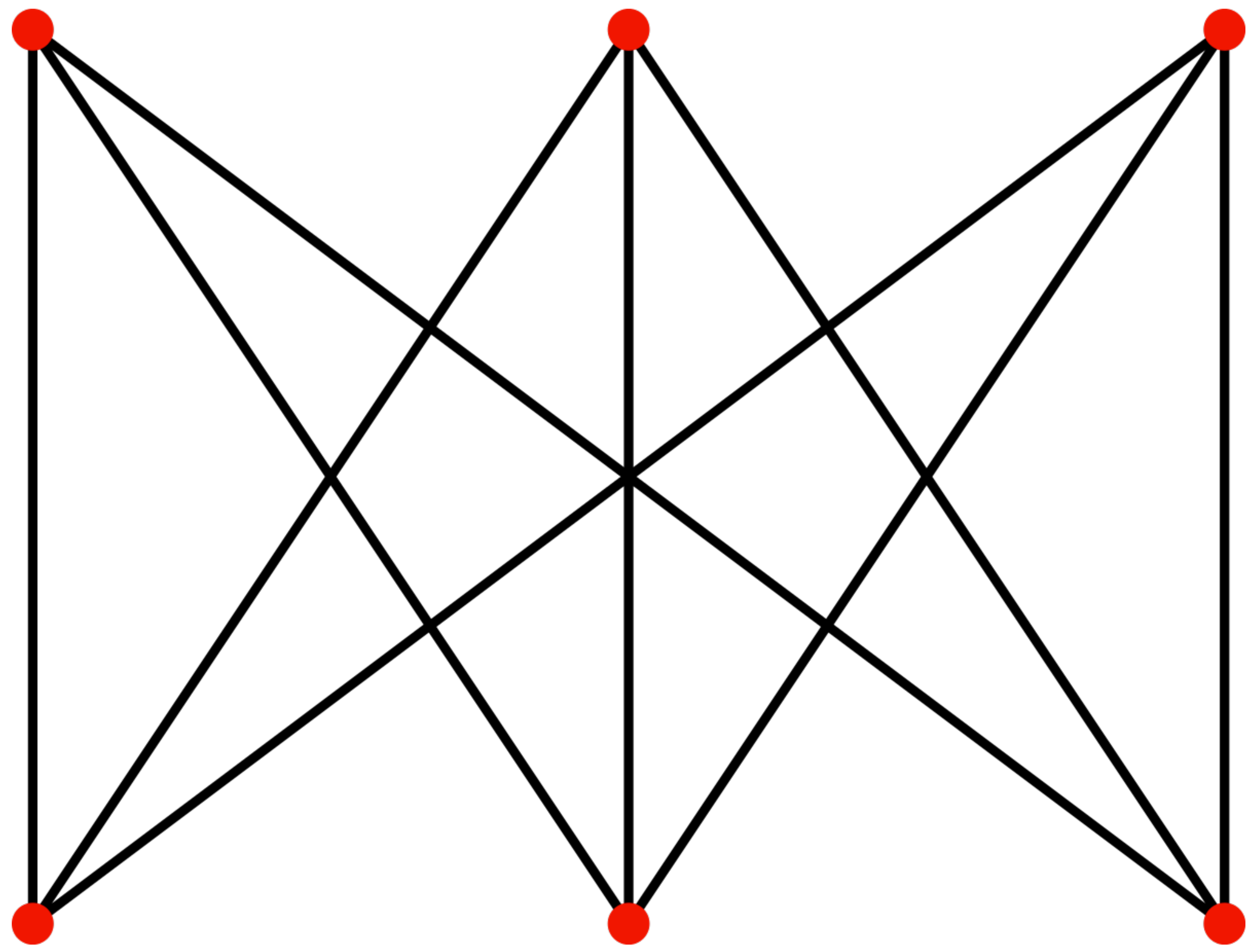
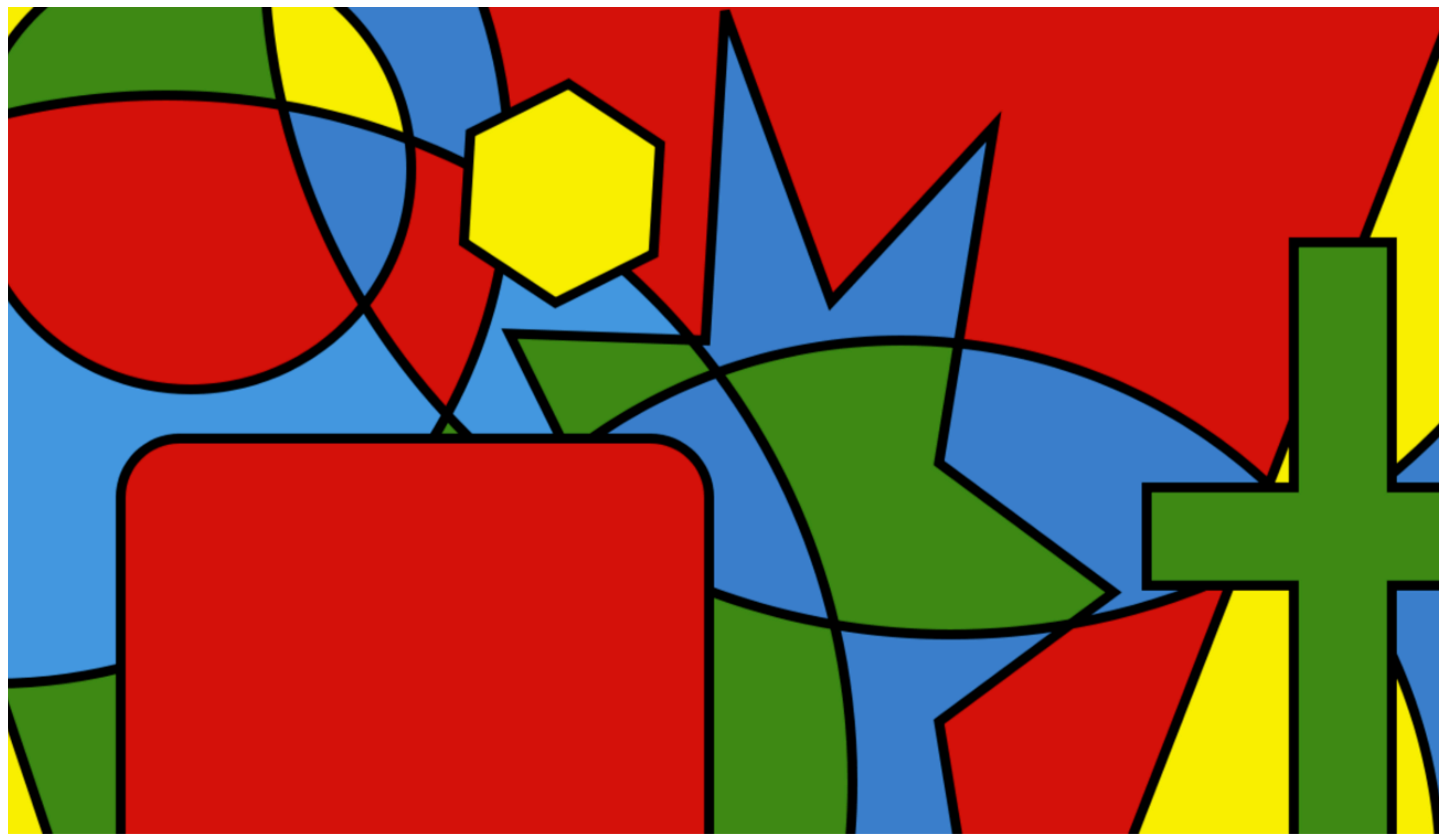
Graph Theory

