Quantitative Morse Theory on Loop Spaces

Regina Rotman

Department of Mathematics, University of Toronto

July, 2022

Theorem of Lusternik and Schnirelmann

Theorem

On any Riemannian 2-sphere there exist at least three simple periodic geodesics.

joint with Y. Liokumovich and A. Nabutovsky

Theorem

Let M be a Riemannian sphere of diameter d and area A. There exist three distinct non-trivial simple periodic geodesics of length at most $20d$. Moreover, there exist three distinct simple periodic geodesics on M of length at most $800d \max\{1, \log \frac{\sqrt{A}}{d}\}$ such that none of these geodesics has index zero.

Summary of the proof of Lusternik and Schnirelmann's theorem

- Consider the space ΠM of non-parametrized simple curves on a 2-dimensional Riemannian sphere. Consider its subspace $\Pi_0 M$ of constant curves of M. Note that $\Pi_0 M$ can be naturally identified with M.

Summary of the proof of Lusternik and Schnirelmann's theorem

- Consider the space ΠM of non-parametrized simple curves on a 2-dimensional Riemannian sphere. Consider its subspace $\Pi_0 M$ of constant curves of M. Note that $\Pi_0 M$ can be naturally identified with M.
- Consider the three relative homology classes of the pair $(\Pi M, \Pi_0 M)$ with coefficients in Z_2

Summary of the proof of Lusternik and Schnirelmann's theorem

- Consider the space ΠM of non-parametrized simple curves on a 2-dimensional Riemannian sphere. Consider its subspace $\Pi_0 M$ of constant curves of M. Note that $\Pi_0 M$ can be naturally identified with M.
- Consider the three relative homology classes of the pair $(\Pi M, \Pi_0 M)$ with coefficients in Z_2
- Lusternik and Schnirelmann constructed a curve shortening flow that should not create self-intersections. Thus, they get stuck on simple closed geodesics.

Proof

Based point loops

We can study based point loops

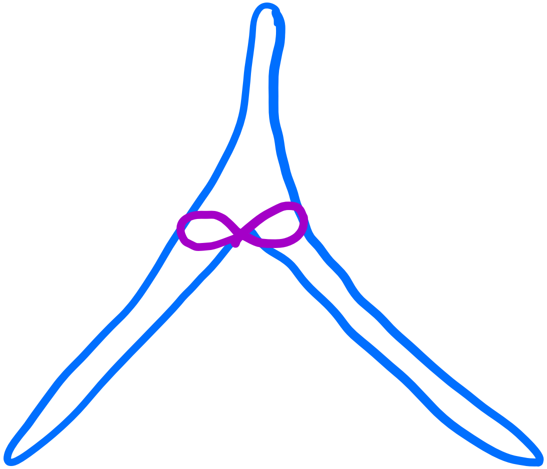
- The length of a shortest geodesic loop on M^n

Based point loops

- The length of a shortest geodesic loop on M^n
- The length of a shortest geodesic loop at each point $p \in M^n$ of a closed Riemannian manifold.

First bounds

- First bounds on *sgl* are due to S. Sabourau

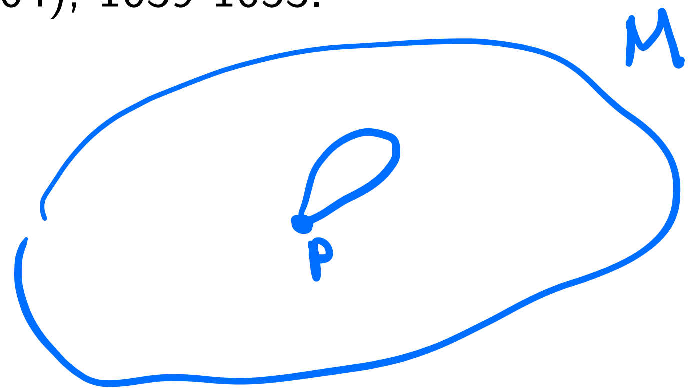


First bounds

- First bounds on *sgl* are due to S. Sabourau
- S. Sabourau, "Global and local volume bounds and the shortest geodesic loop", Communications in Analysis and Geometry, vol. 12 (5) (2004), 1039-1053.

First bounds

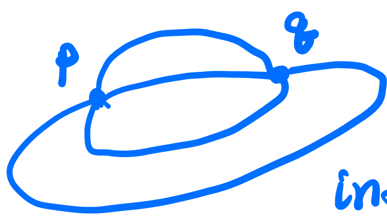
- First bounds on *sgl* are due to S. Sabourau
- S. Sabourau, ”Global and local volume bounds and the shortest geodesic loop”, Communications in Analysis and Geometry, vol. 12 (5) (2004), 1039-1053.
- $l_p(M^n) \leq 2nd$



First bounds

- First bounds on *sgl* are due to S. Sabourau
- S. Sabourau, "Global and local volume bounds and the shortest geodesic loop", Communications in Analysis and Geometry, vol. 12 (5) (2004), 1039-1053.
- $l_p(M^n) \leq 2nd$
- R. R. , "The length of a shortest geodesic loop at a point", JDG, 78 (2008), 497-519

J.-P. Serre's theorem



Take $p=q$ you will get infinitely many geo. loops

Theorem

Given a pair of points on a closed Riemannian manifold, there exist infinitely many geodesics connecting them.

