Periodic Geodesics on Riemannian Surfaces

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Not simply-connected Riemannian Surfaces

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- $8\sqrt{A(M)}$ (A. Nabutovsky, R. R and independently by S. Sabourau)
- area bound improved to $4\sqrt{2}\sqrt{A(M)}$ (R. R.)

Definition

Let M be a complete connected oriented surface. Let γ be a simple closed curve which divides M into two components. Let Ω be one of these components. Then we will say that γ convex to Ω if there exists an $\epsilon > 0$ such that for all $x, y \in \gamma$, with $d(x, y) < \epsilon$, the minimizing geodesic τ from x to y satisfies $\tau \in \bar{\Omega}$. When γ is a piecewise geodesic curve, this condition is equivalent to the the condition that all of the angles of γ are convex to Ω, (see Croke's paper "Area and the length of the shortest closed geodesic", page 5.)

Diameter bound on a Riemannian 2-sphere; proof

Co-area formula

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In our case u is the distance function and level sets are geodesic "spheres" of dimension 1. Thus,
$\int_R Lengths(S_t)dt = \int dA = A$, where A is the area of M

Berger's lemma

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Let M be a compact Riemannian manifold and let $p, q \in M$ be such that $d(p, q) = d$, where d is the diameter of M. Then for all $W \in T_pM$, there exists a minimizing geodesic γ from $p = \gamma(0)$ to q with $< \gamma'(0), W > \geq 0$.

Problem 5

(a) Use Berger's lemma to prove that there exist $x, y \in M$ and minimizing geodesics connecting x and y, $\{\gamma_1, ..., \gamma_n\}$, such that $\gamma_i(0) = x$ and $\gamma_i(1) = y$ and $\gamma_i \cup -\gamma_{i+1}$ is a geodesic digon with both angles $\leq \pi$.

(b) Let γ be a simple closed curve in M separating M into two components. Let us denote one of these components Ω_γ. Let $x \in \Omega_\gamma$ be a point such that $d(x, \gamma) \leq d(y, \gamma)$ for all $y \in \Omega$. Prove that for every $W \in T_x M$, there exists a minimizing geodesic α from x to some point on γ with $< \alpha'(0), W > \geq 0$.

Problem 6 (a)

Let M be a Riemannian 2-sphere of area A, diameter $d > \sqrt{2A}$. Let $x, y \in M$ be two points, such that $d(x, y) = d$. Let $\alpha(t)$ be a minimal geodesic parametrized by its arclength.

Consider the geodesic spheres $S(x, t)$ centered at x of radius t for all $t \in [t_0 - \frac{\sqrt{2A}}{2}, t_0 + \frac{\sqrt{2A}}{2}]$.

(a) Let t_0 be any number in $(\frac{\sqrt{2A}}{2}, d - \frac{\sqrt{2A}}{2})$. Compare the inequality $\int_{t_0-\frac{\sqrt{2A}}{2}}^{t_0+\frac{\sqrt{2A}}{2}} length(S(x, t))dt < A$ and the equality $\int_{t_0-\frac{\sqrt{2A}}{2}}^{t_0+\frac{\sqrt{2A}}{2}} (\sqrt{2A} - 2|t - t_0|)dt = A$ to conclude that for generic $t \in [t_0 - \frac{\sqrt{2A}}{2} + t_0 + \frac{\sqrt{2A}}{2}]$ $length(S(x, t)) < \sqrt{2A} - 2|t - t_0|$

Problem 6 (b)

(b) Show that there exists a simple geodesic loop based at $\alpha(t_0)$ that divides M into two domains, one containing x and another containing y. (Hint: Consider the geodesic spheres $S(x,t)$ centered at x of radius t for all $t \in [t_0 - \frac{\sqrt{2A}}{2}, t_0 + \frac{\sqrt{2A}}{2}]$. (Note that for any generic t there exists a closed curve $\sigma \subset S(x,t)$ with no self-intersections that intersects α transversely at $\alpha(t)$ and this intersection with α is unique. Moreover, this curve divides M into two disks, one of which containing x and another one containing y (see Croke's paper, Section 3)). Consider a loop based at $\alpha(t_0)$ that goes to $\alpha(t)$ along α, then along along σ, next returns to $\alpha(t_0)$ along α. What is the length of this curve? Is this curve contractible in $M - \{x,y\}$?)

Problem 6 (c)

Suppose there exist a simple geodesic loop γ of length at most $\sqrt{2A}$ not contractible in $M - \{x, y\}$ separating M into two domains Ω_x and Ω_y such that $x \in \Omega_x$, $y \in \Omega_y$, based at $\alpha(\frac{\sqrt{2A}}{2})$, convex to Ω_y. Moreover, γ intersects α only at its base point and the tangent vectors to γ at its beginning and end lie on opposite sides of the straight line tangent to α in the plane tangent to M at the base point of γ. Prove that there exists a periodic geodesic in M of length at most $4\sqrt{2A}$. (Hint: use Berger's lemma to further subdivide Ω_x.)

Proof of the first case

Problem 7

Let M be a complete non-compact surface with a finite area A diffeomorphic to a 2-sphere with three punctures. Prove that there exists a periodic geodesic on M. Can you find an area upper bound for its length?

Filling Radius

The notion of Filling Radius of a Riemannian manifold was introduced by M. Gromov in 1983.

