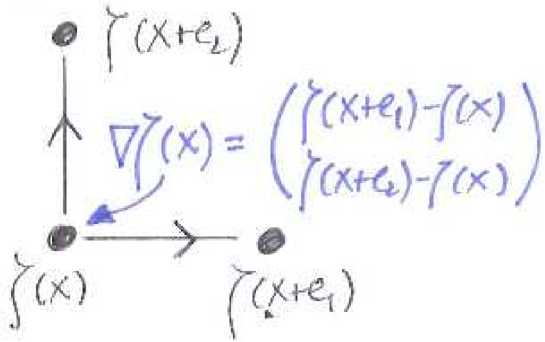
Discrete elliptic equations with random coefficients: setting and qualitative homogenization result

Discrete derivatives

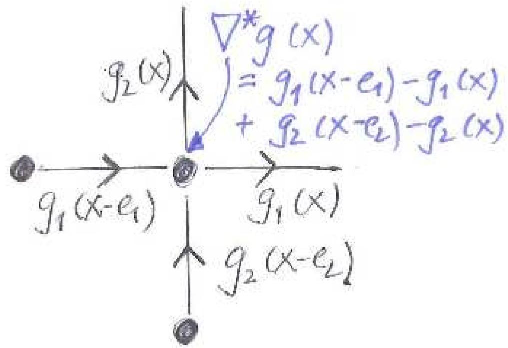
Lattice $\mathbb{Z}^d$,
sites x, y
coordinate directions $e_1, \cdots, e_d$



Gradient ∇.
Scalar field $\zeta(x) \in \mathbb{R}$ $\rightsquigarrow$ vector field $\nabla\zeta(x) \in \mathbb{R}^d$
$\nabla\zeta = (\nabla_1\zeta, \cdots, \nabla_d\zeta)$,
$\nabla_i\zeta(x) = \zeta(x + e_i) - \zeta(x)$

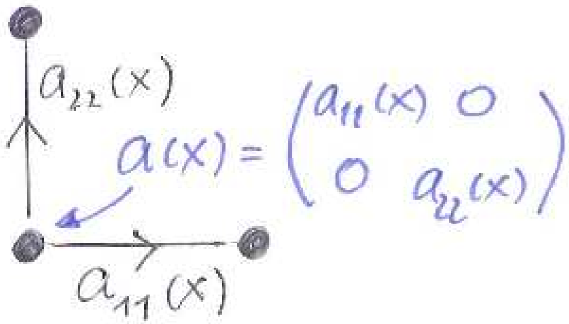


(negative) Divergence ∇^*. ℓ^2-adjoint.
Vector field $g(x) \in \mathbb{R}^d$ $\rightsquigarrow$ scalar field $\nabla^* g(x) \in \mathbb{R}$
$\nabla^* g = \nabla_1^* g_1 + \cdots + \nabla_d^* g_d$,
$\nabla_i^*\zeta(x) = \zeta(x - e_i) - \zeta(x)$



Discrete elliptic operator

Coefficients a.
Tensor field $a(x) \in \mathbb{R}^{d \times d}$,
Diagonal: $a = \text{diag}(a_{11}, \cdots, a_{dd})$,
Uniformly elliptic:
$\exists\, \lambda > 0 \quad \forall x \quad \lambda \leq a(x) \leq 1.$

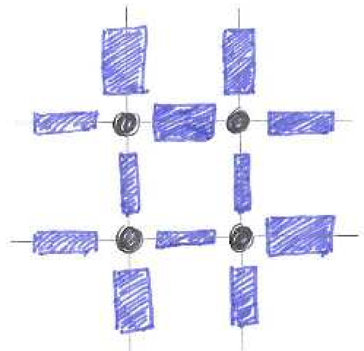


Elliptic operator: $\nabla^* a \nabla$

Network of resistors

a conductivities, a^{-1} resistivities,

u potential, $a\nabla u$ current



Random discrete elliptic operator

Field of coefficients a is random variable,

ensemble average $\langle\cdot\rangle$

Simplest setting: $\{a(x)\}_{x\in\mathbb{Z}^d}$ are **i**ndependent and **i**dentically **d**istributed according to a random variable $A\in\mathbb{R}^{d\times d}$.

Most general setting:

Stationarity: $\forall\, z\in\mathbb{Z}^d \quad a$ and $a(\cdot+z)$ have same distribution

Ergodicity: If $\forall\, z\in\mathbb{Z}^d\ \zeta(a(\cdot+z))=\zeta(a)$ then $\zeta=\langle\zeta\rangle$ a. s.

Qualitative homogenization

Kozlov ['79], Papanicolaou & Varadhan ['79]

$\exists\ a_{hom} \in \mathbb{R}^{d\times d}$, symmetric, $\lambda \le a_{hom} \le 1$ such that:

Given $f_0(\hat{x})$ consider right hand side $f(x) = \frac{1}{L^2} f_0(\frac{x}{L})$

Solve discrete Dirichlet problem : $\begin{cases} \nabla^* a \nabla u = f & \text{in } ((-L,L)\cap\mathbb{Z})^d \\ u = 0 & \text{outside } ((-L,L)\cap\mathbb{Z})^d \end{cases}$

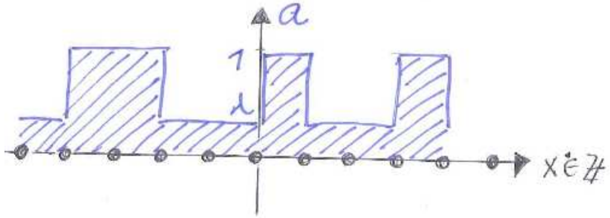
Solve continuum Dirichlet problem : $\begin{cases} -\hat{\nabla}\cdot a_{hom}\hat{\nabla} u_0 = f_0 & \text{in } (-1,1)^d \\ u_0 = 0 & \text{outside } (-1,1)^d \end{cases}$

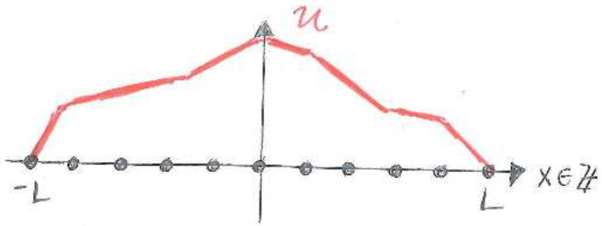
Then $\lim_{L\uparrow\infty} u(L\hat{x}) = u_0(\hat{x})$ **a**lmost **s**urely

