Simple geometrical models for the distribution of domain sizes in martensitic microstructures

Genis Torrents, Xavier Illa, Eduard Vives, Antoni Planes

Departament de Matèria Condensada, Facultat de Física,
Universitat de Barcelona
Martí i Franquès 1, 08028 Barcelona, Catalonia

Research group

Experimental and theoretical studies on martensitic transitions

-Calorimetry and Acoustic emission under external fields/forces
-Applications to caloric effects (elastocaloric, magnetocaloric, barocaloric, etc..)

Permanent: Antoni Planes, Lluís Mañosa, Teresa Castán, E.V.

Present PhD students: Victor Navas, José Reina

Former students and post-docs: G.Torrents, J.Baró, E.Stern, X.Illa…

International collaborations: R.Niemann, D.E.Soto Parra, R.Romero, F.J.Pérez-Reche,…..

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Outline

- Introduction
  - Thermodynamics
  - Experimental results
  - References to existing models
- Simple models:
  - A) Sequential partitioning — Frontera, Goicoechea, Rafols & Vives, PRE 52, 5671 (1995)
  - B) Ball & Planes model — Torrents, Illa, Vives & Planes, submitted to PRE
    - Simulations
    - Dipolar-like interaction
    - Solution in the continuum limit
    - Solution for the discrete case
- Conclusions

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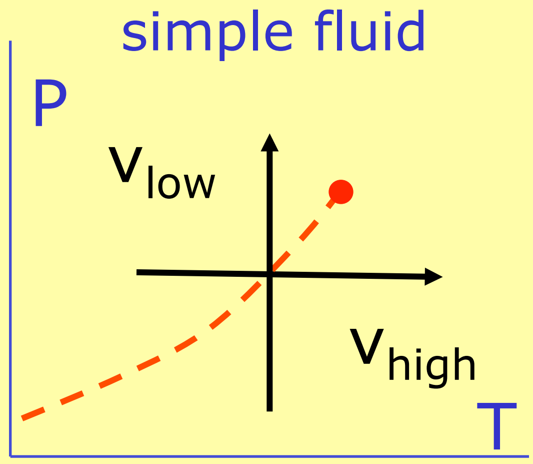
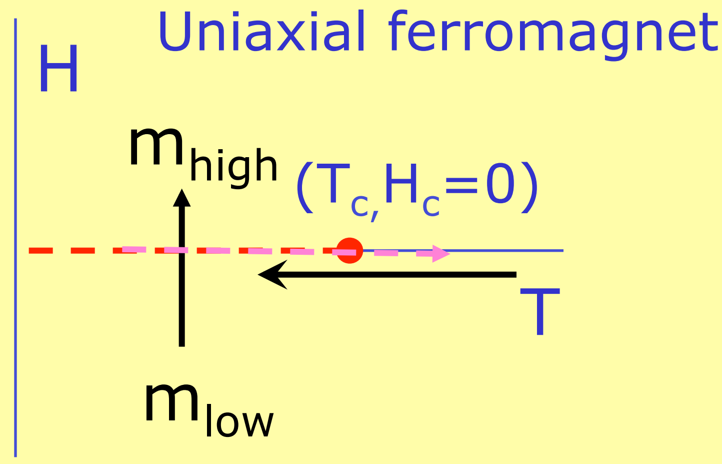
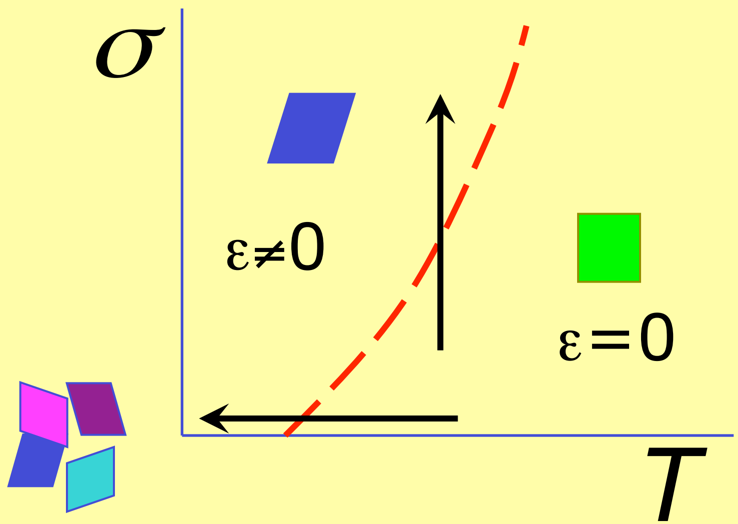
Introduction: first order phase transitions





