Curvature, sweepouts and the length of the shortest closed geodesic

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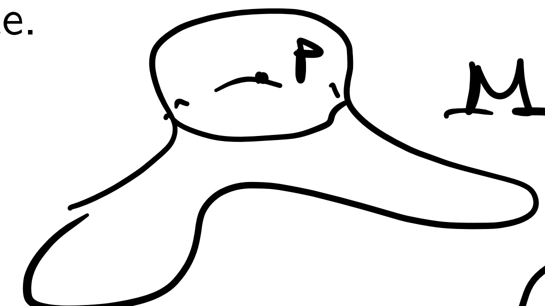
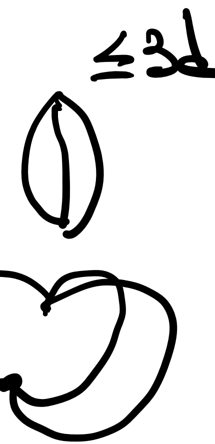
Problem 13

Need to effectively contract curves

Let M be a Riemannian 2-sphere with non-negative curvature. Prove that the length of the shortest periodic geodesic is bounded by $3d$, where d is the diameter of M, thus proving a theorem of I. Adelstein and F. Vargas Pallete.





$\mathcal{M}_{k,v}^{d}$

For any $d > 0$, $v > 0$, k we will let $\mathcal{M}_{k,v}^{d}$ denote the collection of all closed connected Riemannian manifolds of dimension n, such that $K_M \geq k$, $d_M \leq d$, $V_M \geq v > 0$, where d_M denotes the diameter of M, V_M denotes its volume and K_M is its sectional curvature.

can be negative

Length of the shortest periodic geodesic on $\mathcal{M}^d_{v,k}$

Let $M^n \in \mathcal{M}^d_{-1,v}$. Then the length of the shortest geodesic $l(M^n) \leq \exp(\frac{e^{c_1(n)d}}{\min\{1,v\}^{c_2(n)}})$. Here $c_1(n)$ and $c_2(n)$ can be explicitly calculated.

A. Nabutovsky, R. Rotman, "Upper bounds on the length of a shortest closed geodesic and quantitative Hurewicz theorem", Journal of the European Mathematical Society, 5 (2003), 203-244

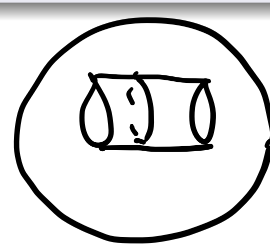
Bounding homotopy types by geometry, K. Grove and P. Peterson

K. Grove, P. Peterson, "Bounding homotopy types by geometry", Annals of Mathematics, vol. 128, no. 2 (1988), 195-206

Theorem

For any $d > 0, v > 0, k, n \geq 2$, $\mathcal{M}^d_{k,v}$ contains only finitely many homotopy types.

Consider metric Balls



& you can estimate the width of the homotopy

Problem 14

Show that the conclusion of the above theorem fails if one drops either the volume lower bound or the diameter upper bound.

The contractibility radius, contractibility function

- Let $\rho : [0, R) \longrightarrow [0, \infty)$ be a function such that amy metric ball of radius ϵ is contractible inside the ball of radius $\rho(\epsilon)$. Then R is called the radius of contractibility, and ρ the contractibility function of M^n.

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- Let $M^n \in \mathcal{M}^d_{-1,v}$ By Grove-Peterson, one can estimate the contractibility radius $r_0 = r_0(n, v, d)$ and the $w(n, v, d)$, so that any closed curve in a ball of radius r_0 or smaller can be contracted to a point by a homotopy of width $\leq w(n, v, d)$. Note that in this case, "small" balls are not necessarily simply connected.

We can assume that every p.w. dif curve of length at most 2d can be contracted to some point without the length increase

Examples

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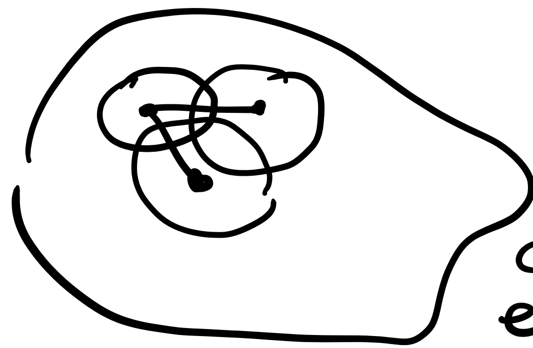
- If $M^n \in \mathcal{M}^d_{-1,v}$, then $\rho(\epsilon) = C(n, v, d)\epsilon$

Consider the space of closed curves of length bounded by $6d$. Will construct a subset $\tilde{J}$ in the space of those curves, s.t. any closed curve of length at most $2d$ can be approximated by a curve in $\tilde{J}$