Qualitative homogenization

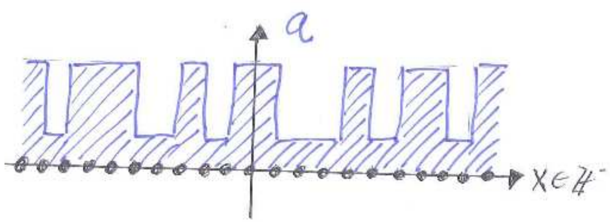
$$d = 1, \quad A = \left\{ \begin{array}{ll} 1 & \text{with probability } \frac{1}{2} \\ \lambda & \text{with probability } \frac{1}{2} \end{array} \right\}, \quad f_0 \equiv 1$$

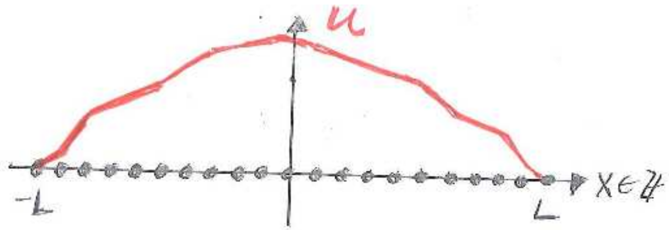
$L = 5$





$L = 10$



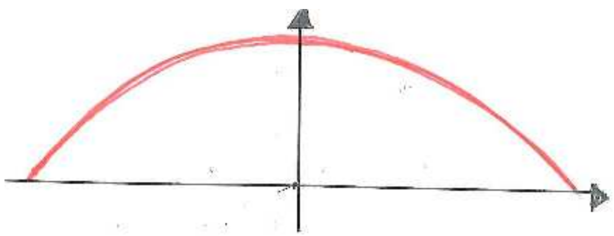


$L = \infty$

$$a_{hom}^{-1} = \tfrac{1}{2}\lambda^{-1} + \tfrac{1}{2}1^{-1}$$

$$-\frac{d}{d\hat{x}} a_{hom} \frac{du_0}{d\hat{x}} = 1 \text{ in } (-1, 1)$$

$$u_0 = 0 \text{ on } \{-1, 1\}$$



Prediction of a_{hom} from distribution of $\{a(x)\}$?

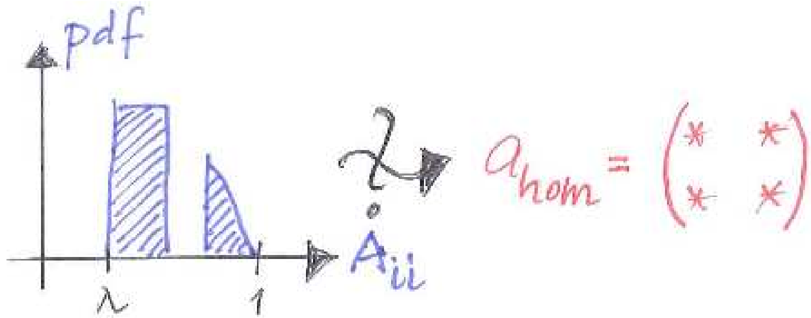
Conductivity^{-1} = Resistance

For d=1: $\langle A^{-1}\rangle^{-1} = a_{hom}$

For any d: $\langle A^{-1}\rangle^{-1} \leq a_{hom} \leq \langle A\rangle$

in series   in parallel

No simple general formula for



A computable approximation of the homogenized coefficient: two approximation errors and their expected scaling

Intuition of homogenized coefficient and corrector

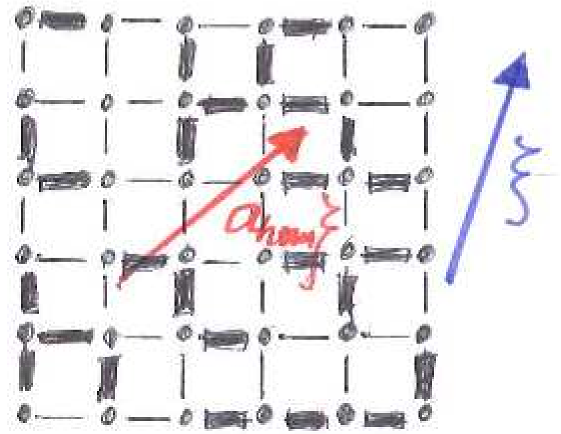
Physical: Consider $\nabla^* a \nabla u = 0$.

Average potential gradient

$$\xi = \lim_{L\uparrow\infty} \frac{1}{L^d} \sum_{x\in[0,L)^d} \nabla u$$

$\rightsquigarrow$ average current

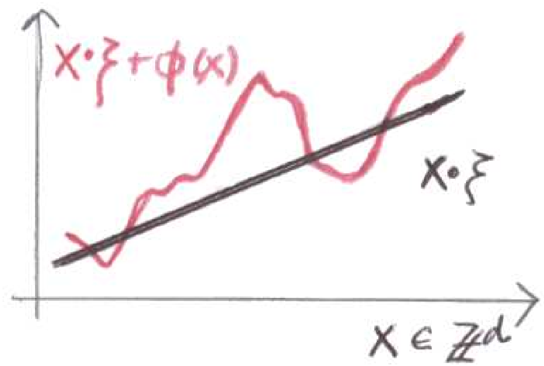
$$a_{hom}\xi = \lim_{L\uparrow\infty} \frac{1}{L^d} \sum_{x\in[0,L)^d} a\nabla u$$



Geometrical: Given affine function $x \cdot \xi$,

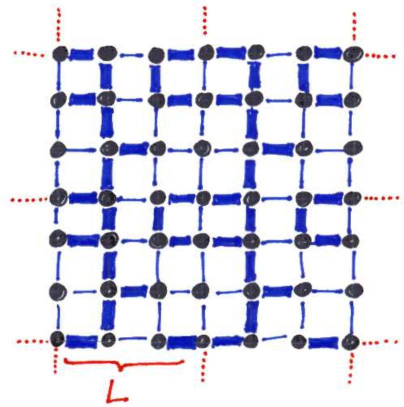
find φ s. t. $\varphi(x) + x \cdot \xi$ is a-harmonic:

$\nabla^* a(\nabla\varphi + \xi) = 0$.



... boundary conditions for corrector φ?

