Introduction

Regina Rotman

Department of Mathematics, University of Toronto

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- given two vectors v_p, w_p at p, there is the a natural inner product $< v_p, w_p >$ in R^3, and so given a smooth surface S immersed into R^3, and two vectors tangent to S at p, there is an inner product that is inherited from R^3.

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- we can define special curves, called geodesics

Geodesics in R^3

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- lines are infinite in both directions

Definition of Geodesic

We take as a definition the property of being straight. Namely, $\gamma : (-1, 1) \longrightarrow S$ such that $\gamma(0) = p$ is geodesic at p if $\frac{Dd\gamma}{dtdt} = 0$ at $t = 0$.

The curve is geodesic if it is geodesic at all of its points.

Examples

Questions

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- Is there always at least one periodic geodesic on a closed Riemannian manifold?
- Are there infinitely many?
- Let $p, q \in M$, where M is a closed Riemannian manifold. Is there infinitely many geodesics connecting these points?

Definition, (see Do Carmo's "Riemannian Geometry", page 254)

A set $\mathcal{L}$ of closed paths in M is called a free homotopy class if given $f \in \mathcal{L}$ and $g : I \longrightarrow M$ such that there exists a homotopy $F : I^2 \longrightarrow M, F(0,t) = f(t), F(1,t) = g(t), F(s,0) = F(s,1)$, then $g \in \mathcal{L}$. We will denote the set of such classes by $C_1(M)$

Cartan's Theorem

Theorem

If M is compact and $\mathcal{L} \in C_1(M)$ is not the constant class, then there exists a periodic geodesic of M in $\mathcal{L}$.

Birkhoff Curve Shortening

C. B. Croke, "Area and the Length of the Shortest Closed Geodesic", JDG 27 (1988), pages 3-7.

Proof of Cartan's Theorem

Definition

Let M be a complete connected oriented surface. Let γ be a simple closed curve on M which divides M into two components. Let Ω be one of these components. Then γ will be called convex to Ω if there is an $\epsilon > 0$ such that for all $x, y \in \gamma$, with $d(x, y) < \epsilon$, the minimizing geodesic τ from x to y satisfies $\tau \in \bar{\Omega}$.

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Let γ be a piecewise geodesic curve. Suppose γ is convex to Ω. Then all the angles of γ are convex to Ω.

Problem 1

(a) Show that the sequence of curves $\{\gamma_i\}$ defined by $\gamma = \gamma_0$ and $\gamma_{i+1} = \beta^N(\gamma_i)$ has a subsequence that converges to a (potentially trivial) periodic geodesic.
(b) Let M be a closed Riemannian manifold of diameter d. Suppose that M is not simply-connected. Prove that the length of the shortest periodic geodesic on M is bounded above by $2d$, where d is the diameter of M.
(c) Let γ be convex to Ω and have length L. Assume $\bar{\Omega}$ is compact and let $N > \frac{L}{inj(\bar{\Omega})}$. Then if we apply B.C.S.P. with N breaks to γ the resulting curves γ_t satisfy:
(1) $\gamma_t \subset \bar{\Omega}$;
(2) γ_t is simple and convex to $\Omega_t = \Omega - \{x \in \gamma_s | 0 \leq s \leq t\}$.

Existence theorem due to L. Lusternik and A. Fet

Theorem

On every closed Riemannian manifold there exists at least one periodic geodesic.

R. Bott, "Lectures on Morse Theory, Old and New", Bulletin of the AMS, Volume 7, Number 2, 1982 (page 335).

Proof by Morse theory on ΛM

Proof

Problem 2

Let M be a closed Riemannian manifold. Suppose $\pi_1(M)$ has infinitely many conjugacy classes (up to powers). Prove that there exist infinitely many periodic geodesics on M.

Infinitely many periodic geodesics

- If the sequence $b_k(\Lambda M)$ is not bounded, then there exist infinitely many geometrically distinct periodic geodesics in M.

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- H.-B. Rademacher, I. A. Taimanov, "Closed geodesics on connected sums and 3-manifolds, JDG, 120 (3), 557-573.

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- H.-B. Rademacher, "On the average indices of closed geodesics", JDG, 29 (1989), 65-83.
- W. Ballmann, V. Bangert, N. Hingston, A. Katok, W. Klingenberg, M. Tanaka, G. Thorbergsson, W. Ziller

Surfaces

For surfaces with a Riemannian metric there is an even stronger result, which combines methods from dynamical systems and Morse theory: For any Riemannian metric on the sphere of dimension 2 there are infinitely many closed geodesics, (Birkhoff, N. Hingston, V. Bangert).
V. Bangert, "On the existence of closed geodesics on two-spheres", Internat. J. Math. 4 (1993).

Result of J. P. Serre

Theorem

Let M be a closed Riemannian manifold. Then for any pair of points $p, q \in M$ there exist infinitely many geodesics connecting them.