J.-P. Serre, "Homologie singuliere des espaces fibres", Annals of Mathematics (1951), 425-505

Is there $f(n,k)$ s.t. $\forall\, p \in M^n$ $\exists$ at least k geodesic loops based at p of length at most $f(n,k)\,d$? d is the diameter

Question

Maybe there always at least k of length at most kd

.

Let $p, q \in M^n$, where M^n is a closed Riemannian manifold of dimension n. Is there a function $f(k, d)$, such that for every k there exist at least k geodesics connecting p and q of length at most $f(k, n)d$, where d is the diameter of M^n?

Curvature-free estimates for the lengths of geodesics

A. Nabutovsky, R. R., "Length of geodesics and quantitative Morse theory on loop spaces", GAFA, 23 (2013), 367-414.

Curvature-free estimates for the lengths of geodesics

A. Nabutovsky, R. R., "Length of geodesics and quantitative Morse theory on loop spaces", GAFA, 23 (2013), 367-414.

Theorem

Let M^n be a closed Riemannian manifold of diameter d. Then for each pair of points $p, q \in M^n$ there exist at least k geodesics of length at most $4nk^2d$

Linear bounds for the length of geodesics on closed Riemannian surfaces

- Let M be a Riemannian 2-sphere of diameter d. Then for each pair of points p, q there exists at least k geodesics of length at most $22kd$, ($20kd$, when $p = q$).

Linear bounds for the length of geodesics on closed Riemannian surfaces

- Let M be a Riemannian 2-sphere of diameter d. Then for each pair of points p, q there exists at least k geodesics of length at most $22kd$, ($20kd$, when $p = q$).
- A. Nabutovsky, R. Rotman, "Linear bounds for lengths of geodesic segments on Riemannian 2-sphere", Journal of Topology and Analysis, 5 (2013), no. 04, 409-438

Linear bounds for the length of geodesics on closed Riemannian surfaces

- Let M be a Riemannian 2-sphere of diameter d. Then for each pair of points p, q there exists at least k geodesics of length at most $22kd$, ($20kd$, when $p = q$).
- A. Nabutovsky, R. Rotman, "Linear bounds for lengths of geodesic segments on Riemannian 2-sphere", Journal of Topology and Analysis, 5 (2013), no. 04, 409-438
- This bound was improved to $8kd$, and to $6kd$, when $p = q$ by Herng Yi Cheng.

Linear bounds for the length of geodesics on closed Riemannian surfaces

- Let M be a Riemannian 2-sphere of diameter d. Then for each pair of points p, q there exists at least k geodesics of length at most $22kd$, ($20kd$, when $p = q$).
- A. Nabutovsky, R. Rotman, "Linear bounds for lengths of geodesic segments on Riemannian 2-sphere", Journal of Topology and Analysis, 5 (2013), no. 04, 409-438
- This bound was improved to $8kd$, and to $6kd$, when $p = q$ by Herng Yi Cheng.
- Herng Yi Cheng, "Curvature-free linear length bounds on geodesics in closed Riemannian surfaces", Transactions of the AMS.

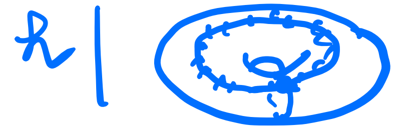
Proof of the Theorem: the starting point

- The starting point of the proof is the existence proof of A. Schwarz of Serre's theorem.

Proof of the Theorem: the starting point

- The starting point of the proof is the existence proof of A. Schwarz of Serre's theorem.
- A. S. Schwarz, "Geodesic arcs on Riemannian manifolds", Uspekhi Matematicheskikh Nauk 13 (1958), no. 6, 181-184.

Proof of the Theorem: the starting point



- The starting point of the proof is the existence proof of A. Schwarz of Serre's theorem.
- A. S. Schwarz, "Geodesic arcs on Riemannian manifolds", Uspekhi Matematicheskikh Nauk 13 (1958), no. 6, 181-184.
- Cartan-Serre's theorem implies the existence of an even-dimensional real cohomology class u of the loop space $\Omega_p M^n$ such that all of its cup powers are non-trivial. Using rational homotopy theory one can prove that there exists such a class u of dimension at most $2n-2$.

$\Omega_p M^n$ = p.w. dif. curves on M^n beginning and ending at p

$H^*(\Omega_p M^n, \mathbb{Q})$

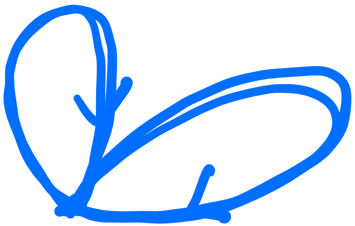
$u, u^2, u^3, \ldots$

Proof of the Theorem: the starting point

- The starting point of the proof is the existence proof of A. Schwarz of Serre's theorem.
- A. S. Schwarz, "Geodesic arcs on Riemannian manifolds", Uspekhi Matematicheskikh Nauk 13 (1958), no. 6, 181-184.
- Cartan-Serre's theorem implies the existence of an even-dimensional real cohomology class u of the loop space $\Omega_p M^n$ such that all of its cup powers are non-trivial. Using rational homotopy theory one can prove that there exists such a class u of dimension at most $2n-2$.
- Apply Morse theory to produce critical points of the length functional on $\Omega_p M^n$ corresponding to cohomology classes u^i.