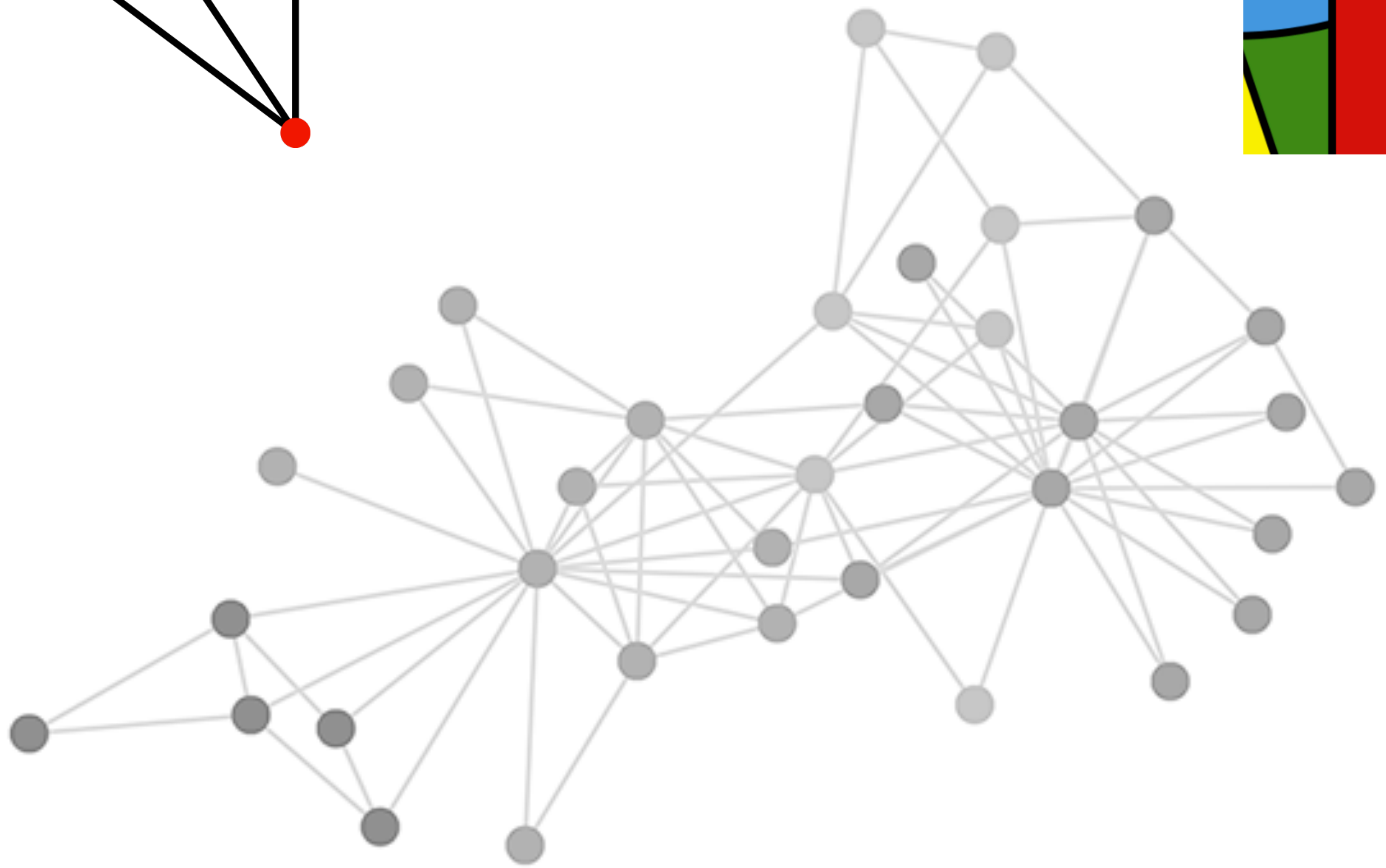
By Jonah

Oxford Online Maths Club

Graph Theory

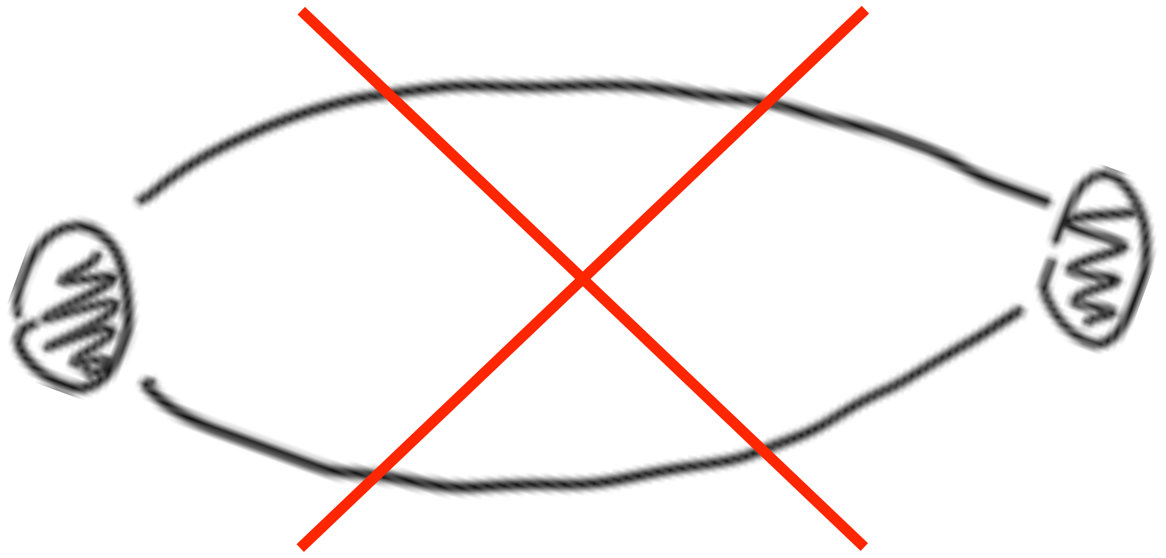


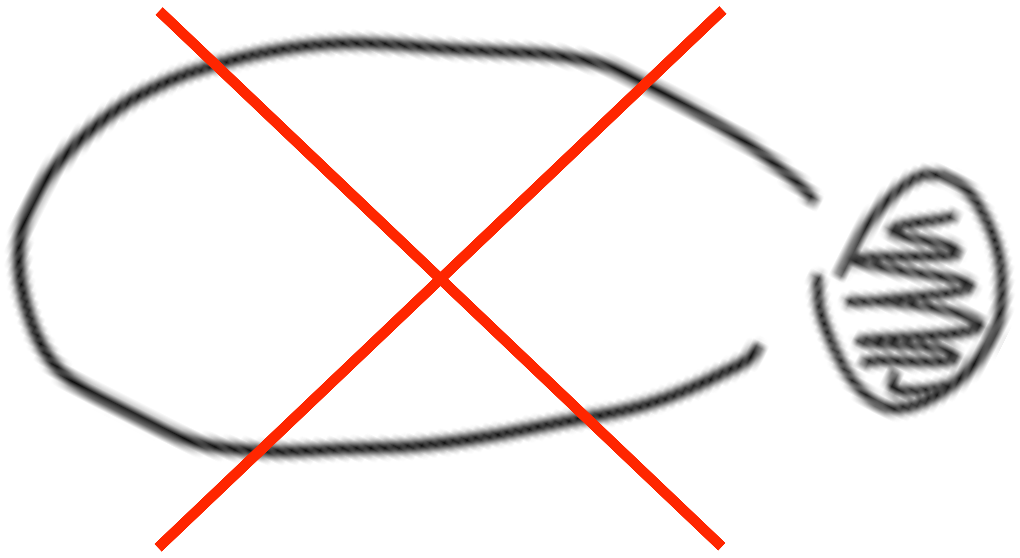




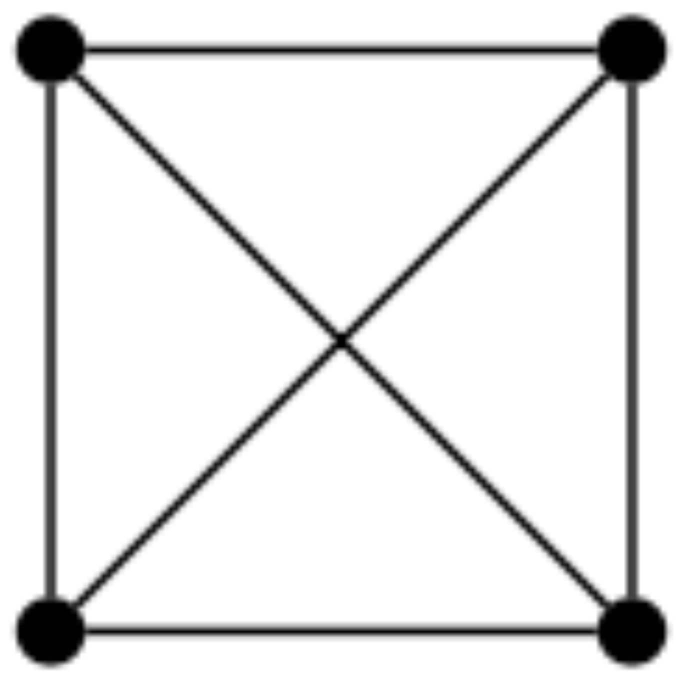
What is a graph?

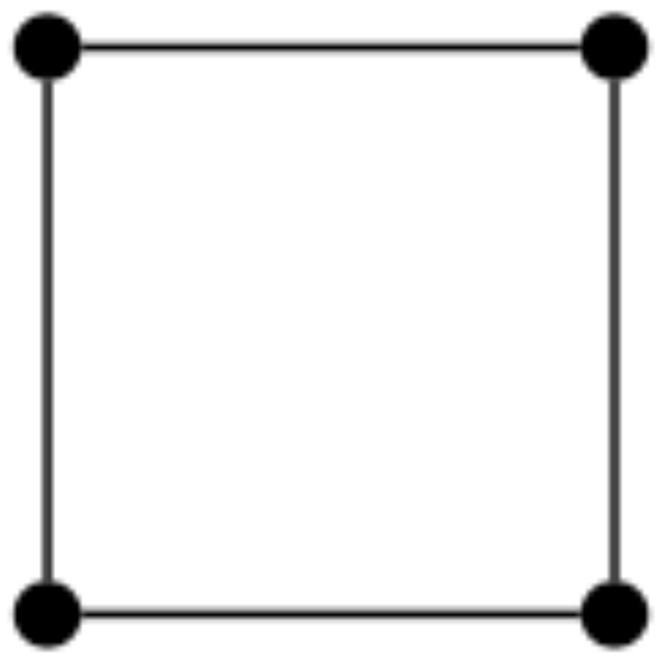
- We start with a couple of points, called vertices or nodes
- Then we connect the vertices by lines, called edges
- There are two rules:
  - Between any two vertices there can be at most one edge
  - No edge can start and end at the same vertex