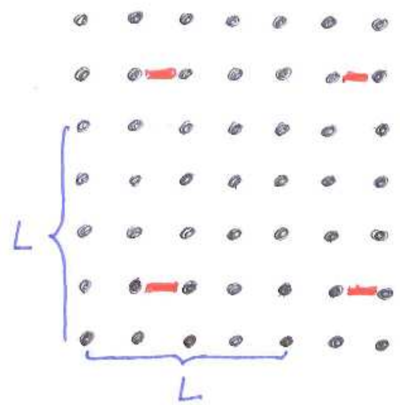
Artificial boundary conditions, modified ensemble



Artificial periodic boundary cond.:

a has period L, i. e. $a(x + Le_i) = a(x)$

$\rightsquigarrow$ φ has period L



Modified, periodic ensemble:

$\langle \cdot \rangle_L := \langle \cdot | a$ has period $L \rangle$

$\langle \cdot \rangle$ i. i. d. $\Longrightarrow$ $\langle \cdot \rangle_L$ i. i. d.

i. e. a has period L,

$\{a(x)\}_{x \in ([0,L) \cap \mathbb{Z})^d}$ independent,

identically distributed like A

... **artificial long-range correlations**

Practical approximation of homogenized coefficient

Pick a according to $\langle\cdot\rangle_L$, **solve** for φ (period L),

compute spatial average $a_{hom,L}\xi := \frac{1}{L^d}\sum_{x\in[0,L)^d} a(\nabla\varphi+\xi)$

Take *random variable* $a_{hom,L}$ **as approximation to** a_{hom}

$\langle$**error**$^2\rangle_L$ = **random**^{2} + **systematic**^{2}:

$$\langle|a_{hom,L}-a_{hom}|^2\rangle_L = \mathrm{var}_{\langle\cdot\rangle_L}[a_{hom,L}] + |\langle a_{hom,L}\rangle_L - a_{hom}|^2$$

Qualitative theory yields:

$$\lim_{L\uparrow\infty}\mathrm{var}_{\langle\cdot\rangle_L}[a_{hom,L}] = 0, \qquad \lim_{L\uparrow\infty}\langle a_{hom,L}\rangle_L = a_{hom}$$

... why is scaling in L of interest?

Number of samples N vs. artificial period L

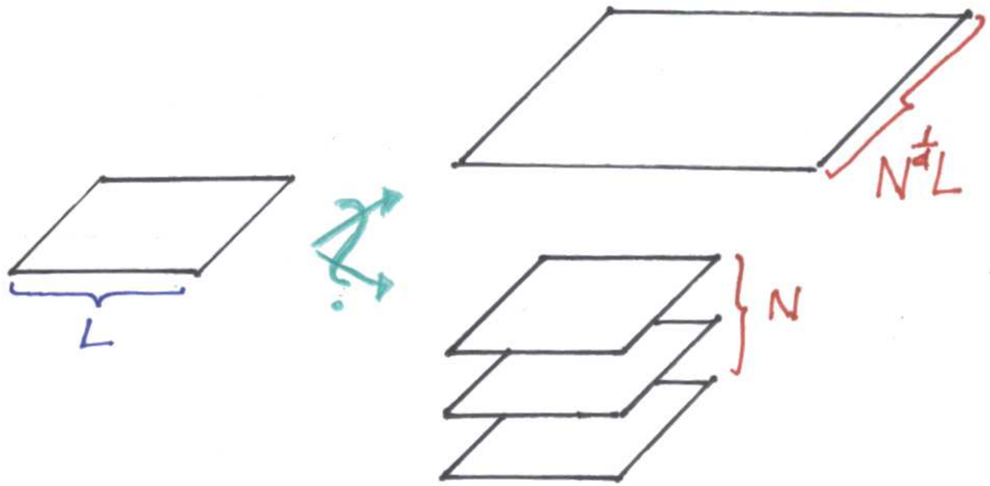
Take **N** samples, i. e. N independent picks $a_1, \cdots, a_N$ from $\langle\cdot\rangle_L$.

Compute empirical mean $\frac{\mathbf{1}}{\mathbf{N}}\sum_{\mathbf{n=1}}^{\mathbf{N}} \frac{1}{L^d} \sum_{x\in[0,L)^d} a_n(\nabla\varphi_n + \xi)$

$\langle \textbf{total error}^2\rangle_L = \frac{1}{\mathbf{N}}\,\text{random error}^2 + \text{systematic error}^2$

L ↑ reduces
systematic error and
random error

N ↑ reduces only
effect of **random error**



… what scaling in L to expect?

Scaling in L for 1-d and for small ellipticity ratio

a periodic, $\{a(x)\}_{x\in([0,L)\cap\mathbb{Z})^d}$ indep. , distributed according to A

d=1; explicit calculation + Central Limit Theorem for $L \gg 1$:

$$\text{random error}^2 \;:=\; \text{var}_{\langle\cdot\rangle_L}[a_{hom,L}] \approx \frac{1}{L}\frac{\text{var}[A^{-1}]}{\langle A^{-1}\rangle^4},$$

$$\text{systematic error} \;:=\; |\langle a_{hom,L}\rangle_L - a_{hom}| \approx \frac{1}{L}\frac{\text{var}[A^{-1}]}{\langle A^{-1}\rangle^2}$$

$1-\lambda \ll 1$; $A = \frac{1+\lambda}{2} + \frac{1-\lambda}{2}B$, second order expansion:

$$\text{random error}^2 \;:=\; \text{var}_{\langle\cdot\rangle_L}[a_{hom,L}\xi] \approx \frac{1}{L^d}\textstyle\sum_{i=1}^d \text{var}[A_{ii}]\xi_i^2.$$

$$\text{systematic error} \;:=\; |(\langle a_{hom,L}\rangle_L - a_{hom})\xi| \approx \frac{1}{L^d}\frac{1}{d}|\textstyle\sum_{i=1}^d \text{var}[A_{ii}]\xi_i|$$

systematic error $\sim L^{-d} \ll$ **random error** $\sim L^{-d/2}$

Multi-d is different from 1-d

$d = 1$: $a_{hom,L}^{-1} = \frac{1}{L}\sum_{x\in[0,L)} a^{-1}$
spatial average of independent variables

$d > 1$: $a_{hom,L}\xi = \frac{1}{L^d}\sum_{x\in[0,L)^d} a(\nabla\varphi + \xi)$
spatial average of correlated variables