- A. Schwarz's proof of Serre's theorem: Consider a product in rational homology group of the loop space induced by the concatenation of loops.

induces
Pontryagin product
for homology



- A. Schwarz's proof of Serre's theorem: Consider a product in rational homology group of the loop space induced by the concatenation of loops.
- Dual real homology class c of u is the homology class of the same dimension such that $< c, u >= 1$. The classes u, c can be chosen so that c is spherical.

- A. Schwarz's proof of Serre's theorem: Consider a product in rational homology group of the loop space induced by the concatenation of loops.
- Dual real homology class c of u is the homology class of the same dimension such that $< c, u >= 1$. The classes u, c can be chosen so that c is spherical.
- Schwarz showed that for every positive i, the ith Pontryagin power of c and a real multiple of u^i are dual up to a multiplicative constant.

- A. Schwarz's proof of Serre's theorem: Consider a product in rational homology group of the loop space induced by the concatenation of loops.
- Dual real homology class c of u is the homology class of the same dimension such that $< c, u >= 1$. The classes u, c can be chosen so that c is spherical.
- Schwarz showed that for every positive i, the ith Pontryagin power of c and a real multiple of u^i are dual up to a multiplicative constant.
- Therefore, critical point corresponding to u^i also corresponds to c^i. We need to choose a representative of c.

- A. Schwarz's proof of Serre's theorem: Consider a product in rational homology group of the loop space induced by the concatenation of loops.
- Dual real homology class c of u is the homology class of the same dimension such that $< c, u >= 1$. The classes u, c can be chosen so that c is spherical.
- Schwarz showed that for every positive i, the ith Pontryagin power of c and a real multiple of u^i are dual up to a multiplicative constant.
- Therefore, critical point corresponding to u^i also corresponds to c^i. We need to choose a representative of c.
- Let L be such that this representative c is contained in the set of loops of length $\leq L$. Then c^i can be represented by a set of loops of length $\leq iL$.

Linear bounds on a sphere

- For the simplicity of exposition we will consider only geodesic loops

Linear bounds on a sphere

- For the simplicity of exposition we will consider only geodesic loops
- In the case of the sphere, the c is of dimension 1. Let $p \in M$ be given. We would like to construct $f : S^1 \longrightarrow \Omega_p M$ that passes through short loops, unless there exist already many sufficiently short loops.

Linear bounds on a sphere

- For the simplicity of exposition we will consider only geodesic loops
- In the case of the sphere, the c is of dimension 1. Let $p \in M$ be given. We would like to construct $f : S^1 \longrightarrow \Omega_p M$ that passes through short loops, unless there exist already many sufficiently short loops.
- We will use the same pseudo-extension technique.

Proof

General dimension

If there are not many geodesic loops, then any sphere in the loop space can be replaced by a homotopic sphere passing through short loops and our result follows.

General dimension

$f : S^k \rightsquigarrow \Omega_p M^n$ if you don't have sufficiently many

If there are not many geodesic loops, then any sphere in the loop space can be replaced by a homotopic sphere passing through short loops and our result follows.

short loops of index 0 then you can homotope your

Theorem

Let M^n be a closed Riemannian manifold of dimension nand diameter d. Then either there exist non-trivial geodesic loops with lengths in every interval $(2(i-1)d, 2d]$ for $i \in \{1, \ldots, k\}$ or all maps $f : S^m \longrightarrow \Omega_p(M^n)$ can be homotoped to a map of loops based at p of length that does not exceed $((4k+2)m + (2k-3))d$, and the length of loops during this homotopy does not increase that much in comparison with the maximal length of loops in the image of f.

sphere to sphere that consists of short loops

Observation

- If there are no geodesic loops of length in some interval $(l, l+2d]$, then any curve of length L can be shortened so that

Observation

- If there are no geodesic loops of length in some interval $(l, l + 2d]$, then any curve of length L can be shortened so that
- the endpoints stay fixed

Observation

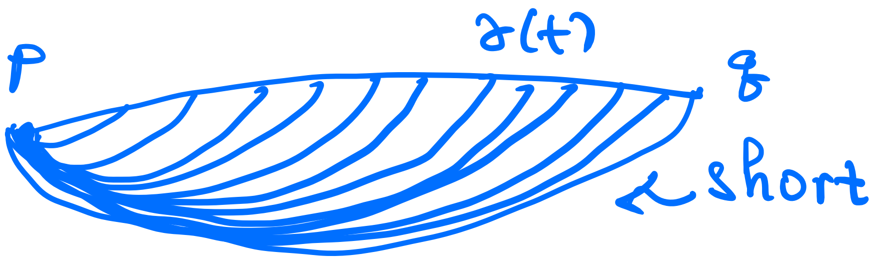
- If there are no geodesic loops of length in some interval $(l, l + 2d]$, then any curve of length L can be shortened so that
- the endpoints stay fixed
- the length of curves in the homotopy is at most $L + 2d$