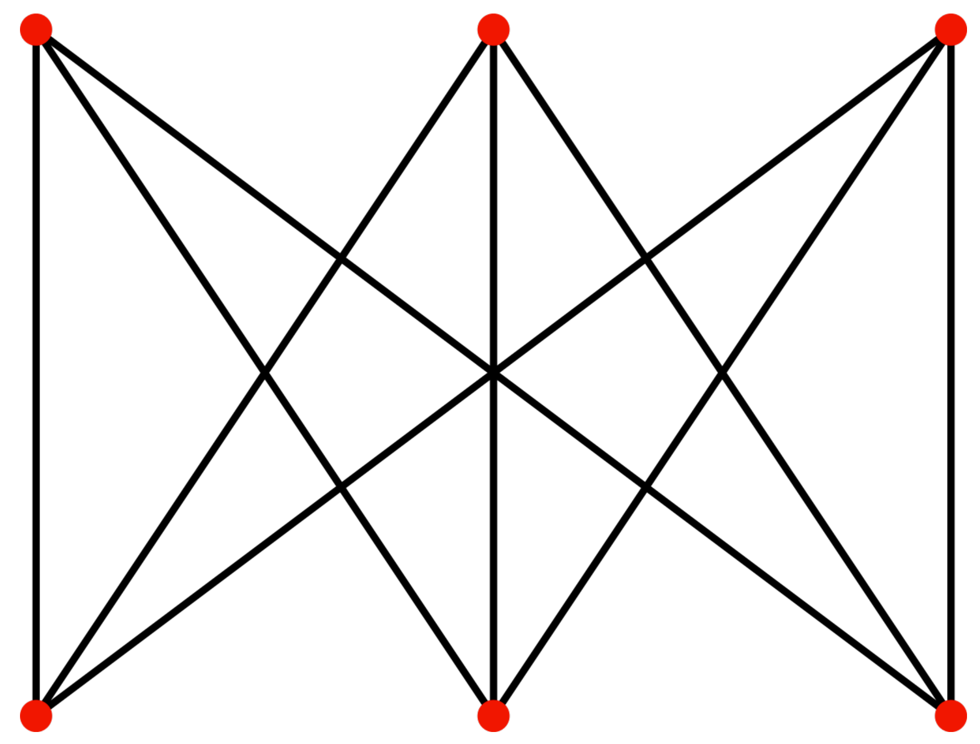


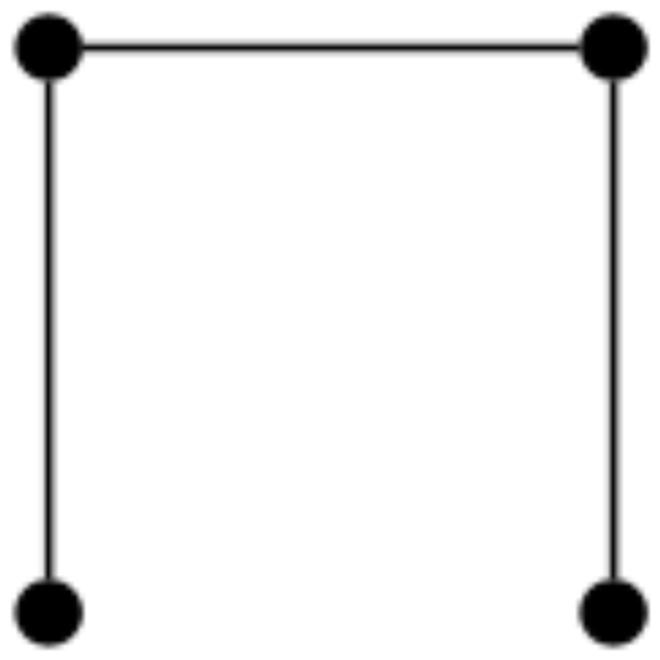


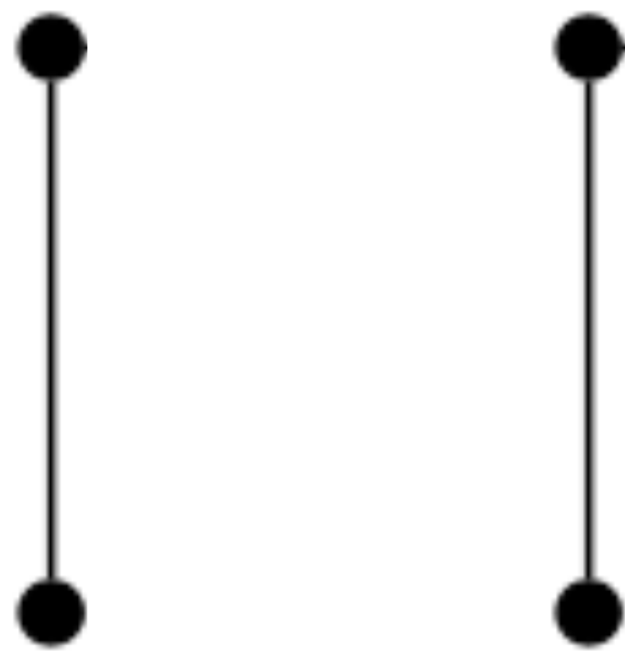
Some Examples of Graphs





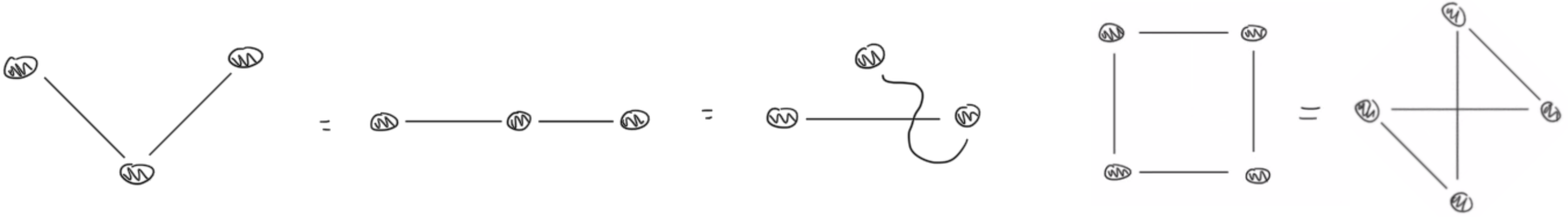






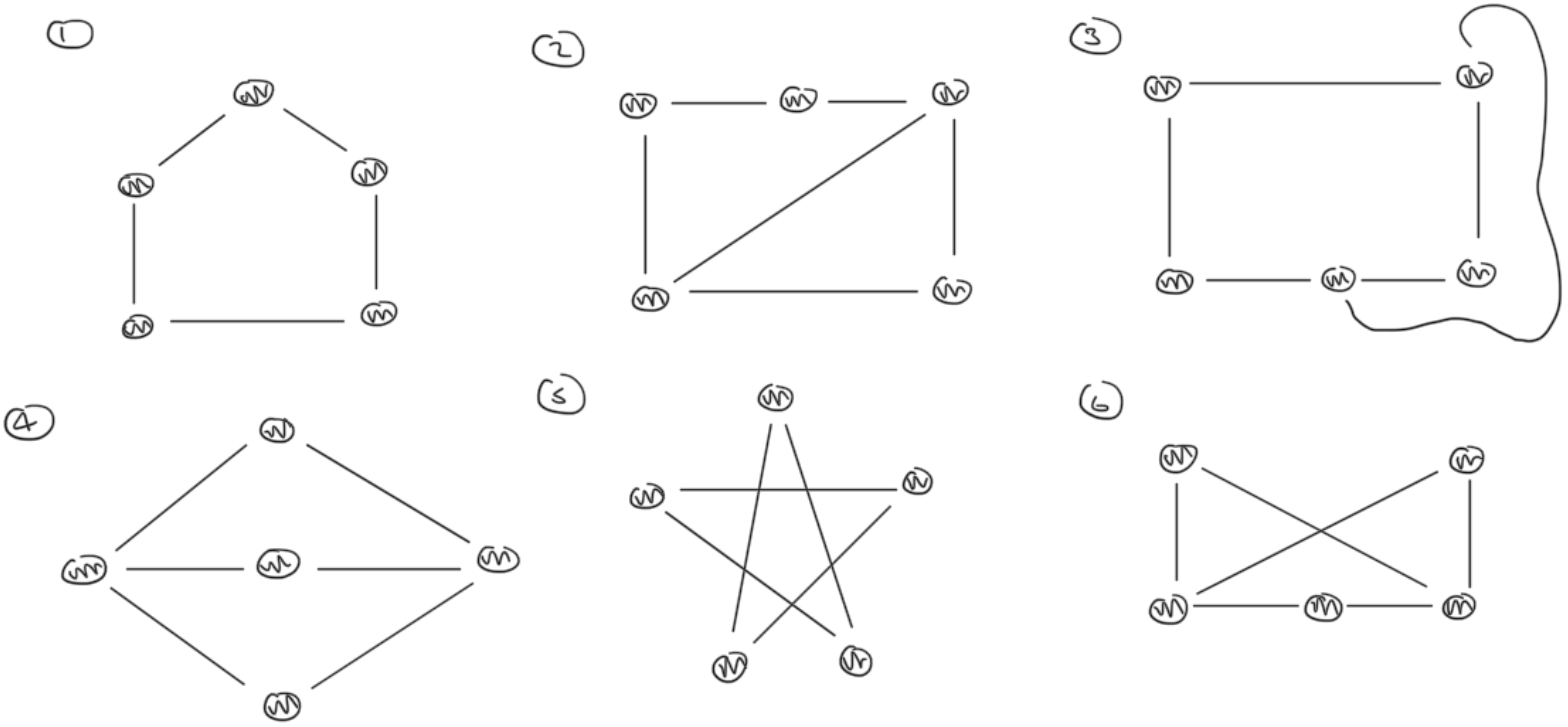
Isomorphic Graphs

- We can draw the same graph in many different ways, but it doesn't change the fact that they're all the same graph
- Moving the vertices around doesn't change the graph as long as the edges stay the same



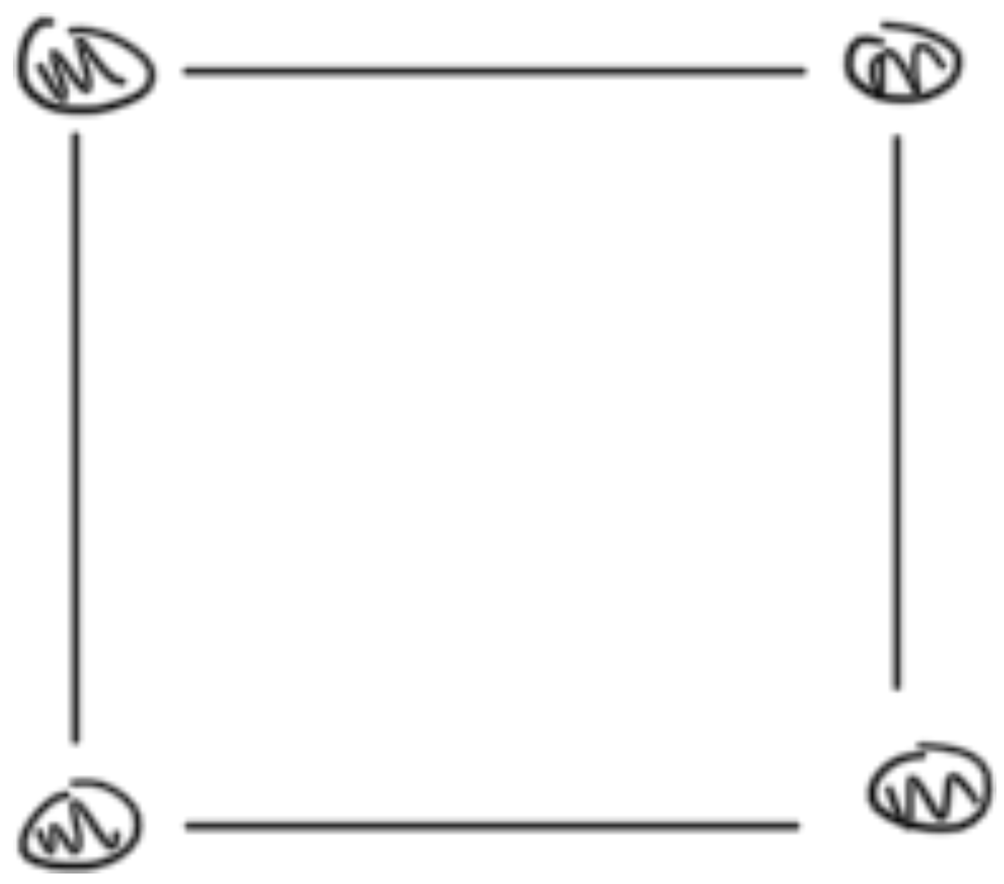
- We call graphs that are drawn differently but are actually the same 'isomorphic'

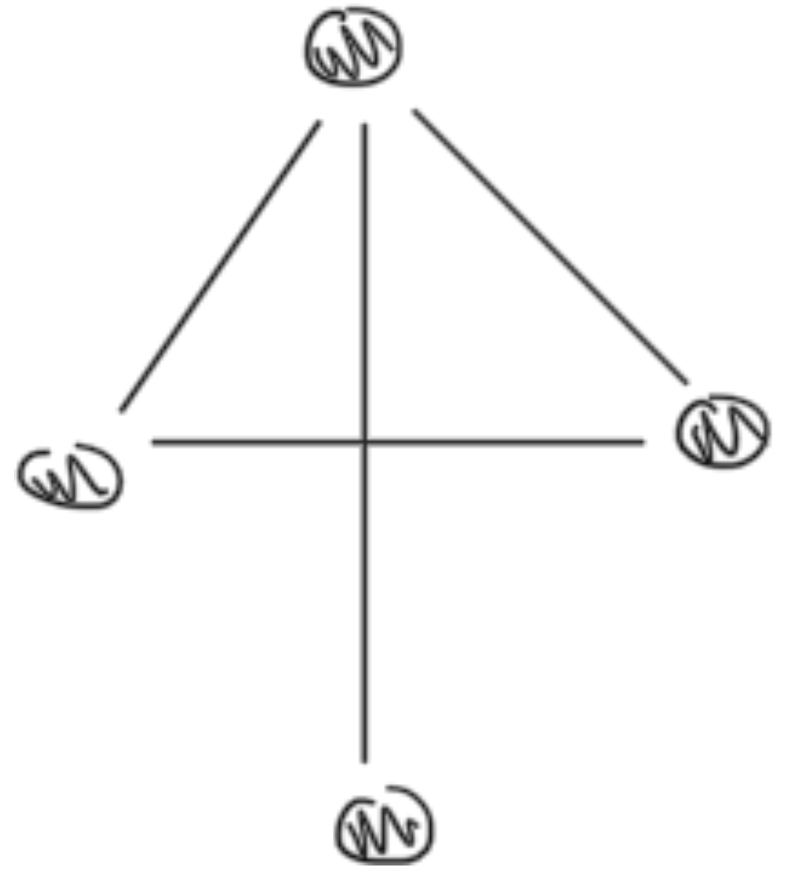
Which of these graphs are isomorphic?