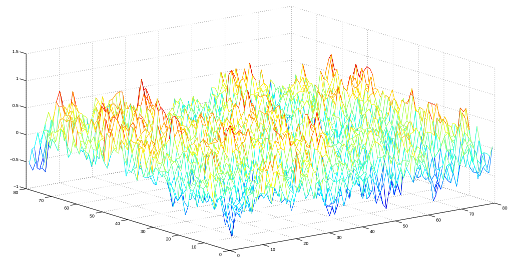
In fact for $1 - \lambda \ll 1$

$$\mathrm{cov}_{\langle\cdot\rangle_L}[a(\nabla\varphi+\xi)(x),\ a(\nabla\varphi+\xi)(y)] \sim \nabla^2 G_L(x-y),$$

in fact $$\mathrm{cov}_{\langle\cdot\rangle_L}[\varphi(x),\varphi(x)] \sim G_L(x-y),$$

where G_L Green's function

for $\nabla^*\nabla$ of period L



... behavior of Gaussian free field, $d = 2$ special

Statement of main result

A near-optimal result

Let $\langle\cdot\rangle_L$ be ensemble of a's with period L, where $\{a(x)\}_{x\in([0,L)\cap\mathbb{Z})^d}$ independent, identically distributed according to A

For a with period L
solve $\nabla^* a(\nabla\varphi + \xi) = 0$ for φ of period L.
Set $a_{hom,L}\xi = \frac{1}{L^d}\sum_{x\in[0,L)^d} a(\nabla\varphi + \xi)$.

Random error$^2 = \mathrm{var}_{\langle\cdot\rangle_L}\left[a_{hom,L}\right] \;\leq\; C(d,\lambda)L^{-d}$

Systematic error $= \left|\langle a_{hom,L}\rangle_L - a_{hom}\right| \;\leq\; C(d,\lambda)L^{-d}\ln^{\frac{d}{2}} L$

References

Random error$^2 = \text{var}_{\langle\cdot\rangle_L}\left[a_{hom,L}\right] \;\le\; C(d,\lambda)L^{-d}$

Systematic error $\;=\left|\langle a_{hom,L}\rangle_L - a_{hom}\right| \;\le\; C(d,\lambda)L^{-d}\ln^{\frac{d}{2}} L$

Yurinskii ('86): suboptimal, no algebraic bounds for $d=2$

Spencer et. al., Conlon et. al.: optimal for $1-\lambda \ll 1$

Random error (besides additional logarithm in $d=2$):

[Gloria&O., Ann.Prob. 2011].

Systematic error

(for $d<4$): [Gloria&O., Ann.Appl.Prob.(accepted)]

(for $d\le 6$): [Gloria&Mourrat, Prob.Theo.Rel.Fields (accepted)]

(general d): [Gloria&Neukamm&O., SIAM PDE Tutorial 2011]

Elements of a calculus: Horizontal derivative, vertical derivative, spectral gap, and moment bounds on the corrector

Stationary random fields and horizontal derivatives

Stationary random field | random variable

$f(a,x)$,

i. e. $f(a,x+z) = f(a(\cdot+z),x)$

$< - - - - - - >$

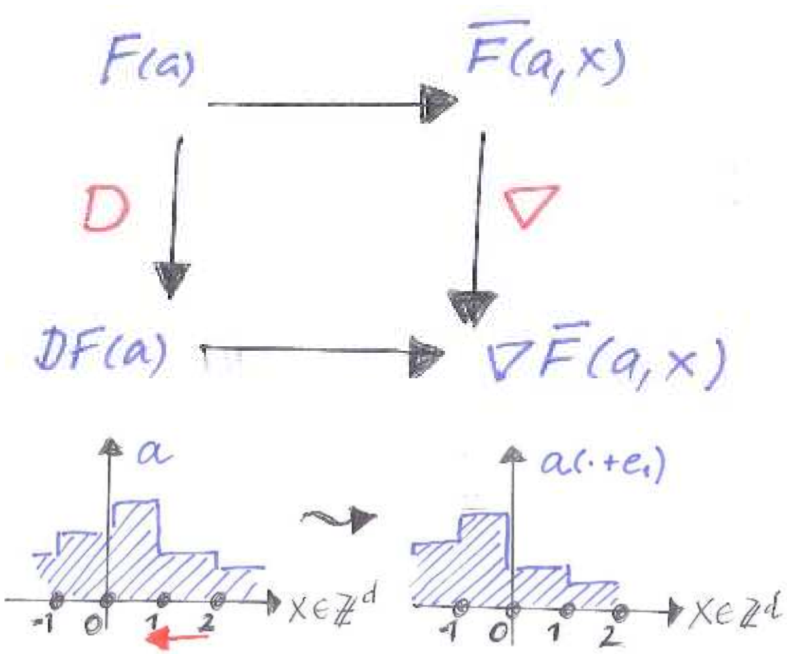
$f(x,a)$

$= F(a(\cdot+x))$

$F(a)$

e. g. $\varphi(a,x)$, $a(x)$

$\phi(a)$, A



Gradient:

$DF = (D_1 F, \cdots, D_d F)$

$(D_i F)(a) = F(a(\cdot + e_i)) - F(a)$

negative **Divergence**:

$D^* G = D_1^* G_1 + \cdots + D_d^* G_d$

$(D_i^* F)(a) = F(a(\cdot - e_i)) - F(a)$

Corrector ϕ and formula for a_{hom}

Gradient:
$DF = (D_1 F, \cdots, D_d F)$
$(D_i F)(a) = F(a(\cdot + e_i)) - F(a)$

negative Divergence:
$D^* G = D_1^* G_1 + \cdots + D_d^* G_d$
$(D_i^* F)(a) = F(a(\cdot - e_i)) - F(a)$

Find random variable $\phi(a) \in \mathbb{R}$ s. t. $\quad D^* A(D\phi + \xi) = 0$

Then $a_{hom}\xi = \langle A(D\phi + \xi)\rangle$, $\langle a_{hom,L}\rangle_L \xi = \langle A(D\phi + \xi)\rangle_L$

... but existence of ϕ for $L = \infty$ not clear. Why?

Riesz: $\exists$ random variable $\chi(a) \in \mathbb{R}^d$ s. t. $\begin{cases} D^* A(\chi + \xi) &= 0 \\ D_i \chi_j - D_j \chi_i &= 0 \\ \langle \chi \rangle &= 0 \end{cases}$

No spectral gap for horizontal derivatives $\{D_i\}_{i=1,\cdots,d}$

Does there exist $\phi(a)$ s. t. $D^*A(D\phi+\xi)=0$?

Yes if $\exists\ \rho>0\quad \forall\ F(a)\qquad \langle(F-\langle F\rangle)^2\rangle\le\frac{1}{\rho}\langle|DF|^2\rangle$.