Observation

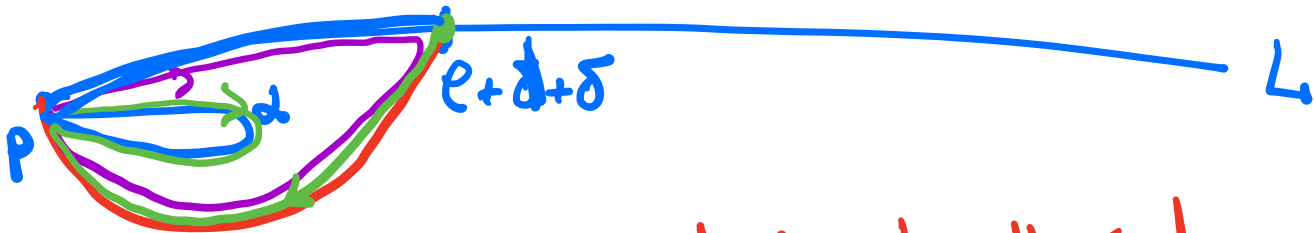
- If there are no geodesic loops of length in some interval $(l, l + 2d]$, then any curve of length L can be shortened so that
- the endpoints stay fixed
- the length of curves in the homotopy is at most $L + 2d$
- the length of the resulting curve is at most $l + d$

Observation

- If there are no geodesic loops of length in some interval $(l, l + 2d]$, then any curve of length L can be shortened so that
- the endpoints stay fixed
- the length of curves in the homotopy is at most $L + 2d$
- the length of the resulting curve is at most $l + d$
- there exist a family of curves β_t which connect p with $\gamma(t)$ and the maximal length of the curves in this family is at most $l + 3d + \delta$



Proof of the observation

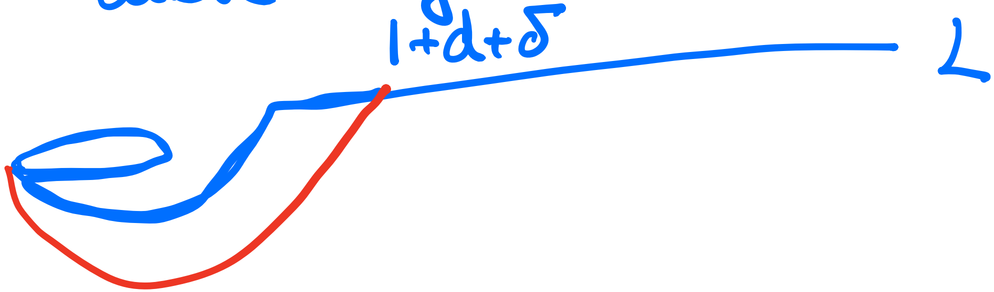


minimizing geodesic, length $\leq d$

length$(\alpha) \leq \ell$

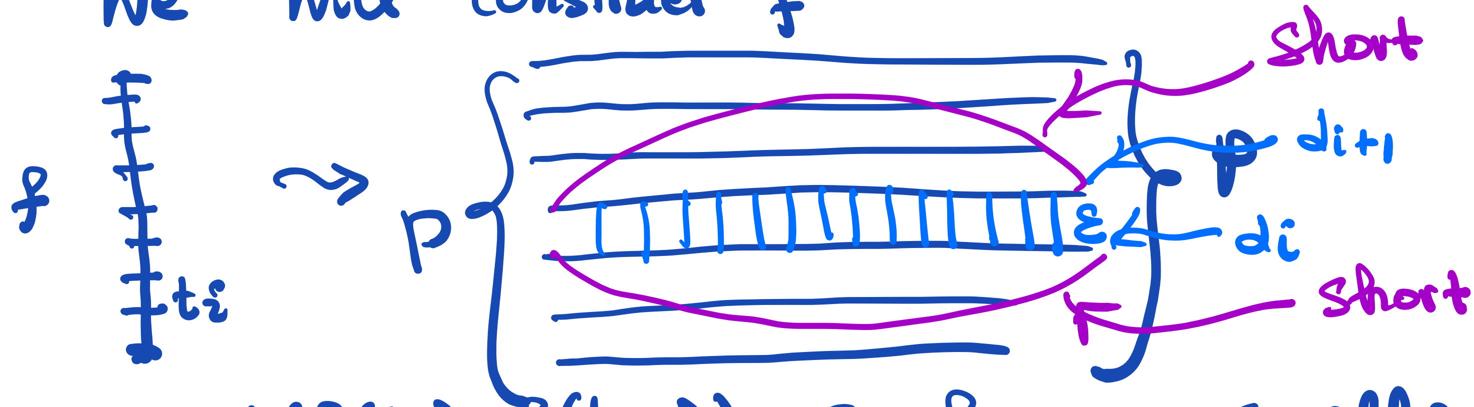
length of green curve is at most $\underline{\ell + d}$