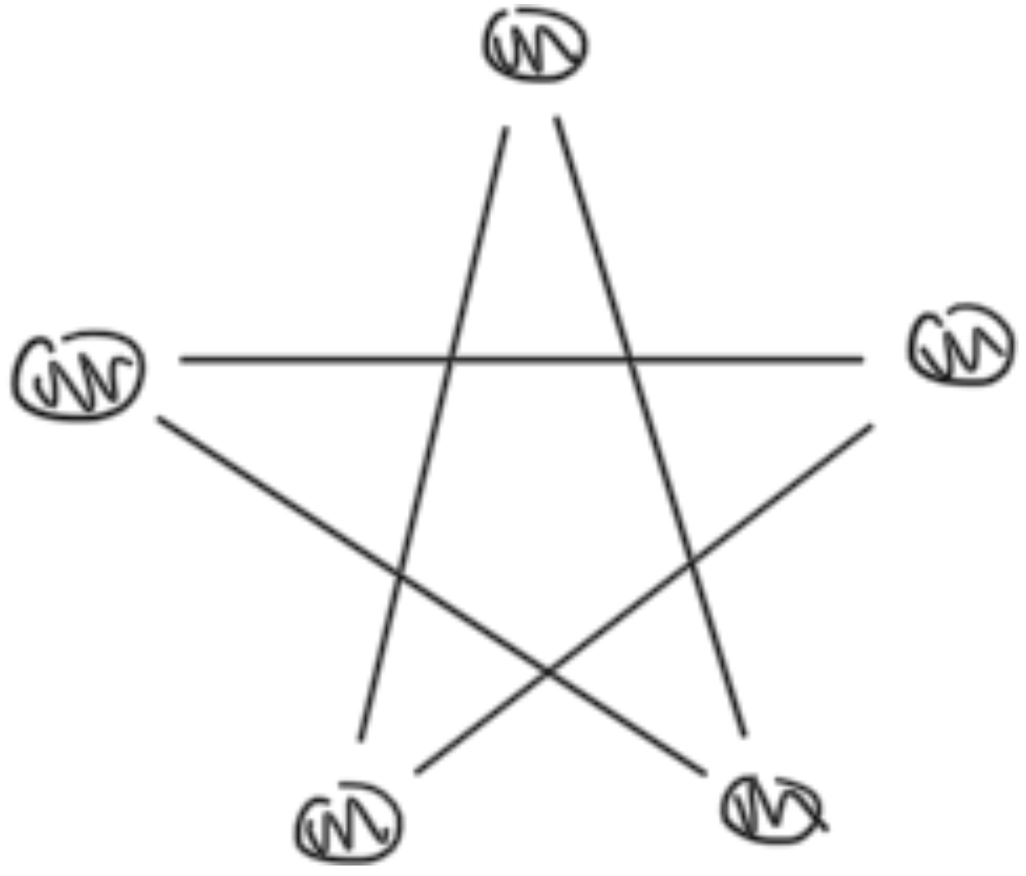


Planar Graphs

- A planar graph is a graph that can be drawn so that no edges over lap

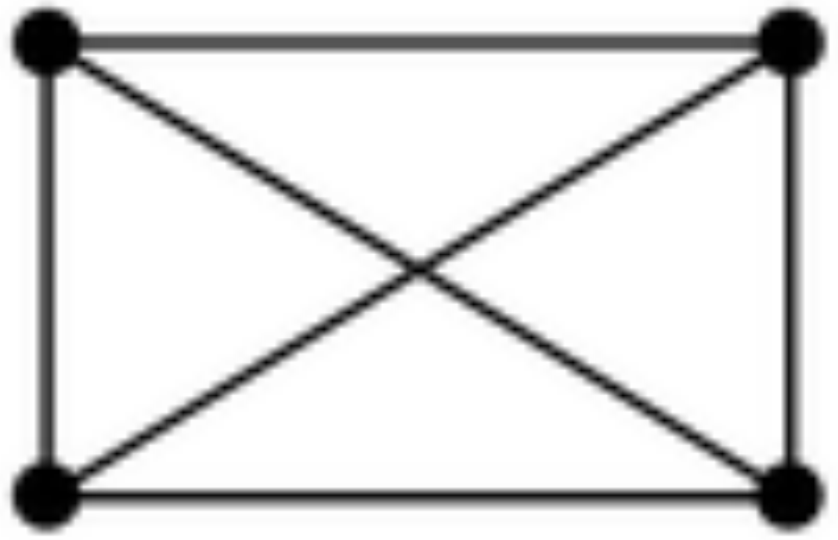


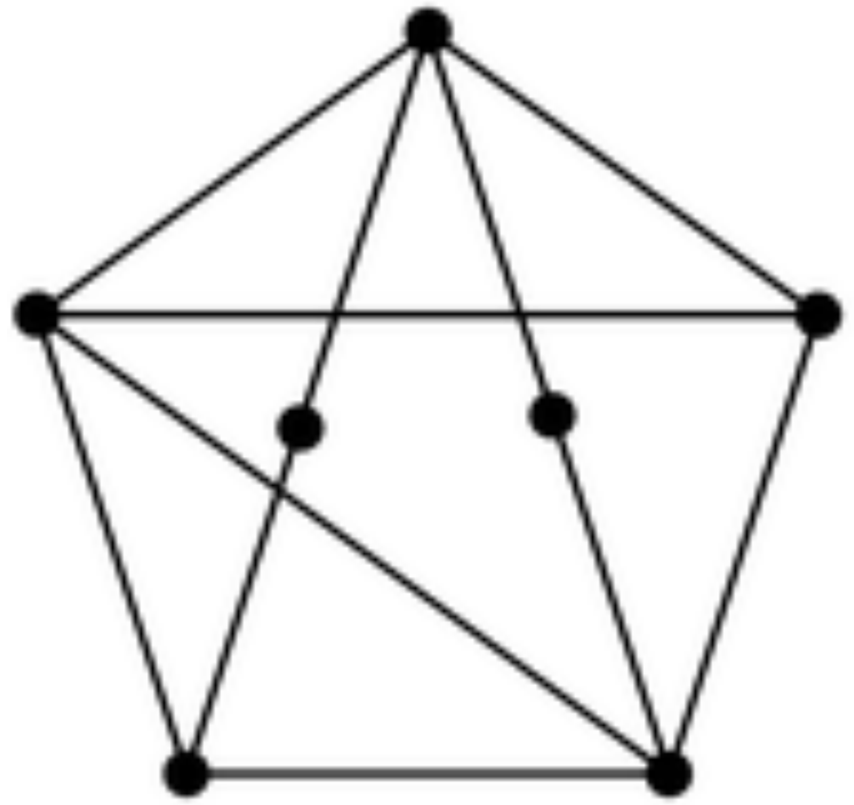


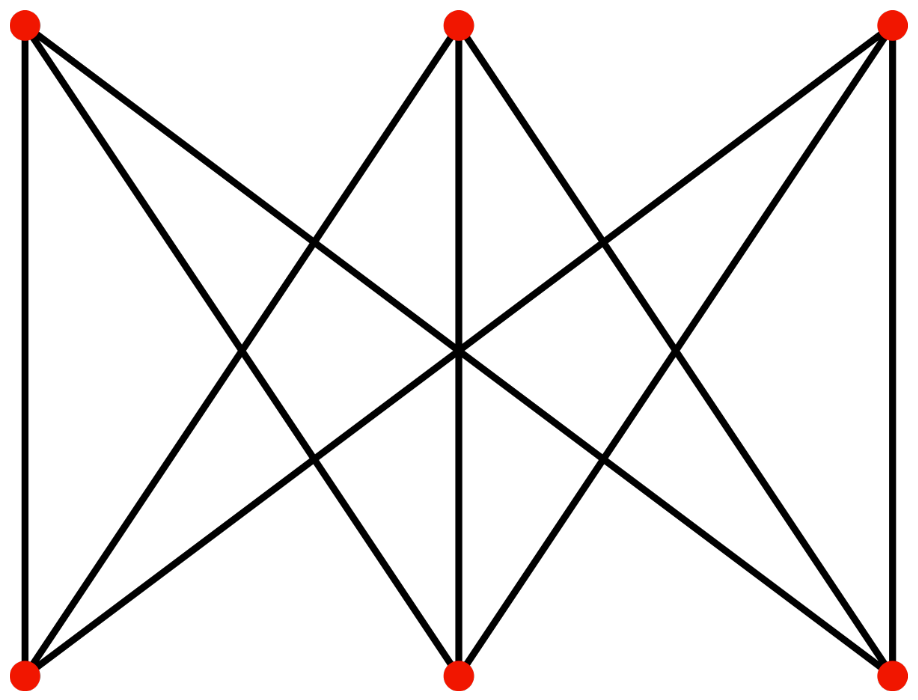


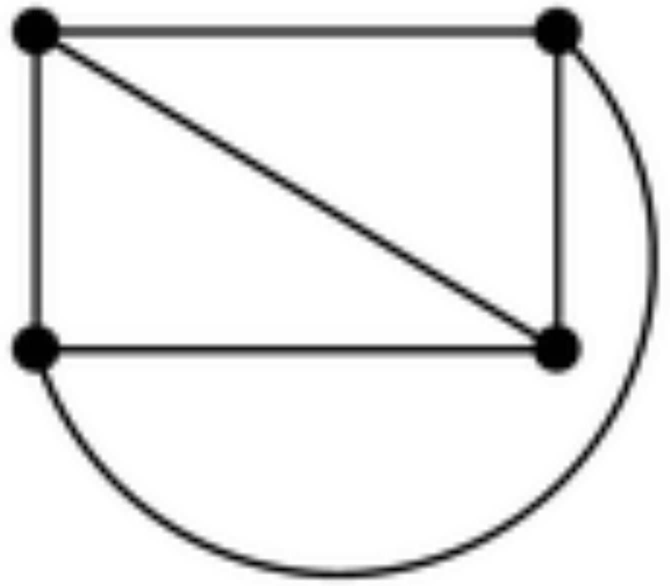
Planar Graphs

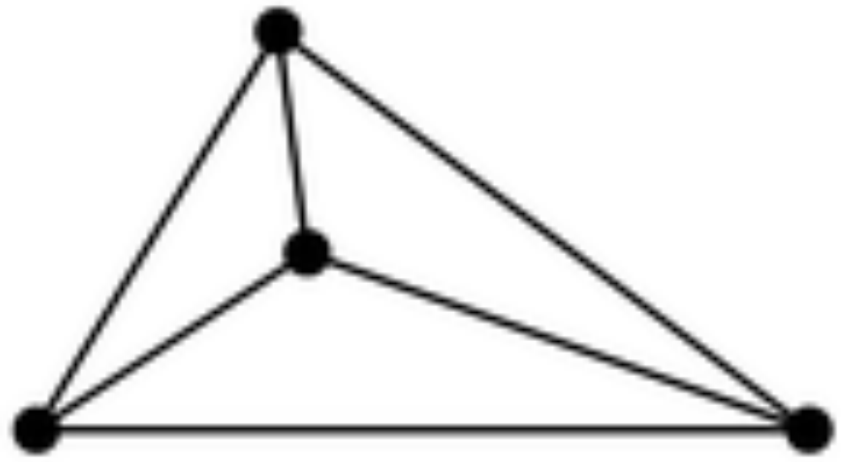
- Which of the following are planar graphs?