Quantitative Geometry

The series of lectures will be focused on some question in Quantitative Topology. There we seek to establish a quantitative version of well-known existence theorems in Riemannian Geometry proven by methods of Algebraic Topology, which ultimately lead to knew geometric inequalities.

Theorem of the three geodesics, aka Lusternik-Schnirelmann theorem

Theorem

Let M be a Riemannian 2-sphere. Then there exist at least three simple periodic geodesics on M.

Quantitative Questions

- Let M^n be a closed Riemannian manifold of dimension n. Is there a constant $c(n)$ such that the length of a shortest closed geodesic on M^n, $l(M^n)$ is bounded above by $c(n) vol(M^n)^{\frac{1}{n}}$? Here $vol(M^n)$ denotes the volume of M^n. (The question is due to

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- Can we bound lengths of Lusternik-Schnirelmann's geodesics in terms of the diameter of M?
- Is there $C(n)$ such that for any pair of points on a closed Riemannian M^n there exist at least k geodesics connecting them of length at most $C(n)kd$?

Connection between closed geodesics and the injectivity radius

See "Riemannain Geometry" by Do Carmo, Chapter 13.

- Let $p \in M$. Suppose there exists a point $q \in C_m(p)$ which realizes the distance from p to $C_m(p)$. Here $C_m(p)$ denotes the cut locus of p. Then:

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- Let ρ be the distance between two closest conjugate points on M. Then $injradM^n = \min\{\rho, \frac{l(M)}{2}\}$.
- Obviously, $injradM^n \leq d$, but also M. Berger proved the isoembolic theorem: $Vol(M) \geq C_n inj(M)^n$. ("Une Borne inferieure pout le volume d'une variete riemannienne en fonction du rayon d'injectivite").

Example of Balacheff, Croke, Katz

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- F. Balacheff, C. Croke, M. Katz, "A Zoll Counterexample to a geodesic length conjecture", GAFA, 19 (2009), 1-10.

Geodesic nets

Let M be a Riemannian manifold, G be a graph. Consider a continuous map $f : G \longrightarrow M$, such that the restriction of f to each edge of G is a piecewise differentiable curve. A variation of G is a continuous mapping $F : (-\epsilon, \epsilon) \times G \longrightarrow M$ such that : $F(0, x) = f(G)$

Each edge can be subdivided into subintervals $[t_i, t_{i+1}]$, such that the restriction of F to each $(-\epsilon, \epsilon) \times [t_i, t_{i+1}]$ is differentiable.

We can speak about variational vector fields along G, by extending the notion of the variational vector fields along curves. We just need to make sure that they all agree at the vertices.

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One can then define length (or energy) of the graph by adding lengths (or, correspondingly energies) of its curves.
Geodesic net will be defined as a critical point of the length (or equivalently energy) functional on the space of graphs.

Problem 3

(a) Prove that a net G is geodesic if and only if every edge of G is geodesic and that the following condition is satisfied at every vertex $q_s \in S$: the sum of the unit vectors tangent to the edges originating at q_s and diverging from q_s equals 0.
(b) Let G be a "figure 8" geodesic net on a surface M, i. e. geodesic net with one vertex and two geodesic loops based at that vertex. Prove that it is a self-intersecting geodesic. Will it be true if the dimension of M equals to 3?

Problem 4

Let G be a geodesic θ-graph, i. e. a geodesic net consisting of two vertices and three edges $\gamma_1, \gamma_2, \gamma_3$ connecting these vertices. We can define a variation of G by defining variations $H_i(t, s)$ of each curve $\gamma_i(t)$ that agree on both vertices. We can define $E(G_s) = \Sigma_{i=1} E((\gamma_i)_s)$, where $(\gamma_i)_s$ is the variational curve of γ_i. Let M be a positively curved Riemannian 2-sphere. Prove that for any geodesic θ-graph there exists a variation, such that $E''(0) < 0$. Conclude that any stationary θ-graph on a positively curved 2-sphere admits directions of decrease for a length shortening flow. (I. Adelstein, F. Vargas Pallete, "The length of the shortest closed geodesic on positively curved 2-spheres").

Hint: Let γ_i be the edges of the θ-graph. for $i \in \{1, 2, 3\}$ define vector fields V_i along γ_i as follows:

$V_1(t) = \frac{1}{\sqrt{3}} \cos(\frac{(t-a_1)\pi}{b_1-a_1})\gamma_1'(t) + (\gamma_1')^{\perp}(t)$

$V_2(t) = \frac{1}{\sqrt{3}} \cos(\frac{(t-a_2)\pi}{b_2-a_2})\gamma_2'(t) - (\gamma_2')^{\perp}(t)$

$V_3(t) = \frac{-2}{\sqrt{3}} \cos(\frac{(t-a_3)\pi}{b_3-a_3})\gamma_3'(t)$

Check that $V_1(a_1) = V_2(a_1) = V_3(a_1)$ and that $V_1(b_1) = V_2(b_2) = V_3(b_3)$, concluding that V_i induce a variation of this θ-graph in the space of θ-graphs. Compute its second variation of energy and verify that $E''(0) < 0$.