We decreased the length of the total curve by δ



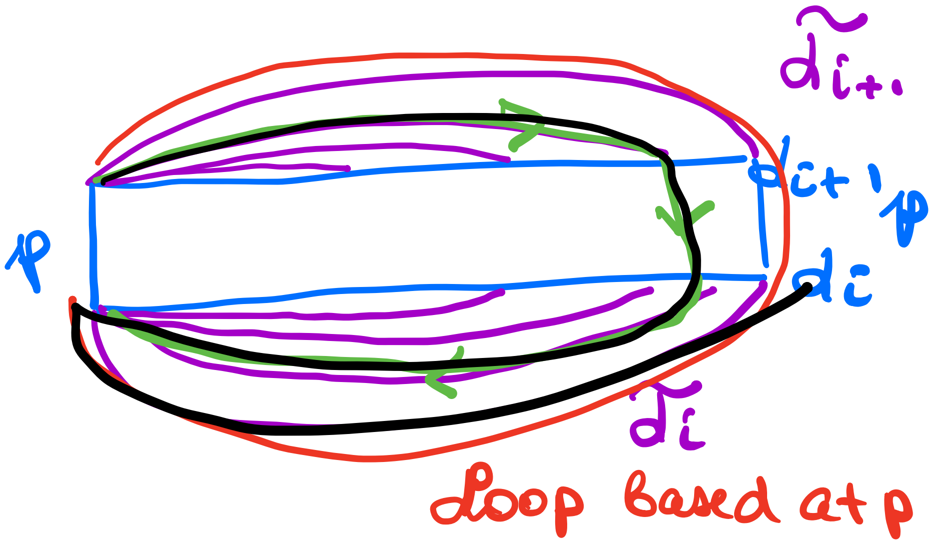
Proof

Let us consider $f : S^1 \to \Omega_p M^n$
We will construct $\tilde{f}$



$d(f(t_i), f(t_{i+1})) < \varepsilon$ for some small ε

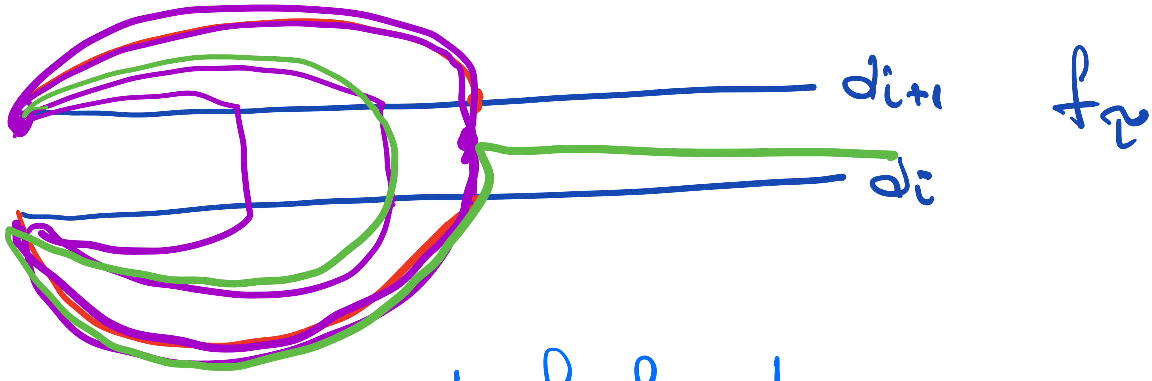
$f(t_i) = d_i$



This means there is a path homotopy between $\tilde{\alpha}_i$ and $\tilde{\alpha}_{i+1}$ over short curves

f $\tilde{f}$, construct is f_τ s.t.

$f_0 = f$, $f_1 = \tilde{f}$



Now generalize to higher dimension

Problem 8

(a) Let M^{2n} be a Riemannian $2n$-sphere for some natural number n with a positive sectional curvature. Let $\gamma : [0,1] \longrightarrow M^{2n}$ be a periodic geodesic on M^{2n}. Show that the index $i(\gamma) \geq 1$.
(b) Let M^n be a closed Riemannian manifold with $Ric \geq (n-1)$ where Ric is the Ricci curvature of M^n. Show that any geodesic loop $\gamma_p \in \Omega_p M^n$ of length $> \pi$ has index $i(\gamma_p) \geq 1$.

Problem 9

Let M^n be a Riemannian manifold, such that $Ric \geq (n-1)H$. Given $r, \epsilon > 0$ and $p \in M^n$ prove that there exists a covering of $B_r(p)$ by balls $B_\epsilon(p_i)$, where $p_i \in B_r(p)$, with the number of balls N bounded in terms of n, H, r, ϵ. Compute some bound on N.

Problem 10

" Critical points of Distance functions and applications to geometry" J. Cheeger — math.uchicago.edu

We will say that a point q on a manifold M^n is critical with respect to p, if for all vectors v in the tangent space T_qM, there exists a minimal geodesic γ from q to p with the absolute value of the angle between $\gamma'(0)$ and v at most $\frac{\pi}{2}$

Let q_1 be critical point with respect to p and let q_2 satisfy $d(p, q_2) \geq \alpha d(p, q_1)$ for some $\alpha > 1$. Let γ_1, γ_2 be minimal geodesics from p to q_1, q_2 respectively, and let θ be the angle between $\gamma_1'(0)$ and $\gamma_2'(0)$. If sectional curvature K_M of a closed Riemannian manifold M is bounded from below by -1 show that

$$\cos\theta \leq \frac{\tanh\frac{d}{\alpha}}{\tanh d}$$

. Here d denotes the diameter of M.