Spectral gap for D w. r. t. $\langle\cdot\rangle_L$ degenerates as $L\uparrow\infty$:

$$\sup_{F(a)}\frac{\langle|DF|^2\rangle_L}{\langle(F-P_{\mathrm{ker}D}F)^2\rangle_L}$$

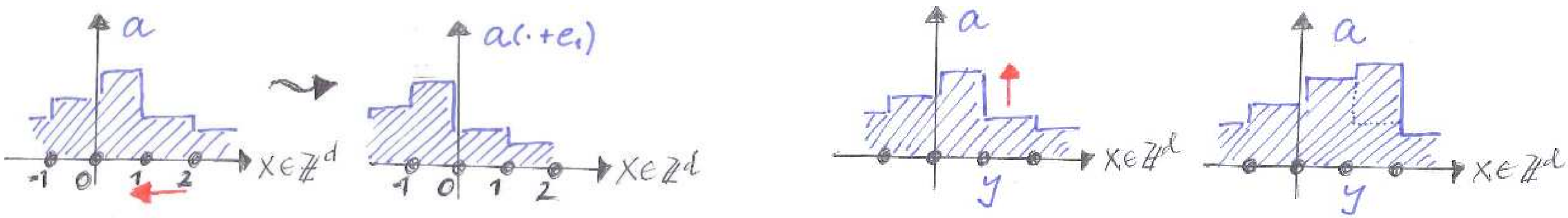
$$\overset{\text{i. i. d. case}}{=}\sup_{f(x)\text{ with period }L}\frac{\sum_{[0,L)^d}|\nabla f|^2}{\sum_{[0,L)^d}(f-L^{-d}\sum_{[0,L)^d}f)^2}=(2\pi)^2L^{-2}$$

Too many variables $(a(x)_{x\in\mathbb{Z}^d})$ – too few derivatives $(D_1,\cdots,D_d)$

Spectral gap for vertical derivatives $\{\frac{\partial}{\partial x}\}_{x\in\mathbb{Z}^d}$

Definition: for $F(a)$ and $x \in \mathbb{Z}^d$, $\frac{\partial F}{\partial x}(a)$ defined by:

$$\frac{\partial F}{\partial x} := F - \langle F | \{a(y)\}_{y\neq x}\rangle \text{ ``}\sim\text{''} \frac{\partial F}{\partial a(x)}$$



$\{a(x)\}_{x\in\mathbb{Z}^d}$ independently distributed

$\Longrightarrow$ Logarithmic Sobolev inequ. for $\{\frac{\partial}{\partial x}\}_{x\in\mathbb{Z}^d}$ with $\rho = 1$

$\Longrightarrow$ Spectral gap for $\{\frac{\partial}{\partial x}\}_{x\in\mathbb{Z}^d}$ with $\rho = 1$, i. e.

$$\forall\ F(a) \quad \langle (F - \langle F\rangle)^2\rangle \le \left\langle \sum_{x\in\mathbb{Z}^d} \left(\frac{\partial F}{\partial x}\right)^2 \right\rangle$$

... why can it help?

Corrector: Existence and moment bounds

Example of usefulness: $SG(\rho)$ for $\{\frac{\partial}{\partial x}\}_{x\in\mathbb{Z}^d} \overset{\text{easy}}{\Longrightarrow}$

$$\langle (F-\langle F\rangle)^2\rangle \ \le\ C(d,\rho)\ \langle |DF|^2\rangle^{\frac{d}{d+2}} \Big(\sum_x \langle (\tfrac{\partial F}{\partial x})^2\rangle^{\frac12}\Big)^{\frac{4}{d+2}}.$$

$\overset{\text{i. i. d. case}}{\Longrightarrow}$ Nash-Aronson, i. e. $\sum f^2 \ \le\ C(d)\,(\sum|\nabla f|^2)^{\frac{d}{d+2}}\,(\sum|f|)^{\frac{4}{d+2}}$.

Theorem A [GO, GNO]

i) Let $d>2$, suppose $\{\frac{\partial}{\partial x}\}$ satisfies $SG(\rho)$. Then

$\forall\, q<\infty \quad \langle \phi^{2q}\rangle^{\frac1q} \ \le\ C(d,\lambda,\rho,q)$.

ii) Let $d=2$, suppose $\{\frac{\partial}{\partial x}\}$ satisfies $LSI(\rho)$. Then

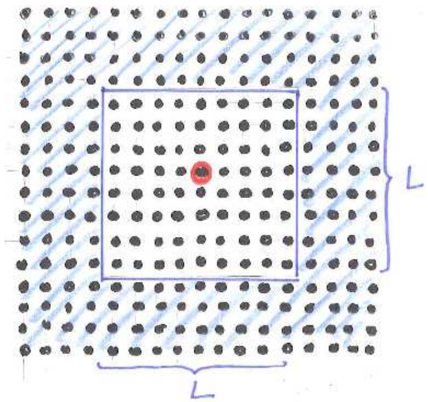
$\forall\, q<\infty \quad \langle \phi^{2q}\rangle_L^{\frac1q} \ \le\ C(d,\lambda,\rho,q)\,\ln L$.

... same behavior as Gaussian free field

Systematic error and algebraic semi group decay

Systematic error and modified corrector

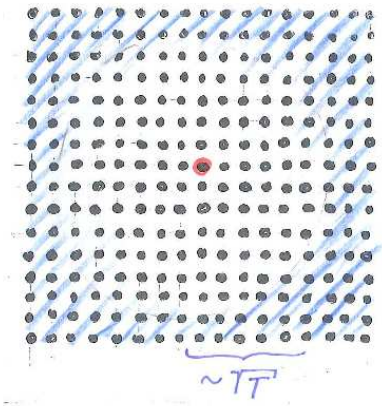
Recall systematic error $|\langle a(\nabla\varphi+\xi)(0)\rangle_L - \langle a(\nabla\varphi+\xi)(0)\rangle|$



Recall $\nabla^* a(\nabla\varphi + \xi) = 0$;

how sensitively does

$\nabla\varphi(0)$ depend on $\{a(x)\}_{x\notin(-\frac{L}{2},\frac{L}{2})^d}$?



Consider modified corrector φ_T:

$T^{-1}\varphi_T + \nabla^* a(\nabla\varphi_T + \xi) = 0$ with $T \sim L^2$;

how close is $\nabla\varphi(0)$ to $\nabla\varphi_T(0)$?