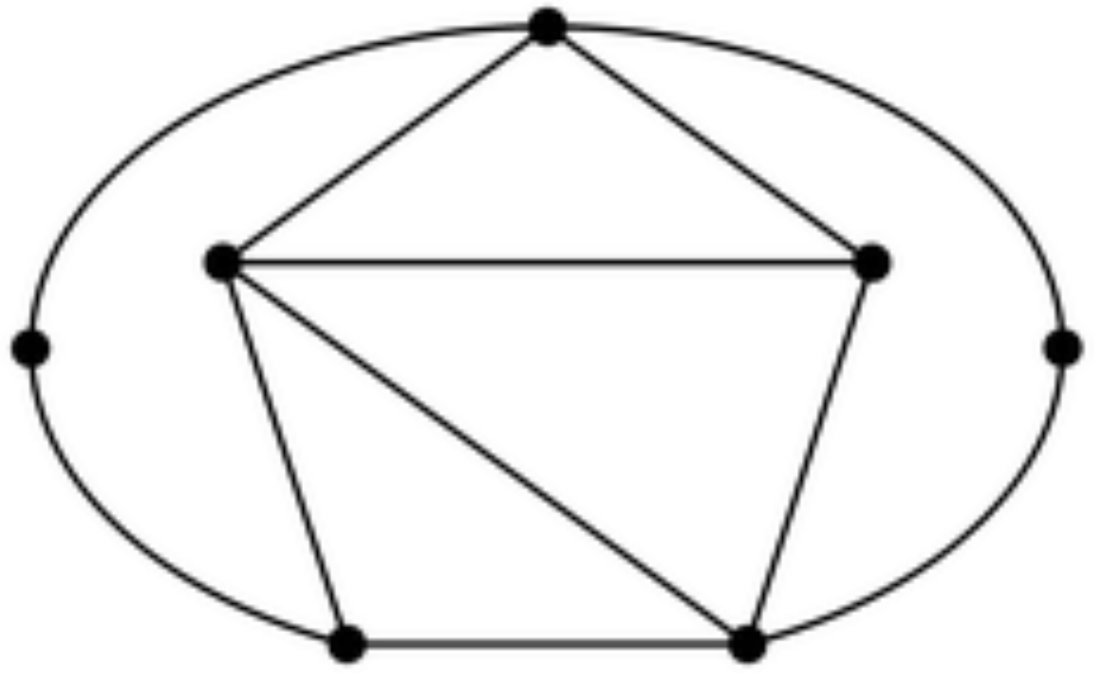






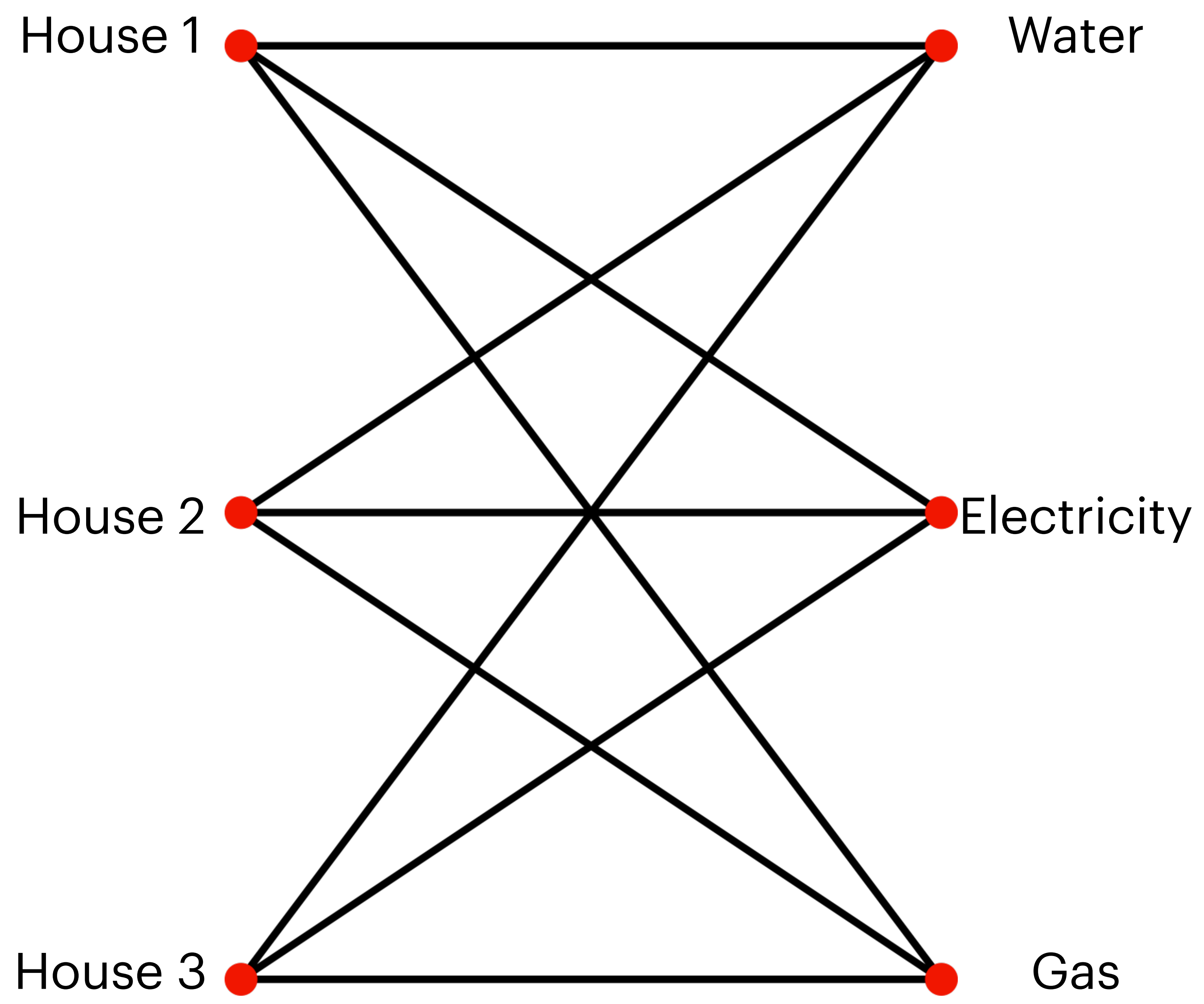






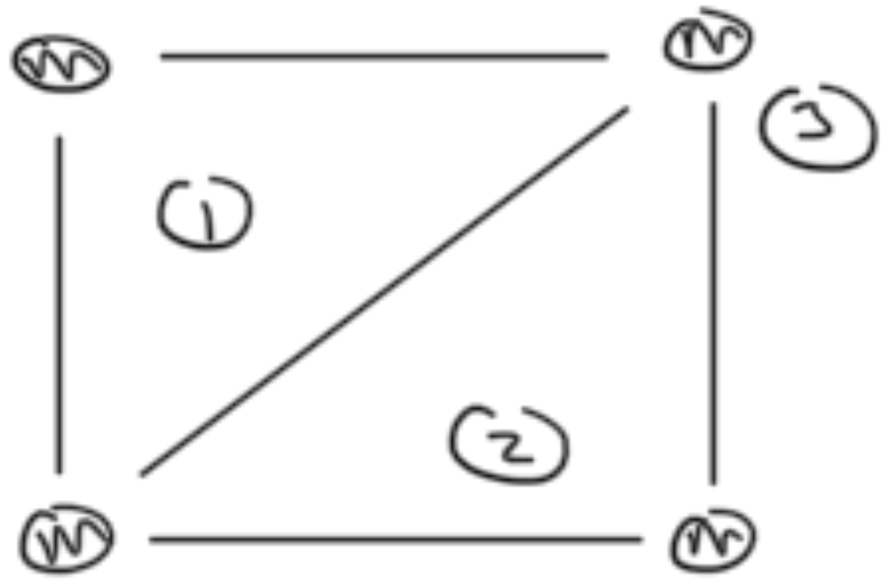
The Three Utilities Problem

- Suppose three houses each need to be connected to the water, gas, and electricity companies, with a separate line from each house to each company
- Is there a way to make all nine connections without any of the lines crossing each other?



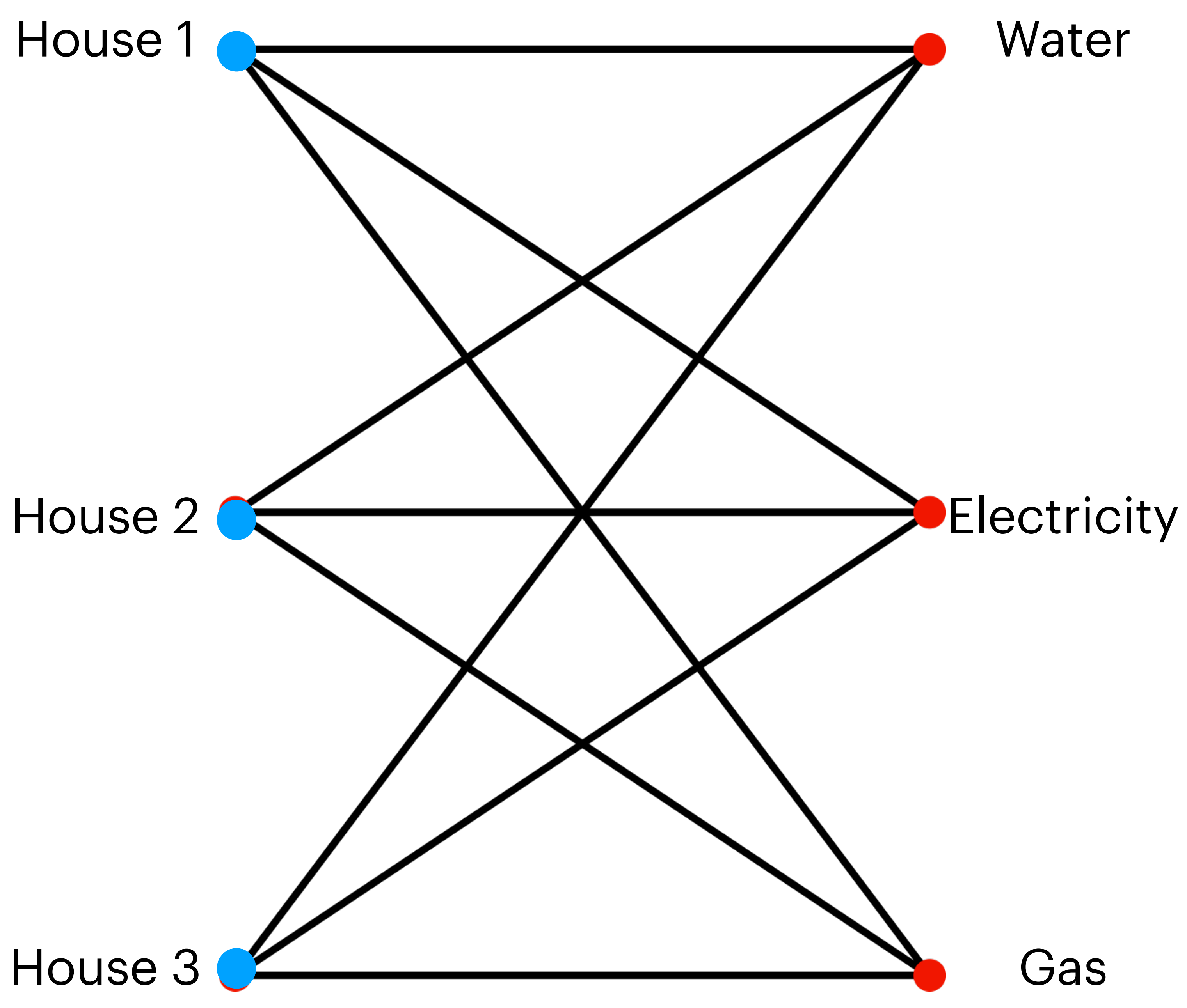
Euler's Formula

- Let G be a connected (i.e. every vertex can be reached from every other vertex by travelling along edges) planar graph with V vertices, E edges and F faces
- The number of faces F is the number of regions the graph divides the plane into