Hint: Use Toponogov comparison theorem twice.

Geodesic nets on closed Riemannian manifolds

- Geodesic nets that are cycles are called geodesic cycles

Geodesic nets on closed Riemannian manifolds

- Geodesic nets that are cycles are called geodesic cycles
- $\alpha(M^n) \leq \frac{(n+2)!d}{4}$

Geodesic nets on closed Riemannian manifolds

- Geodesic nets that are cycles are called geodesic cycles
- $\alpha(M^n) \leq \frac{(n+2)!d}{4}$
- $\alpha(M^n) \leq (n+2)!(n+1)n^n\sqrt{(n+1)!}vol(M^n)^{\frac{1}{n}}$

Geodesic nets on closed Riemannian manifolds

- Geodesic nets that are cycles are called geodesic cycles
- $\alpha(M^n) \leq \frac{(n+2)!d}{4}$
- $\alpha(M^n) \leq (n+2)!(n+1)n^n\sqrt{(n+1)!}vol(M^n)^{\frac{1}{n}}$
- A. Nabutovsky, R. Rotman, "Volume, diameter and the minimal mass of a stationary 1-cycle", GAFA, 14 (2004), 748-790

- "How many geodesic nets are there?"

- "How many geodesic nets are there?"
- Y. Liokumovich, B. Staffa, "Generic density of geodesic nets"

- "How many geodesic nets are there?"
- Y. Liokumovich, B. Staffa, "Generic density of geodesic nets"
- Let M^n be a closed manifold, $\mathcal{M}^k$ be the space of C^k Riemannian metrics on M, $3 \leq k \leq \infty$. For a generic subset of $\mathcal{M}^k$ the union of the images of all embedded stationary geodesic nets in (M, g) is dense.

- "How many geodesic nets are there?"
- Y. Liokumovich, B. Staffa, "Generic density of geodesic nets"
- Let M^n be a closed manifold, $\mathcal{M}^k$ be the space of C^k Riemannian metrics on M, $3 \leq k \leq \infty$. For a generic subset of $\mathcal{M}^k$ the union of the images of all embedded stationary geodesic nets in (M, g) is dense.
- K. Irie, "Dense existence of periodic Reeb orbits and ECH spectral invariants", J. Mod. Dyn. 9 (2015)

- Similar density result for periodic geodesics on surfaces.

- Similar density result for periodic geodesics on surfaces.
- K. Irie, F. C. Marques, A. Neves, "Density of minimal hypersurfaces for generic metrics", Ann. Math. Vol. 187 (3), following:

- Similar density result for periodic geodesics on surfaces.
- K. Irie, F. C. Marques, A. Neves, "Density of minimal hypersurfaces for generic metrics", Ann. Math. Vol. 187 (3), following:
- Y. Liokumovich, F. C. Marques, A. Neves, "Weyl law for the volume spectrum, Ann. Math., vol. 187 (3) (2018), 933-961.

- Similar density result for periodic geodesics on surfaces.
- K. Irie, F. C. Marques, A. Neves, "Density of minimal hypersurfaces for generic metrics", Ann. Math. Vol. 187 (3), following:
- Y. Liokumovich, F. C. Marques, A. Neves, "Weyl law for the volume spectrum, Ann. Math., vol. 187 (3) (2018), 933-961.
- Similar result for minimal hypersurfaces, $3 \leq n \leq 7$.

- One can study shapes of geodesic nets on Riemannian manifolds

- One can study shapes of geodesic nets on Riemannian manifolds
- J. Hass, F. Morgan, "Geodesic nets on the 2-sphere", Proceedings of the AMS 124 (1996), no. 12, 3843-3850

- One can study shapes of geodesic nets on Riemannian manifolds
- J. Hass, F. Morgan, "Geodesic nets on the 2-sphere", Proceedings of the AMS 124 (1996), no. 12, 3843-3850
-

- One can study shapes of geodesic nets on Riemannian manifolds
- J. Hass, F. Morgan, "Geodesic nets on the 2-sphere", Proceedings of the AMS 124 (1996), no. 12, 3843-3850

Theorem

Let S be a 2-sphere with a smooth Riemannian metric with positive curvature. There exists a geodesic net G partitioning S into three components shaped either as a θ-graph, "figure 8", or "glasses".

-

- One can study shapes of geodesic nets on Riemannian manifolds
- J. Hass, F. Morgan, "Geodesic nets on the 2-sphere", Proceedings of the AMS 124 (1996), no. 12, 3843-3850

Theorem

Let S be a 2-sphere with a smooth Riemannian metric with positive curvature. There exists a geodesic net G partitioning S into three components shaped either as a θ-graph, "figure 8", or "glasses".

-
- Given a graph, a positively curved 2-sphere and target total curvatures, there is a length-minimizing graph (θ, figure 8, glasses), which divides the two-sphere into regions with those total curvatures.

- One can study shapes of geodesic nets on Riemannian manifolds
- J. Hass, F. Morgan, "Geodesic nets on the 2-sphere", Proceedings of the AMS 124 (1996), no. 12, 3843-3850

Theorem

Let S be a 2-sphere with a smooth Riemannian metric with positive curvature. There exists a geodesic net G partitioning S into three components shaped either as a θ-graph, "figure 8", or "glasses".