Modified corrector and algebraic semi group decay

Recall modified corrector $T^{-1}\varphi_T + \nabla^* a(\nabla\varphi_T + \xi) = 0$

Consider "horizontal" version $(T^{-1}+D^*AD)\phi_T = -D^*A\xi$;
how close is $D\phi_T$ to $D\phi$ for $T \gg 1$?

Consider semi group $U(t) = -\exp(-tD^*AD)D^*A\xi$, i. e.
$\frac{dU}{dt} + D^*ADU = 0, \quad U(t=0) = -D^*A\xi$;
how quickly does $\langle (U(t))^2\rangle$ decay for $t \gg 1$?

Theorem B [GNO]

Suppose $\{\frac{\partial}{\partial x}\}$ satisfies $SG(\rho)$. Then for all $q < \infty$

$$\langle |\exp(-tD^*AD)D^*g|^{2q}\rangle^{\frac{1}{q}} \le C(d,\lambda,\rho,q)\, t^{-(\frac{d}{2}+1)} (\sum_x \langle |\frac{\partial g}{\partial x}|^{2q}\rangle^{\frac{1}{2q}})^2$$

Spectral gap, semi group decay, and ergodicity

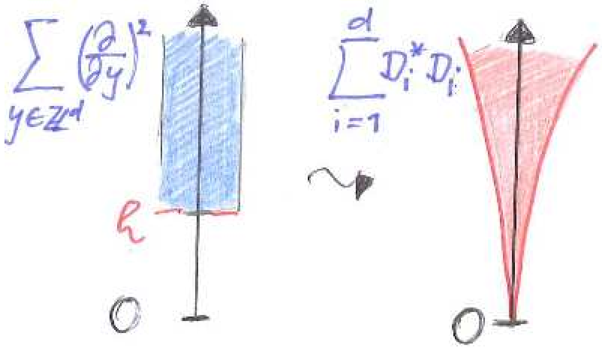
$\{a(x)\}_x$ identically distributed

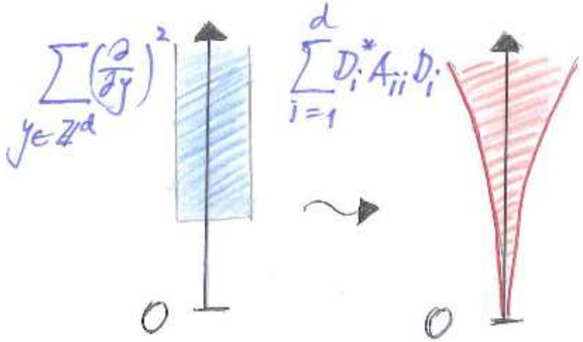
$\Longrightarrow$ $\{\frac{\partial}{\partial x}\}_x$ satisfies SG (with $\rho = 1$)

$\Longleftrightarrow$ $\exp(-t\sum_x(\frac{\partial}{\partial x})^2)$ decays exponentially

$\Longrightarrow$ $\exp(-tD^*D)$, $\exp(-tD^*D)D$, $\exp(-tD^*AD)$, and $\exp(-tD^*AD)D$ decay *algebraically*

$\Longrightarrow$ distribution of $\{a(x)\}_x$ is ergodic under shift





... via estimates on *gradient* of Green's function for $\nabla^* a \nabla$

Mathematical context:

Yurinskii's work and random walks in random environments

Naddaf & Spencer's work and Gradient Gibbs measures

Yurinskii ('86) result

Estimate for *modified corrector* $\frac{1}{T}\phi_T + D^*A(D\phi_T + \xi) = 0$:

system. error $\overset{d\lesssim 4}{\sim}$ $\langle\, |D\phi_T - D\phi|^2 \,\rangle \lesssim T^{-\frac{d-2}{4+d}+}$ $\quad\left(T^{-\min\{d/2,2\}} \right)$

... relies on suboptimal **variance estimate** for φ_T

$$\mathrm{var}\left[L^{-d}\textstyle\sum_{x\in[0,L)^d}\varphi_T\right] \lesssim T\left(TL^{-d}\right)^{\frac{1}{2}-} \qquad \left(L^{2-d} \right)$$

... relies on **random walk decomposition**

$$\varphi_T = \sum_{S\subset\{\text{edges}\}} \varphi_{T,S}$$

with

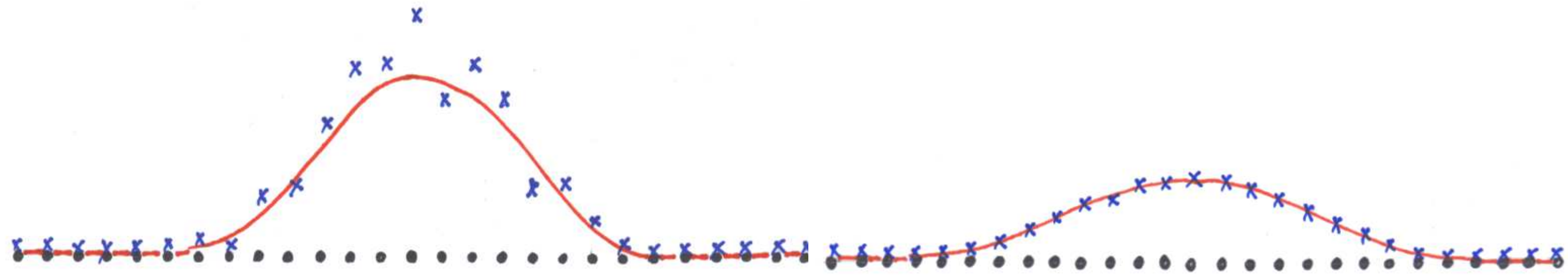
$$\varphi_{T,S} = \varphi_{T,S}(\{a(e)\}_{e\in S})$$

Random walk in random environment I

Large scale behavior of parabolic Green's function $G(a,t,x)$

$$\partial_t G + \nabla^* a \nabla G \;=\; 0 \quad G(t=0,x) \;=\; \left\{ \begin{array}{ll} 1 & \text{for } x = 0 \\ 0 & \text{for } x \neq 0 \end{array} \right\}$$

is Gaussian with variance a_{hom} : $G_{hom}(t,x) = \frac{c_d}{t^{d/2}} \exp(-\frac{x \cdot a_{hom} x}{4t})$



Almost sure ("quenched") *qualitative* result:

Corrector ϕ provides harmonic coordinates, Martingale argument

[Sidoravicius & Sznitman 04]