Martensitic transformation (MT)

Shape memory alloys



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Microstructures: optical microscopy

Transformed sample of Cu-Zn-Al (after cooling, no external stress)

Optical microscope with polarized light

3mm x 2mm

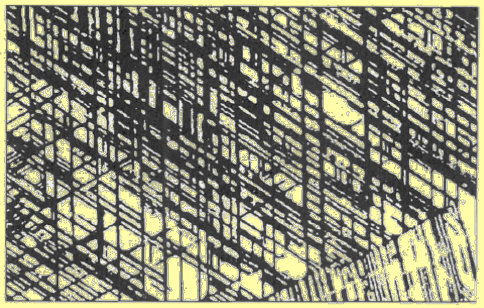
Absence of a characteristic scale of the transformed domains

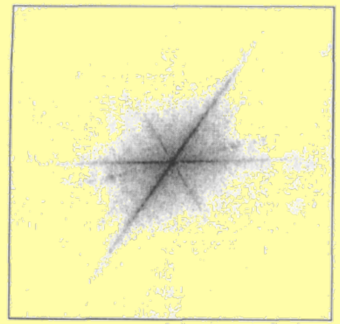
Microstructures: lack of charactersitic scales

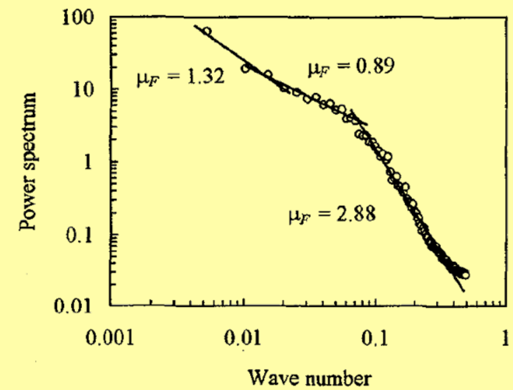
Radially averaged Fourier power spectrum:power-law behaviour
<u>Large scale region</u>: related to the impossibility of the different variants to penetrate each other

A.Y.Pasko, A.A.Likhachev, Y.N.Koval, V.I.Kolomystev, J.PHYS. IV France **7** Colloque C5,435 (1997).