-
- Given a graph, a positively curved 2-sphere and target total curvatures, there is a length-minimizing graph (θ, figure 8, glasses), which divides the two-sphere into regions with those total curvatures.
- F. Morgan, "Soap bubbles in R^2 and in surfaces", Pc. J. Math 96 (1989), 333-348

Questions

- Do all metrics on the 2-sphere contain a geodesic net homeomorphic to a θ-graph?

Questions

- Do all metrics on the 2-sphere contain a geodesic net homeomorphic to a θ-graph?
- Does the conclusion of Theorem 1 hold for arbitrary metrics on the 2-sphere?

Problem 11

Let M be a Riemannian 2-sphere. Let $\mathcal{N}$ be a geodesic net modelled either on θ-graph, figure 8, or glasses, that subdivides M into three regions $R_i, i = \{1, 2, 3\}$.
(a) Evaluate the total curvature K_i of each of R_i, when $\mathcal{N}$ is either a θ-graph or glasses;
(b) Find the bounds for K_i, when $\mathcal{N}$ is a figure 8 with vertex angles $\frac{\pi}{3} \leq t \leq \frac{2\pi}{3}$.

Poincare problem and related questions

- Let M be a convex two-dimensional surface, then the curve of smallest length that splits the total curvature of M into two pieces of equal curvature is a periodic geodesic.

Poincare problem and related questions

- Let M be a convex two-dimensional surface, then the curve of smallest length that splits the total curvature of M into two pieces of equal curvature is a periodic geodesic.
- C. B. Croke, Poincare's problem on the shortest closed geodesic on a convex hypersurface, JDG 17 (1982), 595-634

Poincare problem and related questions

- Let M be a convex two-dimensional surface, then the curve of smallest length that splits the total curvature of M into two pieces of equal curvature is a periodic geodesic.
- C. B. Croke, Poincare's problem on the shortest closed geodesic on a convex hypersurface, JDG 17 (1982), 595-634
- Find length bounds on closed curves that subdivide surfaces into pieces of comparable areas.

Poincare problem and related questions

- Let M be a convex two-dimensional surface, then the curve of smallest length that splits the total curvature of M into two pieces of equal curvature is a periodic geodesic.
- C. B. Croke, Poincare's problem on the shortest closed geodesic on a convex hypersurface, JDG 17 (1982), 595-634
- Find length bounds on closed curves that subdivide surfaces into pieces of comparable areas.
- P. Papazoglou, "Cheeger constants of surfaces and isoperimetric inequalities", Trans. Amer. Math. Soc. 361 (2009), no. 10, 5139-5162.

Problem 12

Let M be a Riemannian 2-sphere of area A and diameter d.
(a) Show that for any $\delta > 0$ there exist a closed curve of length at most $2d$ that subdivides M into two pieces of area at least $\frac{A}{3} - \delta$.
(b) Besicovitch Lemma. Let D be a Riemannian 2-disk. Consider a subdivision of ∂D into four consecutive sub-arcs with disjoint interiors, i.e. $\partial D = a \cup b \cup c \cup d$. Let l_1 denote the length of a minimizing geodesic between a and c, l_2 denote the length of a minimizing geodesic between b and d. Then the area of D, $A \geq l_1 l_2$.
Use Besicovitch Lemma to show that there exists a closed curve of length at most $4\sqrt{A}$ that subdivides M into two pieces of area at least $\frac{A}{4}$.

Subdividing n-dimensional manifolds into pieces of comparable volume

- Question of P. Papazoglou: Let M be a Riemannian 3-disk with diameter d, boundary area A, volume V. Is there a function $f(d, A, V)$ such that there exists a homotopy S_t contracting the boundary to a point, so that the area of S_t is bounded by $f(d, A, V)$? Is it possible to subdivide M by a disk D into two regions of volume at least $\frac{M}{4}$ so that the area of D is bounded by $h(d, A, V)$?

Subdividing n-dimensional manifolds into pieces of comparable volume

- Question of P. Papazoglou: Let M be a Riemannian 3-disk with diameter d, boundary area A, volume V. Is there a function $f(d, A, V)$ such that there exists a homotopy S_t contracting the boundary to a point, so that the area of S_t is bounded by $f(d, A, V)$? Is it possible to subdivide M by a disk D into two regions of volume at least $\frac{M}{4}$ so that the area of D is bounded by $h(d, A, V)$?
- The answer is NO.