- Let us first consider a simple loop C in the plane. Its filling radius is the largest radius R of a circle that fits inside C.
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- Let us define it in another way: consider the ϵ-neighborhood of the loop C, $N_\epsilon C \subset R^2$. $FillRad(C \subset R^2) = \inf \epsilon > 0$, such that C contracts to a point in $N_\epsilon C$.

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- Next let us consider a closed Riemannian manifold M embedded in Euclidean space R^N. We define $FillRad(M \subset R^N)$ as $\inf \epsilon > 0$ such that M can be homotoped to something smaller dimensional. $FillRad(M \subset R^N) = \inf\{\epsilon > 0 | i_{\epsilon_*}([M]) = 0 \in H_n(N_\epsilon M)\}$.

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- Thus defined, the Filling Radius of M depends on the embedding.

Absolute Filling Radius

Consider Kuratowski embedding. We will embedd M into $L^\infty(M)$. Here $L^\infty(M)$ is the Banach space of bounded Borel functions on M equipped with the sup norm $|| \cdot ||$. $x \in M \longrightarrow f_x \in L^\infty(M)$, where $f_x(y) = d(x, y)$ for all $y \in M$. This is an isometric embedding. Thus, $FillRad(M) = FillRad(M \subset L^\infty(M))$.

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- A. Nabutovsky, "Linear bounds for constants in Gromov's systolic inequality and related results"

Filling Radius and the Length of the shortest periodic geodesic

S. Sabourau, "Filling Radius and short closed geodesic of the 2-sphere", Bulletin de la Societe Mathematique de France, Tome 132 (2004), no. 1, 105-136.

- Gromov's inequality for essential manifolds

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- Gromov's inequality for essential manifolds
- Filling Radius and the length of the shortest periodic geodesic on a Riemannian 2 -sphere.

Related topics: contracting based point loops, width of homotopies

Lemma

Let $p, q \in M$. Let e_1, e_2 be two segments connecting p and q. Let l_i be the length of e_i for $i = 1, 2$. Suppose $e_1 \star \bar{e}_2$ is contractible to p over loops based at p of length at most l_3. Then e_1. is path homotopic to e_2 over paths of length at most $l_3 + \min\{l_1, l_2\}$.

Proof

Homotopies vs. Isotopies

Let M be a Riemannian 2-disk, $|\partial M| = L$. Suppose there exists a homotopy of ∂M to p over curves of length at most L. Is there a homotopy of ∂M to some point in M over simple closed curves of length at most L that also don't intersect each other?

Quantitative version of theorem of R. Baer and D. B. A. Epstein

G. Chambers, Y. Liokumovich, "Converting homotopies to isotopies and dividing homotopies in half in an effective way", GAFA, vol. 24 (20014), 1080-1100

Theorem

Let M be a 2-dimensional Riemannian manifold with or without boundary, and let γ_0, γ_1 be non-contractible simple closed curves which are homotopic through curves bounded in length by L via a homotopy γ. Then for any $\epsilon > 0$ there exists an isotopy from γ_0 to γ_1 through curves of length at most $L + \epsilon$.

Can one construct an embedded geodesic via min – max methods on $\Lambda(S^2, g)$? (M. Friedman, 1980)

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Motivation: Consider a similar problem for families of 2-dimensional spheres in a homotopy sphere M. If we could replace a sweepout of M by immersed 2-spheres with a sweepout by embedded 2-spheres, then by ambient isotopy theorem it follow that M is diffeomorphic to S^3, thus, implying the Poincare conjecture.

E. W. Chambers, G. Chambers, A. de Mesmay, T. Ophelders, R. Rotman, "Constructing monotone homotopies and sweepouts", JDG, 119 (3), 383-401

E. W. Chambers, G. Chambers, A. de Mesmay, T. Ophelders, R. Rotman, "Constructing monotone homotopies and sweepouts", JDG, 119 (3), 383-401

Theorem

Suppose that (D, g) be a Riemannian disc, and suppose there is a contraction of ∂D through curves of length less than L. Then there exist a monotone contraction of ∂D ghrough curves of length less than L.

Can we control lengths of curves in homotopies?

- Let D be a Riemannian 2-disk of area A and $|\partial D| = L$. Is there a constant, $c(L, A)$ such that there exist a homotopy between ∂D and some point in D such that lengths of curves in the homotopy are bounded by $c(L, A)$?

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- Is there $k(L, d)$, (where d is the diameter of D), such that lengths of curves in the homotopy is bounded by $k(L, d)$?

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- Is there $k(L, d)$, (where d is the diameter of D), such that lengths of curves in the homotopy is bounded by $k(L, d)$?
- How about if we can control L, d and A?

Example of S. Frankel and M. Katz

S. Frankel, M. Katz, "The Morse Landscape of a Riemannian disk", Annales de l'Inst. Fourier, 43 (1993), no. 2, 503-507.

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Theorem

One can construct a sequence of metrics g_n on D, such that $|\partial D| = 1, d(D, g_n) \leq 2$ and a function $f(n)$ tending to infinity with n, such that every homotopy of S^1 to a point in (D, g_n) contains an intermediate curve of length bigger than $f(n)$.

Imbed the binary tree T_n in the disk D. Consider a homotopy of ∂D to a point, such that each intermediate curve passes through at most 1 vertex of T_n. Frankel and Katz have shown that some intermediate curve meets at least $O(\frac{n}{\log n})$ edges of T_n