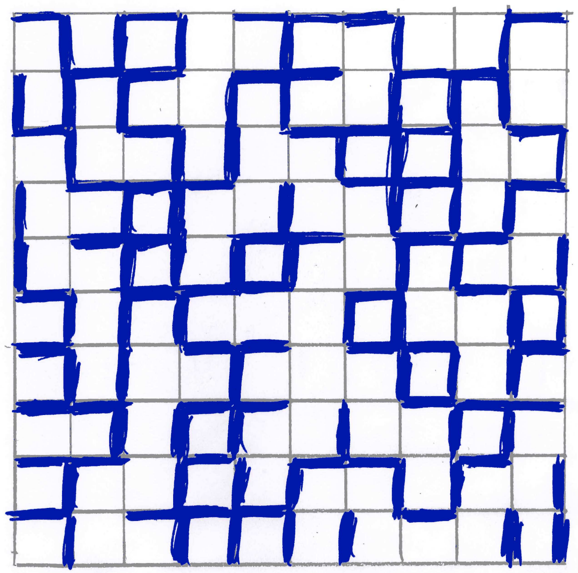
Random walk in random environment II

Random walk on

percolation cluster

Bernoulli, supercritical

i. e. $\lambda = 0$



Almost surely behaves like Brownian motion on large scales:

Sidoravicius & Sznitman '04 ($d \geq 4$),
Mathieu & Piatnitski '07 ($d \geq 2$),
Berger & Biskup '07 ($d \geq 2$)

} *sub-linear* growth of corrector ϕ

Naddaf&Spencer (unpublished) result on random error

Spectral gap estimate: $\mathrm{var}[F] \;\le\; \frac{1}{\rho}\Big\langle \sum_{x\in[0,L)^d} \left(\frac{\partial F}{\partial x}\right)^2\Big\rangle$

Vertical derivative of energy density:

Have $\xi\cdot a_{hom,L}\xi = L^{-d}\sum_{x\in[0,L)^d}(\nabla\varphi+\xi)\cdot a(\nabla\varphi+\xi)$

Get $\frac{\partial}{\partial x}\xi\cdot a_{hom,L}\xi \sim -L^{-d}|\nabla\varphi(x)+\xi|^2$.

Meyer's estimate for $\nabla^* a\nabla\varphi = -\nabla^* a\xi$:

$\exists\, p(d,\lambda) > 2$ s. t. $\sum|\nabla\varphi|^p \le C(d,\lambda)\sum|a\xi|^p$

For small ellipticity ratio $1-\lambda \ll 1$: $\sum|\nabla\varphi|^4 \le C(d)\sum|a\xi|^4$

Optimal conclusion for $1-\lambda \le c(\mathrm{d})$:

$\mathrm{var}[a_{hom,L}] \;\le\; C(d)L^{-d}$

Gradient Gibbs measures (Funaki & Spohn '97)

From *quenched* $\min_{\varphi} \sum_x (\xi+\nabla\varphi(x)) \cdot a(x)(\xi+\nabla\varphi(x))$

to *annealed* $\langle\langle\cdot\rangle\rangle := \frac{1}{Z} \exp\Big(-\sum_x \sum_i V(\xi_i+\nabla_i\varphi(x))\Big) \Pi_x d\varphi(x)$

Naddaf & Spencer '97, Giacomin & Olla & Spohn '01, Conlon & Spencer '11:

For $\lambda \le V''(\eta) \le 1$ like Gaussian free field

$$\langle\langle\varphi(x)\varphi(x')\rangle\rangle \overset{\text{H.-S.}}{=} \int_0^\infty \langle\langle\langle G(t,x,x')\rangle\rangle\rangle\, dt \approx G_{hom}(x-x')$$

where $G(t,x,x')$ parabolic Green's function
for *time-dependent* coefficients $a_{ii}(t,x) := V''(\nabla_i\varphi(t,x))$
and where $\varphi(t,x)$ Glauber dynamics w. r. t. $\langle\langle\cdot\rangle\rangle$

Elements of the proof, connection to classical elliptic regularity theory

A Poincaré and reverse Hölder for corrector

Recall $D^*A(D\phi+\xi)=0$, $T^{-1}\phi_T+D^*A(D\phi_T+\xi)=0$

Proposition A1 [GO]. Suppose $\{\frac{\partial}{\partial x}\}_x$ satisfy $SG(\rho)$.
For $d>2$ and $q\ge C(d,\lambda)$:

$$\langle|\phi|^{2q}\rangle^{\frac{1}{q}} \le C(d,\lambda,\rho,q)\,\langle|D\phi+\xi|^{2q}\rangle^{\frac{1}{q}}$$

For $d=2$ and $q\ge C(d,\lambda)$:

$$\langle|\phi_T|^{2q}\rangle^{\frac{1}{q}} \le C(d,\lambda,\rho,q)\,(\ln T)\,\langle|D\phi_T+\xi|^{2q}\rangle^{\frac{1}{q}}$$ like Gaussian free field

Proposition A2 [GNO]

Suppose $\{\frac{\partial}{\partial x}\}_x$ satisfy $LSI(\rho)$. Then for $q<\infty$:

$$\langle|D\phi+\xi|^{2q}\rangle^{\frac{1}{q}} \le C(d,\lambda,\rho,q)\,\langle|D\phi+\xi|^{2}\rangle \le C(d,\lambda,\rho,q)$$

Proof of A1: Spectral gap & Green's function

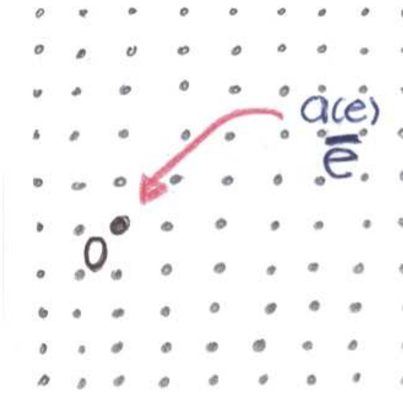
Spectral gap estimate $(L^{2q}$-version)

$$\langle|\phi|^{2q}\rangle = \langle|\varphi(0)|^{2q}\rangle \lesssim \left\langle\left(\sum_x\left|\frac{\partial\varphi(0)}{\partial x}\right|^2\right)^q\right\rangle$$

from $\nabla^* a(\nabla\varphi+\xi) = 0$

Green's function

$$\frac{\partial\varphi(0)}{\partial x} \text{ “}=\text{''} -\nabla G(0,x)\cdot(\nabla\varphi+\xi)(x)$$



Get

$$\langle|\phi|^{2q}\rangle \lesssim \langle(\sum_x|\nabla G(0,x)|^2\,|(\nabla\varphi+\xi))(x)|^2\,)^q\rangle$$