Subdividing n-dimensional manifolds into pieces of comparable volume

- Question of P. Papazoglou: Let M be a Riemannian 3-disk with diameter d, boundary area A, volume V. Is there a function $f(d, A, V)$ such that there exists a homotopy S_t contracting the boundary to a point, so that the area of S_t is bounded by $f(d, A, V)$? Is it possible to subdivide M by a disk D into two regions of volume at least $\frac{M}{4}$ so that the area of D is bounded by $h(d, A, V)$?
- The answer is NO.
- P. Glynn-Adey, Z. Zhu, "Subdividing three-dimensional Riemannian disks", Journal of Topology and Analysis, vol. 09 (2017), no. 03, 533-550

Subdividing n-dimensional manifolds into pieces of comparable volume

- Question of P. Papazoglou: Let M be a Riemannian 3-disk with diameter d, boundary area A, volume V. Is there a function $f(d, A, V)$ such that there exists a homotopy S_t contracting the boundary to a point, so that the area of S_t is bounded by $f(d, A, V)$? Is it possible to subdivide M by a disk D into two regions of volume at least $\frac{M}{4}$ so that the area of D is bounded by $h(d, A, V)$?
- The answer is NO.
- P. Glynn-Adey, Z. Zhu, "Subdividing three-dimensional Riemannian disks", Journal of Topology and Analysis, vol. 09 (2017), no. 03, 533-550
- based on D. Burago, S. Ivanov, "On asymptotic constant of tori", GAFA 8 (1998), no. 5, 783-787

- P. Papazoglou, E. Swenson, "A surface with discontinuous isoperimetric profile and expander manifolds", Geometriae Dedicata, 206 (2020), 43-54

- P. Papazoglou, E. Swenson, "A surface with discontinuous isoperimetric profile and expander manifolds", Geometriae Dedicata, 206 (2020), 43-54
- For any $\epsilon, M > 0$ there exists a Riemannian 3-sphere S of volume 1 such that any, not necessarily connected surface separating S into two regions of volume $> \epsilon$ has area greater than M

Geodesic flowers and wide loops

- R. Rotman, "Flowers on Riemannian manifolds", Mathematische Zeitschrift, 269 (2011), 543-554

Geodesic flowers and wide loops

- R. Rotman, "Flowers on Riemannian manifolds", Mathematische Zeitschrift, 269 (2011), 543-554
- Let M^n be a closed Riemannian manifold. There exists a geodesic net with one vertex and at most $(2n-1)$ geodesic loops of length at most $2n!d$, where d is the diameter of M^n

Geodesic flowers and wide loops

- R. Rotman, "Flowers on Riemannian manifolds", Mathematische Zeitschrift, 269 (2011), 543-554
- Let M^n be a closed Riemannian manifold. There exists a geodesic net with one vertex and at most $(2n-1)$ geodesic loops of length at most $2n!d$, where d is the diameter of M^n
- There exists a geodesic net with one vertex and at most $3^{(n+1)^2}$ geodesic loops of total length at most $2(n+1)!^{\frac{5}{2}}3^{(n+1)^3}(n+1)n^n vol(M^n)^{\frac{1}{n}}$, where $vol(M^n)$ is the volume of M^n

Geodesic flowers and wide loops

- R. Rotman, "Flowers on Riemannian manifolds", Mathematische Zeitschrift, 269 (2011), 543-554
- Let M^n be a closed Riemannian manifold. There exists a geodesic net with one vertex and at most $(2n-1)$ geodesic loops of length at most $2n!d$, where d is the diameter of M^n
- There exists a geodesic net with one vertex and at most $3^{(n+1)^2}$ geodesic loops of total length at most $2(n+1)!^{\frac{5}{2}}3^{(n+1)^3}(n+1)n^n vol(M^n)^{\frac{1}{n}}$, where $vol(M^n)$ is the volume of M^n
- R. Rotman, "Wide short geodesic loops on closed Riemannian manifolds"

Geodesic flowers and wide loops

- R. Rotman, "Flowers on Riemannian manifolds", Mathematische Zeitschrift, 269 (2011), 543-554
- Let M^n be a closed Riemannian manifold. There exists a geodesic net with one vertex and at most $(2n-1)$ geodesic loops of length at most $2n!d$, where d is the diameter of M^n
- There exists a geodesic net with one vertex and at most $3^{(n+1)^2}$ geodesic loops of total length at most $2(n+1)!^{\frac{5}{2}}3^{(n+1)^3}(n+1)n^n vol(M^n)^{\frac{1}{n}}$, where $vol(M^n)$ is the volume of M^n
- R. Rotman, "Wide short geodesic loops on closed Riemannian manifolds"
- Let $\epsilon > 0$ be given. There exists a geodesic loop with angle $\pi - \epsilon \leq \theta \leq \pi$ of length at most $2n!a^n d$.

Geodesic flowers and wide loops

- R. Rotman, "Flowers on Riemannian manifolds", Mathematische Zeitschrift, 269 (2011), 543-554
- Let M^n be a closed Riemannian manifold. There exists a geodesic net with one vertex and at most $(2n-1)$ geodesic loops of length at most $2n!d$, where d is the diameter of M^n
- There exists a geodesic net with one vertex and at most $3^{(n+1)^2}$ geodesic loops of total length at most $2(n+1)!^{\frac{5}{2}}3^{(n+1)^3}(n+1)n^n vol(M^n)^{\frac{1}{n}}$, where $vol(M^n)$ is the volume of M^n
- R. Rotman, "Wide short geodesic loops on closed Riemannian manifolds"
- Let $\epsilon > 0$ be given. There exists a geodesic loop with angle $\pi - \epsilon \leq \theta \leq \pi$ of length at most $2n!a^n d$.
- Also there exists a wide geodesic loop of length at most $n(n+1)!a^{(n+1)^3} vol(M^n)^{\frac{1}{n}}$