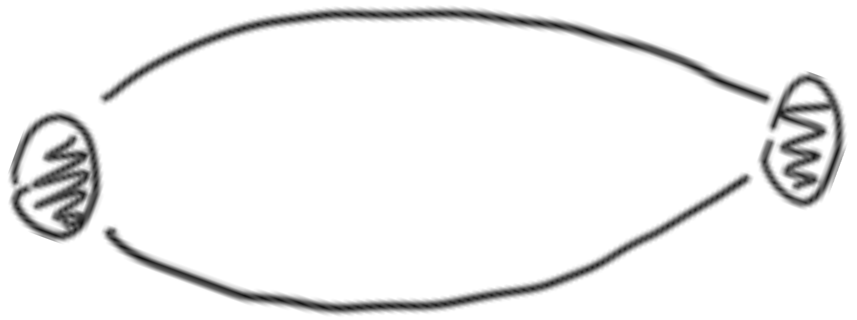


- Then $V - E + F = 2$

The Three Utilities Problem

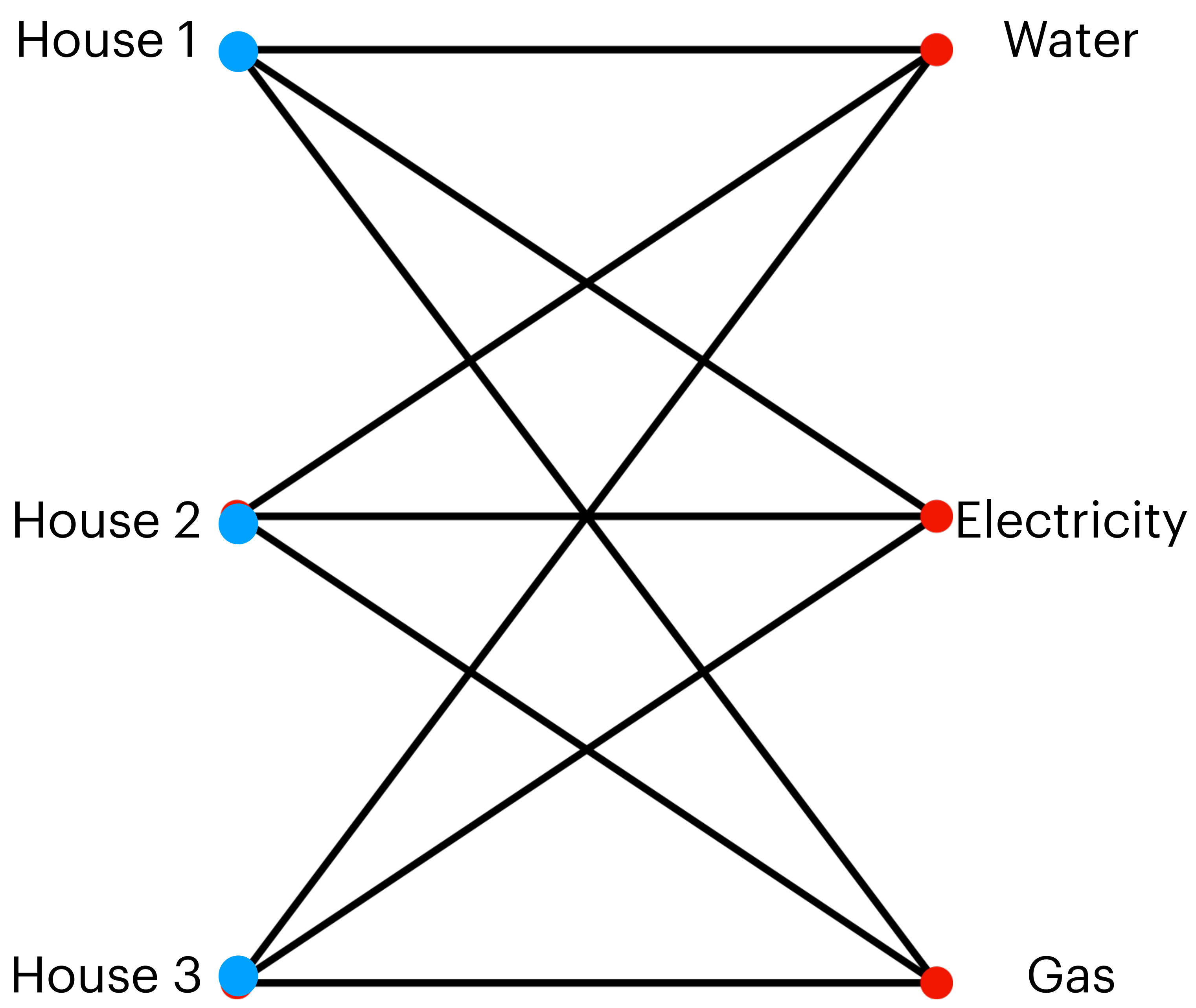


- If this graph is planar, then $V - E + F = 2$
- Clear $V = 6$ and $E = 9$, but what does F equal?
- We know no face can have two edges or we would get a face that looks like this



- Moreover, if we colour in the vertices as above, we can see that each face must have at least 4 vertices, hence at least 4 edges

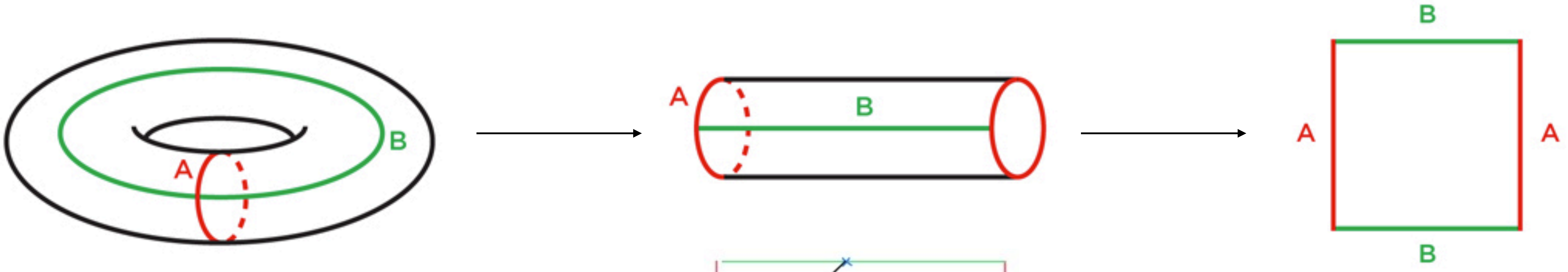
The Three Utilities Problem

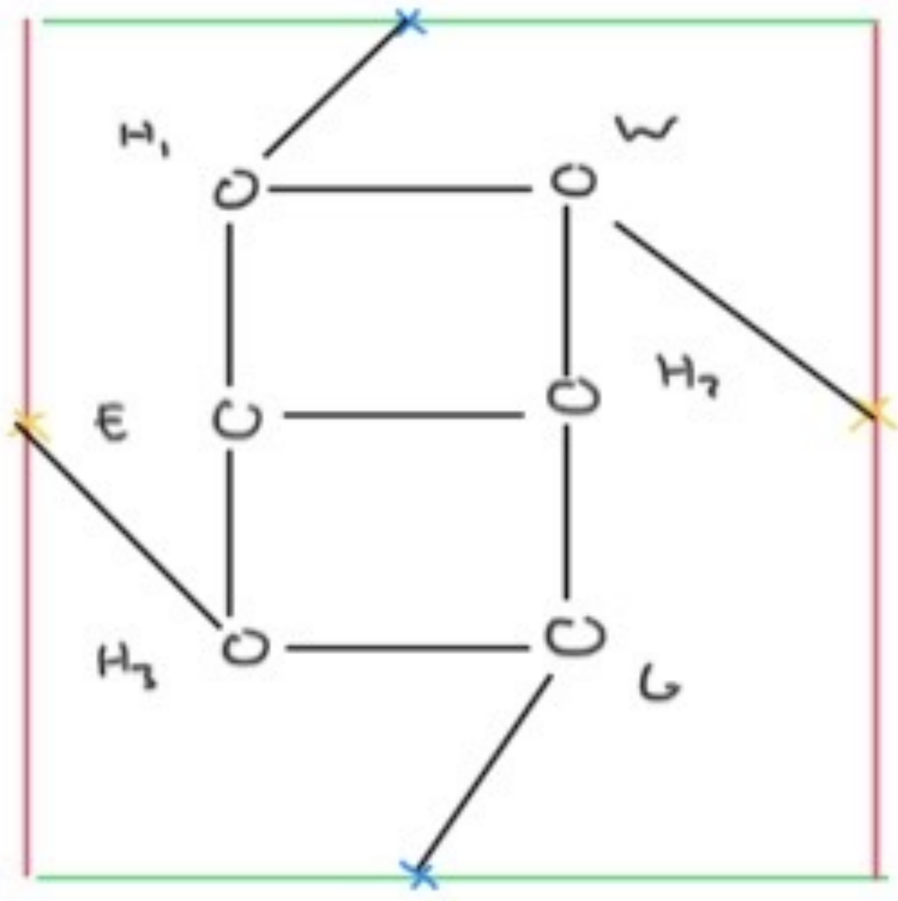


- Each edge belongs to exactly two faces
- Hence $F \leq \dfrac{2E}{4} = \dfrac{E}{2} = 4.5$
- So $V - E + F \leq 6 - 9 + 4.5 = 1.5 < 2$
- Hence by Euler's Formula, the graph is not planar

The Three Utilities Problem on a Torus

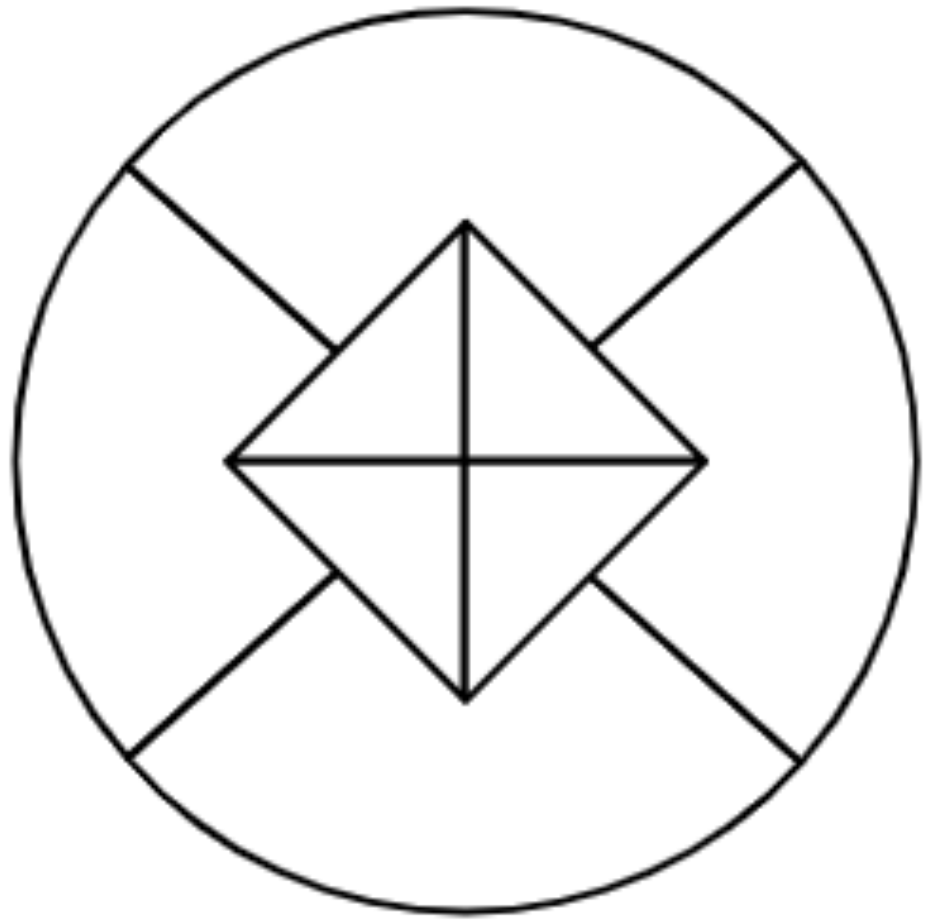
- We know we can’t draw this graph on the plane, but could we draw it on a torus (doughnut shape)?