Need uniform control of Green's function

Recall

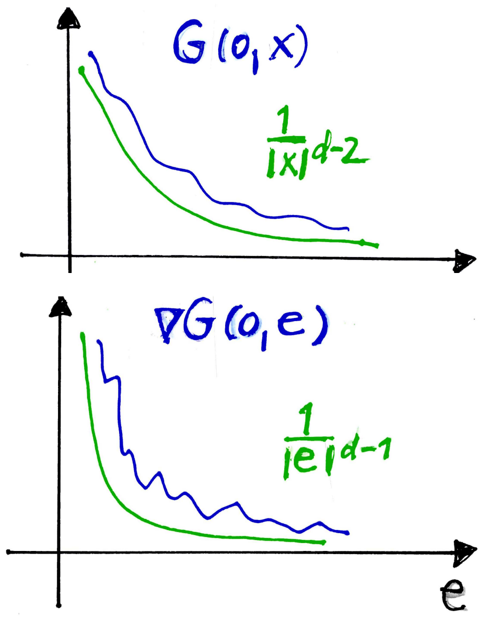
$$\langle |\phi|^{2q} \rangle \lesssim \langle \big(\sum_x |\nabla G(0,x)|^2 \, |(\nabla \varphi + \xi)(x)|^2 \big)^q \rangle$$

In which sense

$$G(0,x) \sim \frac{1}{|x|^{d-2}}$$

$$|\nabla_x G(0,x)| \sim \frac{1}{|x|^{d-1}}$$

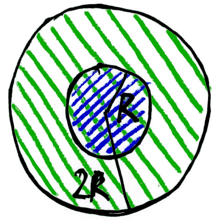
uniformly in $\{a(x)\}_x \subset [\lambda, 1]$



Need elliptic regularity (discrete version)

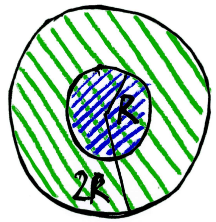
De Giorgi: For $\nabla^* a \nabla u = 0$ in $\{|x| \le 2R\}$

$$\sup_{|x|\le R} |u(x)| \lesssim \frac{1}{R^d} \sum_{|x|\le 2R} |u(x)|$$



Caccioppoli: For $\nabla^* a \nabla u = 0$ in $\{|x| \le 2R\}$

$$\sum_{|x|\le R} |\nabla u(x)|^2 \lesssim \frac{1}{R^2} \sum_{|x|\le 2R} |u(x)|^2$$



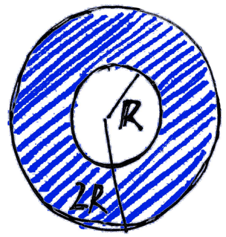
Meyers: $\exists\, p(d,\lambda) > 2$ s. t. for $\nabla^* a \nabla u = \nabla^* g$

$$\sum_x |\nabla u(x)|^p \lesssim \sum_x |g(x)|^p$$

Get optimal gradient estimate of Green's function

Get for some $p > 1$

$$\left(\frac{1}{R^d}\sum_{R\le|x|<2R}|\nabla G(0,x)|^{2p}\right)^{\frac{1}{2p}} \lesssim \frac{1}{R^{d-1}}$$



... spatially averaged "quenched estimate"

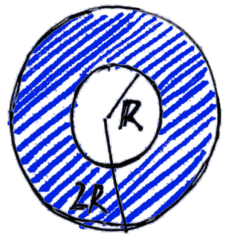
vs. pointwise "annealed estimate" ...

$$\langle|\nabla G(0,x)|^2\rangle^{\frac{1}{2}} \lesssim \frac{1}{|x|^{d-1}}$$ [Delmotte & Deuschel '05].

Apply gradient estimate on Green's function

for some
$p > 1$

$$\sum_{R \le |x| < 2R} |\nabla G(0,x)|^{2p} \lesssim R^d \left(\frac{1}{R^{d-1}}\right)^{2p}$$



Get

$$\left\langle \Big(\sum_{R \le |x| < 2R} |\nabla G(0,x)|^2 |(\nabla\varphi+\xi)(x)|^2 \Big)^q \right\rangle$$

$\frac{1}{p} + \frac{1}{q} = 1$
hence $p \approx 1$
for $q \gg 1$

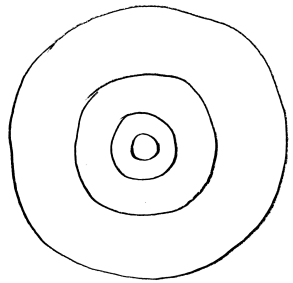
$$\le \left\langle \Big(\sum |\nabla G(0,x)|^{2p} \Big)^{q-1} \sum |(\nabla\varphi+\xi)(x)|^{2q} \right\rangle$$

$$\lesssim \Big(R^d \big(\frac{1}{R^{d-1}}\big)^{2p} \Big)^{q-1} \left\langle \sum_{R \le |x| < 2R} |(\nabla\varphi+\xi)(x)|^{2q} \right\rangle$$

$$\sim \Big(\frac{1}{R^{d-2}}\Big)^q \langle |D\phi+\xi|^{2q} \rangle$$

Conclusion

Dyadic annuli $\quad \sum_x \rightsquigarrow \sum_{\ell=0}^{\infty} \sum_{2^\ell \le |x| < 2^{\ell+1}}$



Triangle inequality in $L^q(\langle\cdot\rangle)$: $\quad \left\langle \left(\sum_x \cdot\right)^q \right\rangle^{\frac{1}{q}} \le \sum_{\ell=0}^{\infty} \left\langle \left(\sum_{2^\ell \le |x| < 2^{\ell+1}} \cdot \right)^q \right\rangle^{\frac{1}{q}}$

Get estimate of ϕ in terms of $D\phi$:

$$\left\langle |\phi|^{2q} \right\rangle^{\frac{1}{q}} \lesssim \left\langle |D\phi + \xi|^{2q} \right\rangle^{\frac{1}{q}} \underbrace{\sum_{\ell=0}^{\infty} \frac{1}{(2^\ell)^{d-2}}}$$

finite for $d > 2$

logarithm for $d = 2$

Future directions

Allow for (fast decaying) **correlations** [ok]

Extension to **continuum** elliptic equations [ok]

J Nolan: Random error is approx. **Gaussian** [ok]

JC Mourrat: Optimal error in path-wise convergence [ok]

Extension to **systems** (elasticity) [ok for random, ?]

Extension to **percolation** clusters [ok for toy model, ?]