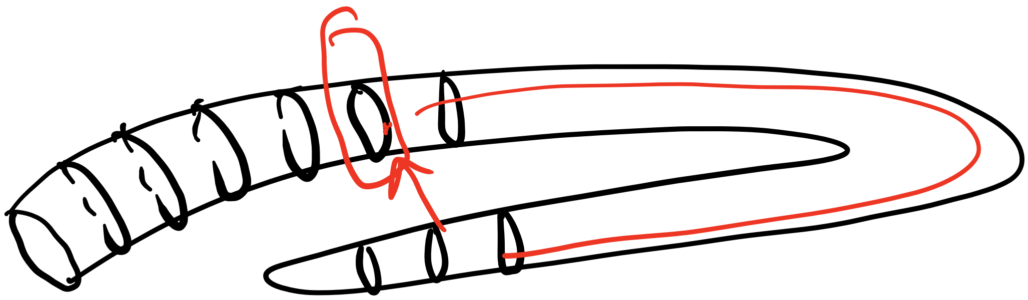
$|\alpha| \le 2d$ I can find γ $|\gamma| \le 6d$

$$d(\alpha, \gamma) \le \frac{r_0}{100}$$

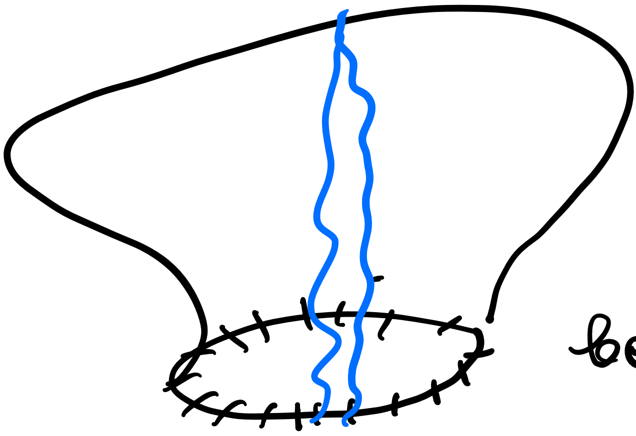


M

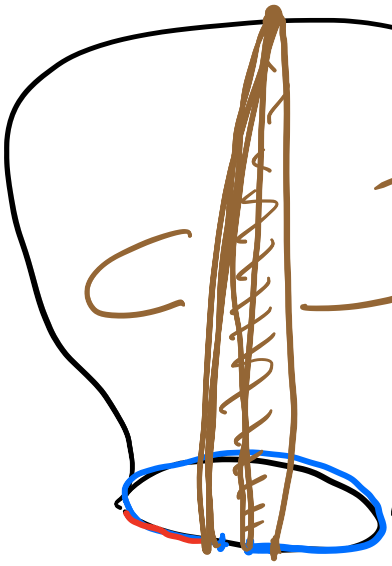
N can be estimated



obtain a new homotopy with bounded width



∃ homotopy #3 in which both length & width can be controlled.



$\leq$ length of original curve $+ 2W$

Homotopies over short loops

Let γ be a curve of length at most $2d$ that can be contracted to a point without the length increase. Then there exists a homotopy of γ to a point over "short" loops based at $p = \gamma(t*)$.

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- Show that for any such pair of $\gamma, \tilde{\gamma}$ there exists a homotopy H, such that its width can be bounded in terms of $w(n, v, d)$.
- Let F_t be a homotopy contracting γ to a point in M^n over curves of length at most $2d$. Show that there exists a homotopy $\tilde{F}_t$, such that its width can be bounded above in terms of $w(n, v, d)$ and $\tilde{N}$.

Width implies length

Bounded Ricci Curvature

Jeff Cheeger and Aaron Naber, "Regularity of Einstein manifolds and the codimension 4 conjecture", Ann. Math. (2), 182 (3) (2015), 1093-1165

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Theorem

The collection of 4-manifolds (M^4, g) *with* $|Ric_{M^4}| \leq 3$, $Vol(M^4) > v > 0$, *and* $diam(M^4) \leq D$ *contains at most a finite number of diffeomorphism classes.*

Nan Wu, Zhifei Zhu, "Length of a shortest closed geodesic in manifolds of dimension 4", to appear in JDG.

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Theorem

Let M be a closed 4-dimensional simply-connected Riemannian manifold with Ricci curvature $|Ric| \leq 3$, volume $vol(M) > v > 0$ and diameter $diam(M) \leq D$. Then the length of a shortest closed geodesic on M is bounded by a function $F(v, D)$.

Positive Ricci Curvature

$Ric \geq (n-1)$

Bonnet-Myers Theorem

Positive Ricci Curvature

Bonnet-Myers Theorem

Theorem

Let M^n be a complete Riemannian manifold. Suppose that the Ricci curvature of M satisfies $Ric_p(v) \geq \frac{n-1}{r^2} > 0$ for all $p \in M$ and all $v \in T_pM^n$. Then M^n is compact and the diameter of M is $\leq \pi r$. In particular, the fundamental group $\pi(M^n)$ is finite.

Length of a shortest closed geodesic on manifolds with positive Ricci curvature.