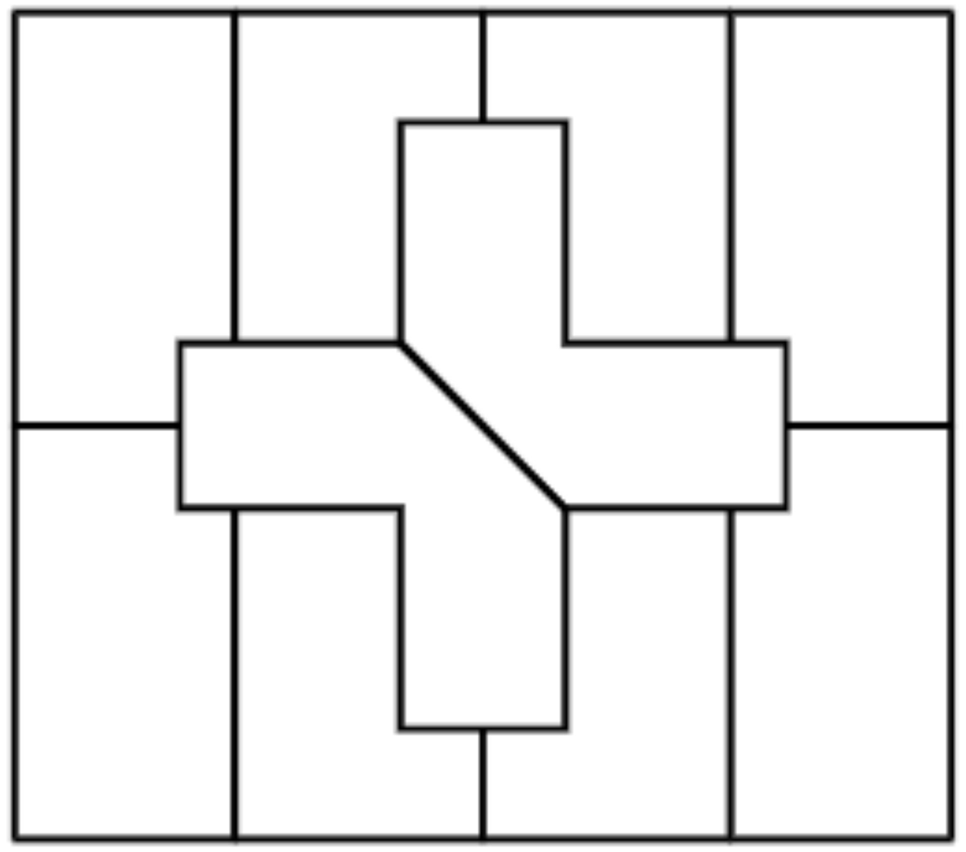


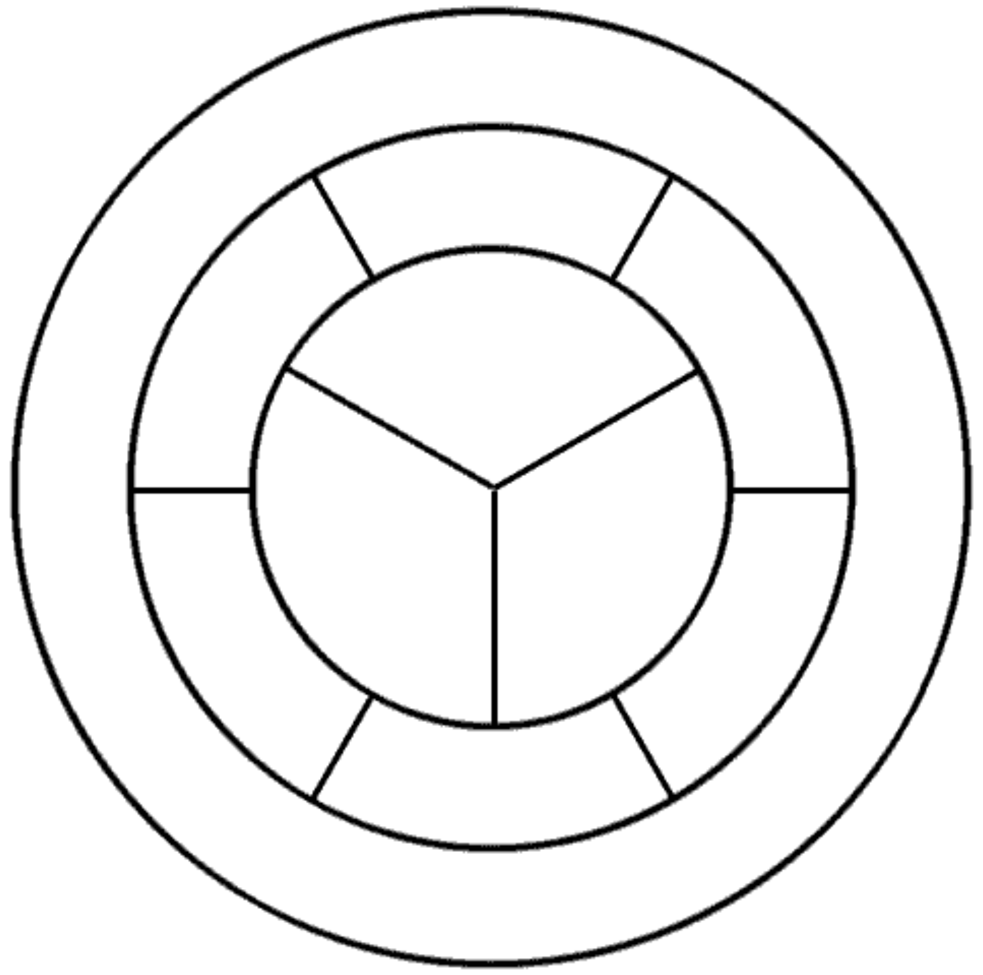


Map Colouring

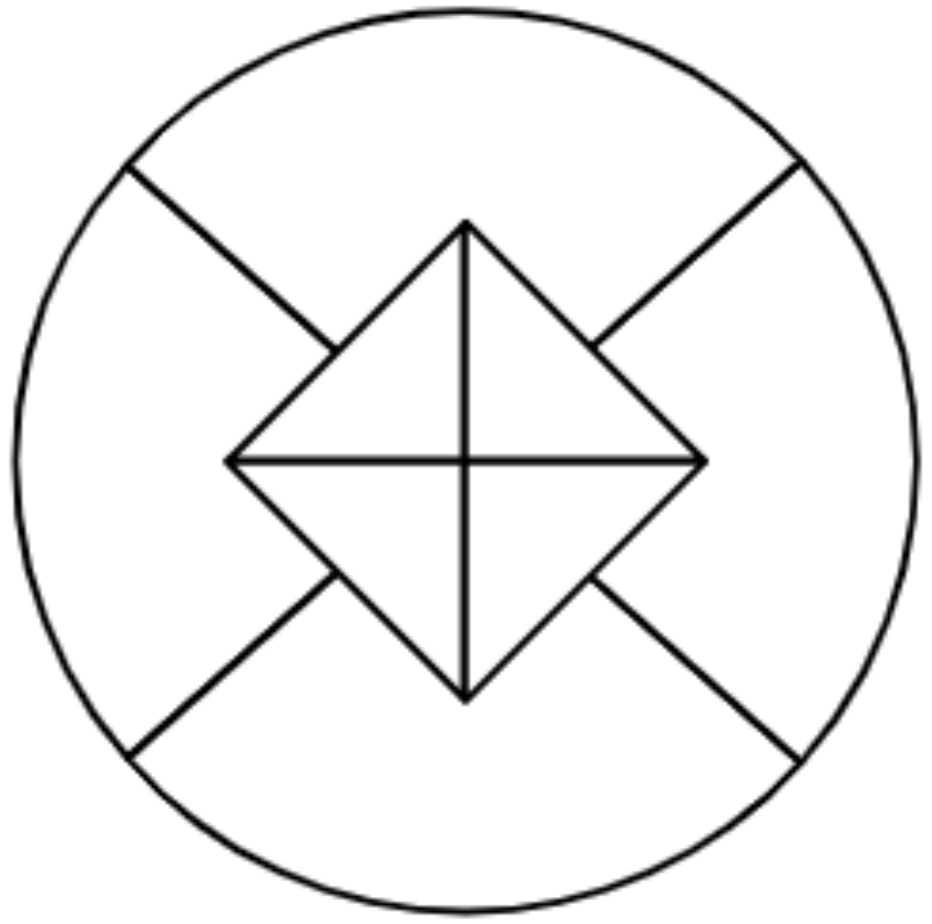
- Suppose we want to colour a map so that no countries which share a border are the same colour
- What is the minimum number of colours we would need?
- What is the minimum number of colours needed to colour the following maps in so that no ‘countries’ bordering each other have the same colour?

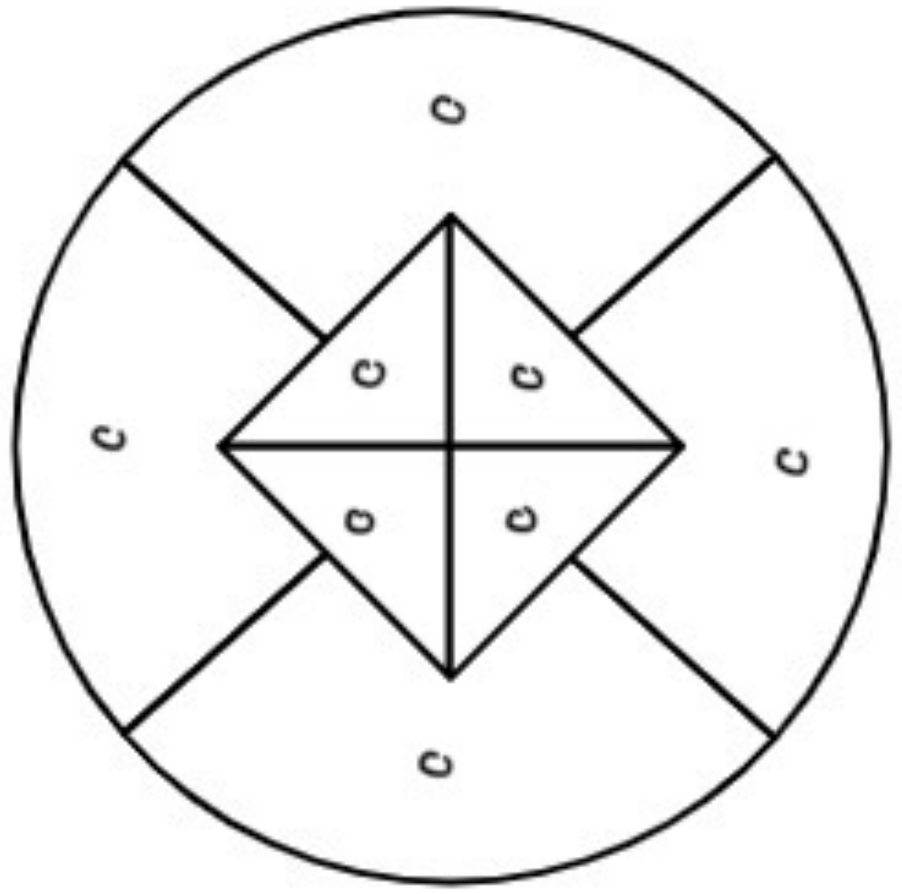


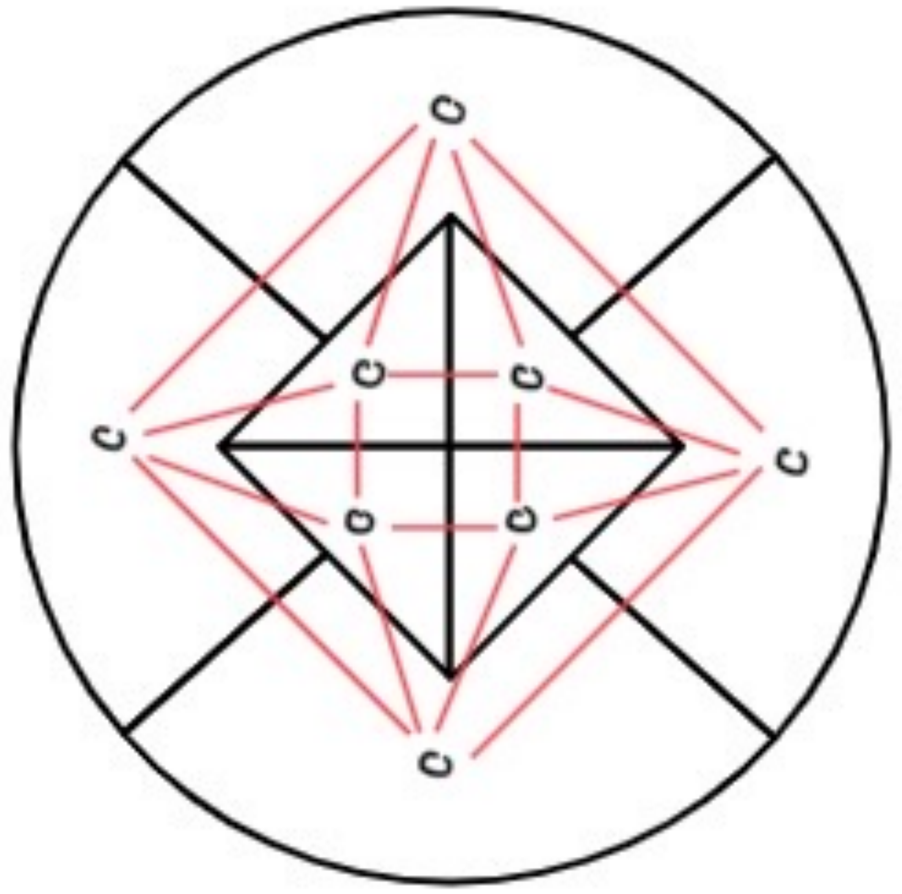




Converting Maps to Graphs

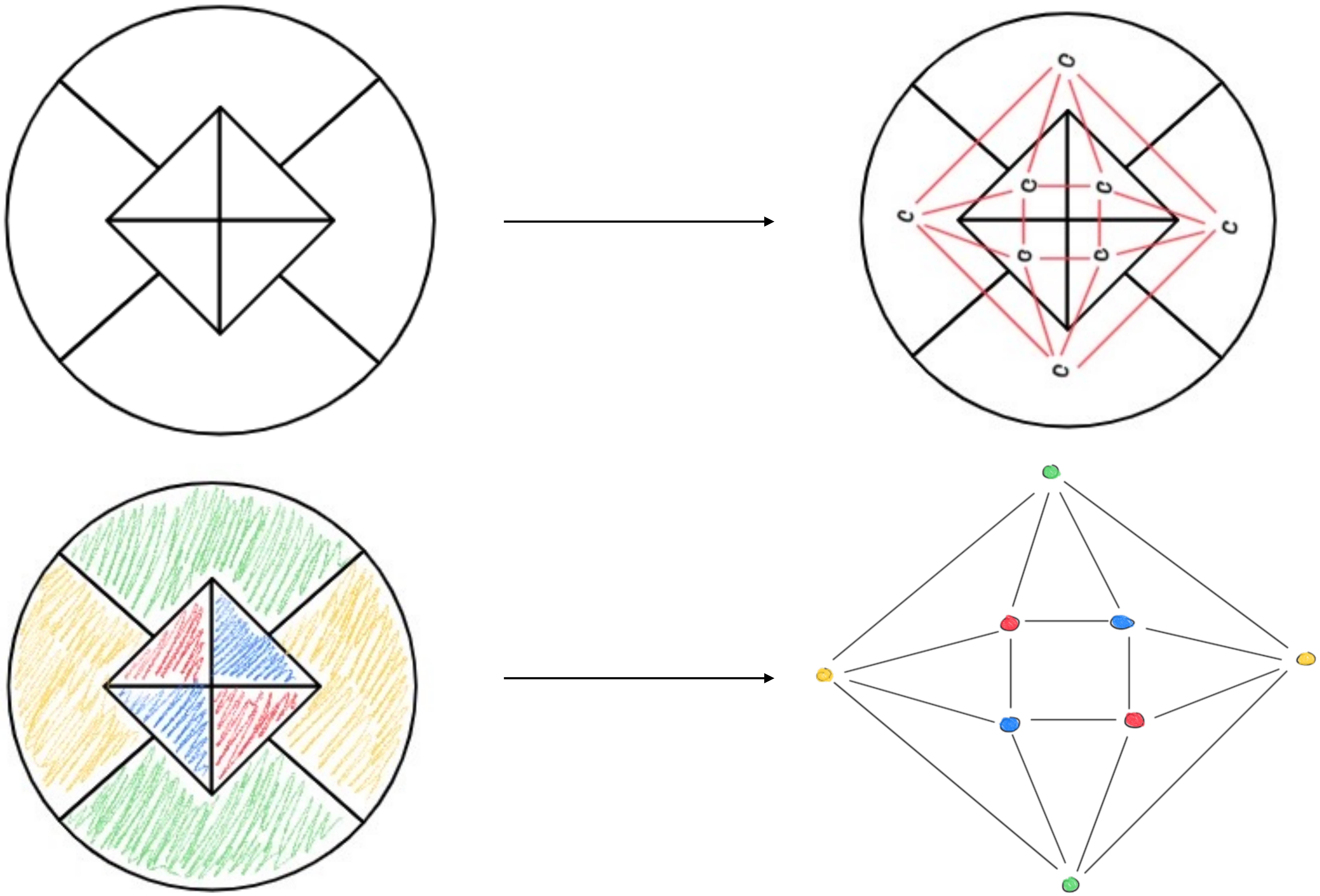






- Suppose we have this map, how we can we phrase our colouring question as a question about graphs?
- For each section of our map we can assign a vertex
- And then we draw an edge between two vertices if their corresponding sections of the map are bordering each other
- Then we assign colours to the vertices
- Our rule ‘two sections of the map can’t be the same colour if they share a border’ becomes ‘two vertices can’t be the same colour if there’s an edge between them’
- Can you see why all graphs of this form are planar?

Converting Maps to Graphs

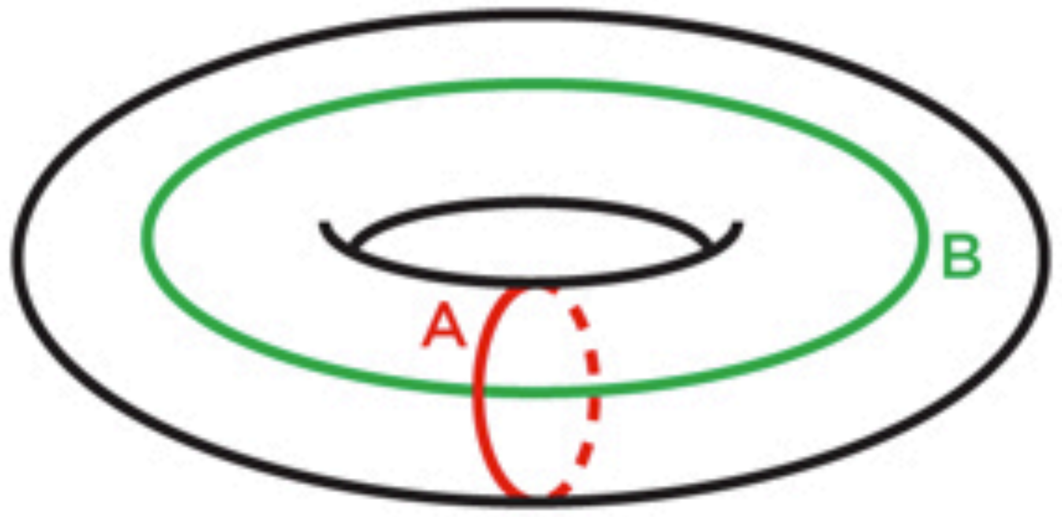


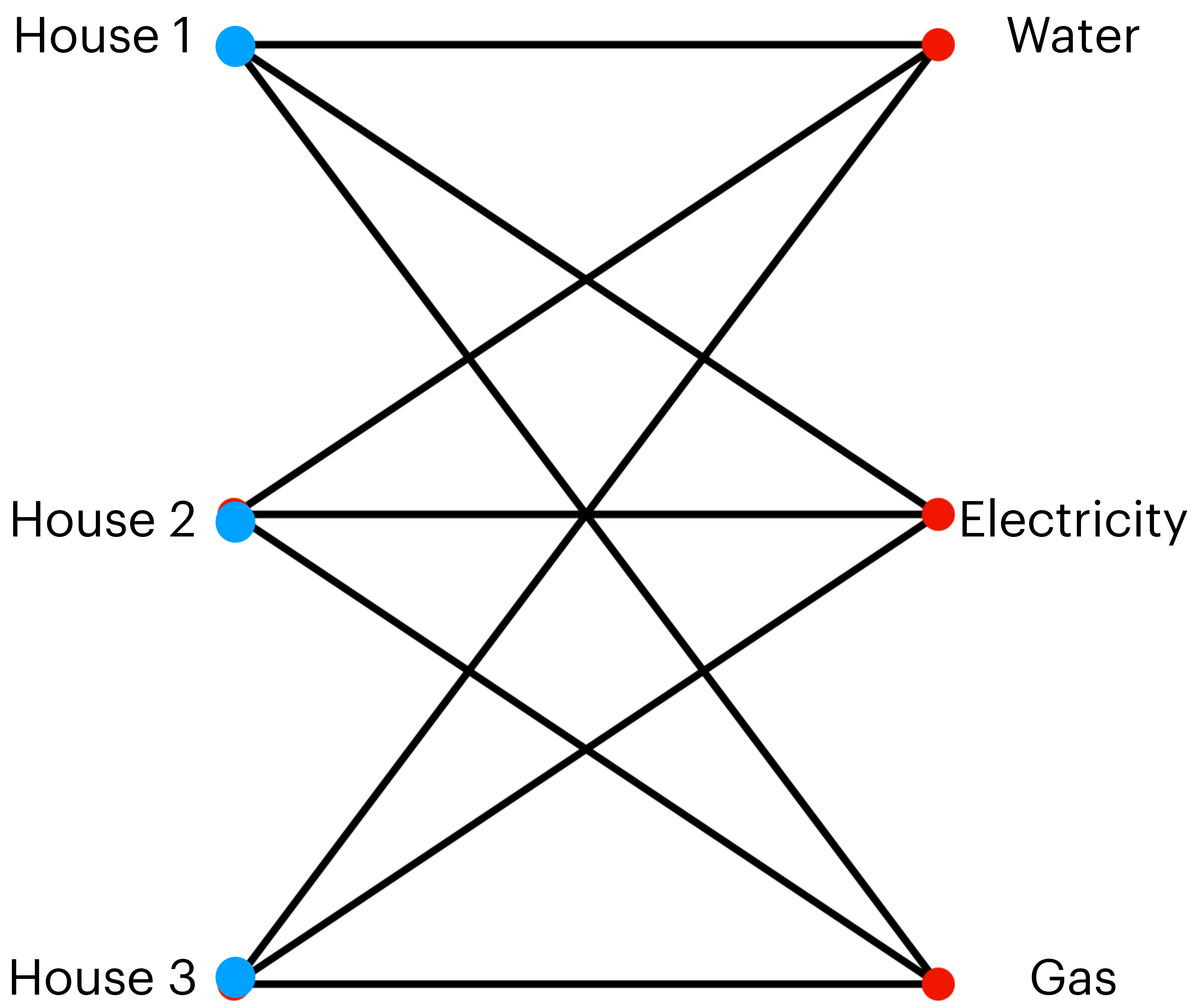
Four Colour Theorem

- The Four Colour Theorem states that any map can be coloured by four colours in such a way that no regions which share a border are the same colour
- Converting to graph theory, the Four Colour Theorem states that any planar graph can be coloured by four colours such that no vertices which share an edge are the same colour
- This theorem has only ever been proven computationally
- It is currently unknown if a mathematical proof exists (though it is debatable as to whether a computational proof counts as a ‘proper’ proof)

Seven Colour Theorem on a Torus

- Suppose now we want to colour the vertices of a planar graph on a torus such that no two vertices joined by an edge are the same colour
- As seen by the utilities problem graph, there are more planar graphs on a torus than on a plane





- Interestingly, we need 7 colours if we wish to colour any planar graph on a torus
- There are further Colour Theorems for doughnuts with more holes

$$\left\lfloor \frac{7+\sqrt{1+48g}}{2} \right\rfloor$$

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