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- Now, let us consider a sphere $f : S^q \longrightarrow \Omega_p M^n$. Let us assume that the energy functional on $\Omega_p M^n$ is Morse. Let us deform this sphere until it is stuck on a critical point. Then the index of this critical point is at most the dimension of the sphere. Thus, its length is at most πq. This gives us an upper bound for the length of the shortest periodic geodesic.

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- . Thus, $l(M^n) \leq (n-1)\pi$

Positive Scalar Curvature

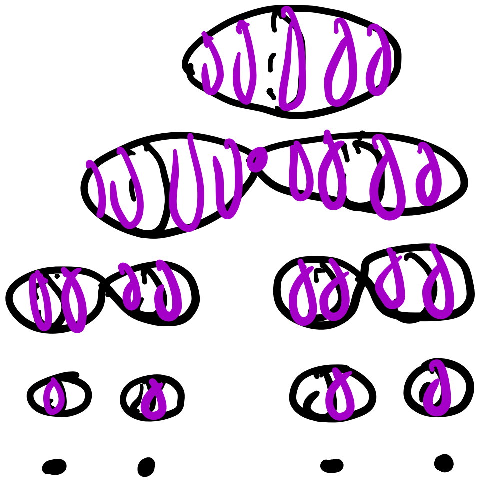
Y. Liokumovich, D. Maximo, "Wast inequality for 3-manifolds with positive scalar curvature"

Theorem

Let (M^3, h) be a compact three-manifold with positive scalar curvature $R_h \geq \Lambda_0 > 0$. Then there is a Morse function $f : M^3 R$ such that for all $x \in R$ and each connected component Σ of $f^{-1}(x)$ we have
(a) Area $(\Sigma) \leq \frac{192\pi}{\Lambda_0}$
(b) diameter $(\Sigma) \leq \frac{40\pi}{\sqrt{\Lambda_0}}$
(c) genus $(\Sigma) \leq 13$
If M^n is a Riemannian 3-sphere, then genus (Σ) is zero.

Proof

Let us assume that M is a Riemannian 3-sphere

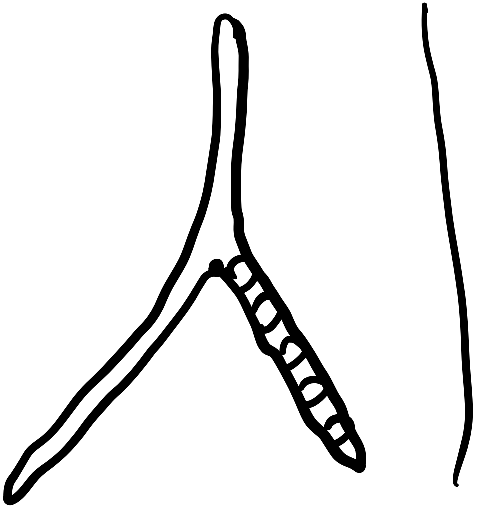


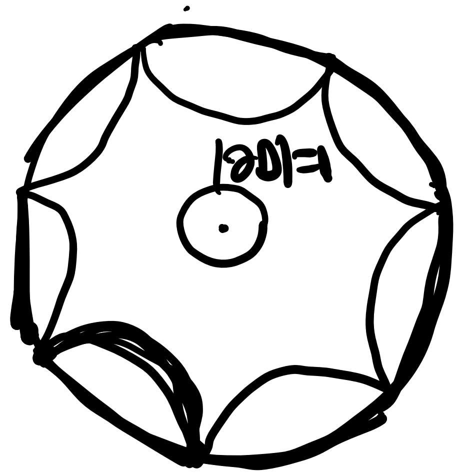
Out of this we would like to obtain a sweepout by short curves

Let us consider a 2-dimensional sphere M of area A. Can we bound "the best" sweepout of M by $c\sqrt{A}$

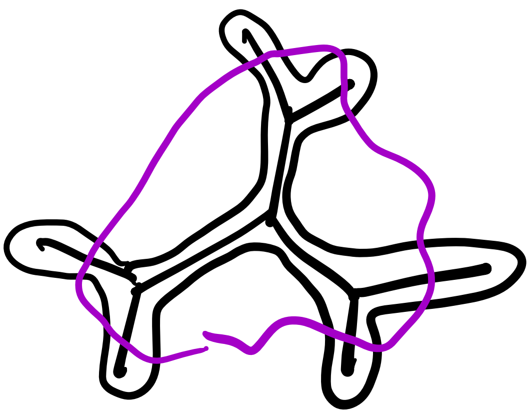
Can one bound the best sweepout of M by cd

S. Frankel, M. Katz





$-c^2$



joint with Y. Liokumovich, A. Nabutovsky

you can control maximal length of curves in the "best sweepout" in terms of $d_1 A$

Also possible to do it continuously wrt to a sphere

Work in progress with Y. Liokumovich, D. Maximo.

$$f : D^2 \to \Lambda M$$
$$\partial D^2 \to M$$