Fe-Mn-Si

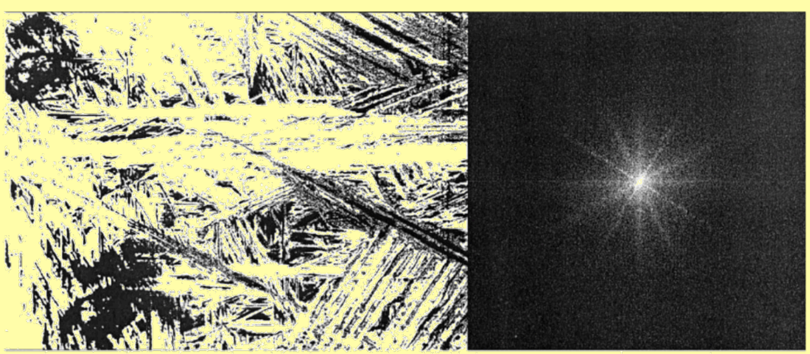




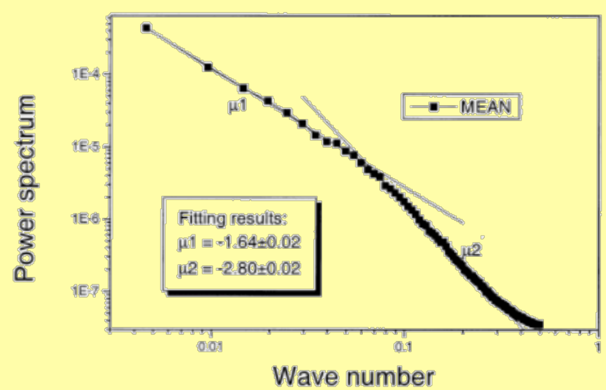


A.A.Likhachev, J.Pons, E.Cesari, A.Yu.Pasko, V.I. Kolomytsev , Scripta Materialia **43**, 765 (2000)

Cu-Al-Ni



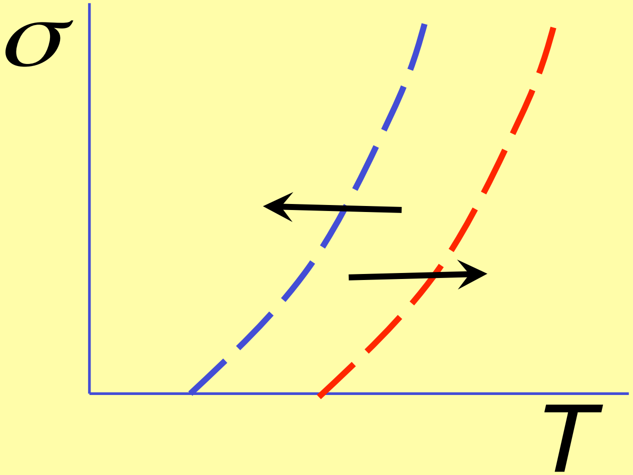
(a)



Metastability and hysteresis

First-order phase transitions hardly occur in equilibrium:

- low T: high energy barriers

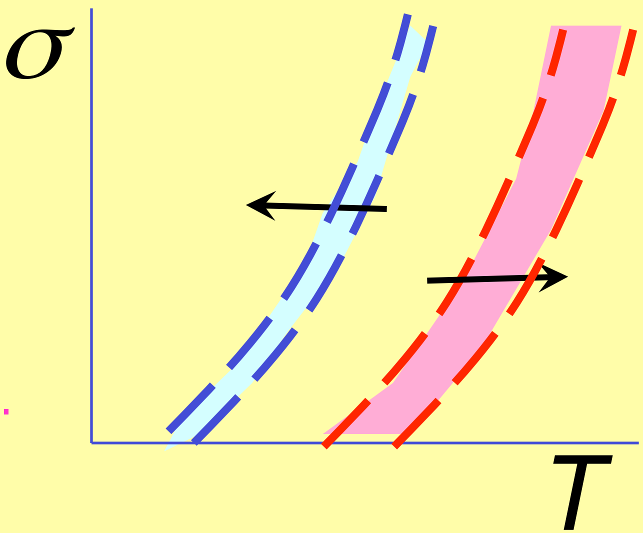


Hysteresis

J.Ortín & A.Planes, Chapter 5, in "The Science of Hysteresis", edited by G.Bertotti & I.Mayergoyz, Acad. Press. (2005).

- disorder & long range elastic forces

Inhomogeneous behaviour



Extended FOPT

Thermoelastic equilibrium

G.B.Olson & M.Cohen, Scripta Metall **9**, 1247 (1975).
J.Ortín & A.Planes, Acta Metall **37**, 1433 (1989)

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Martensitic phase transition is a sequential process

The microstructure is not built instantaneously as assumed by some models that are based on energy minimization. On the contrary it is built sequentially.

The final microstructure is not the one that minimizes some energy functional, but the one that results from the "sequential path" that minimizes the energy at every instant of time.

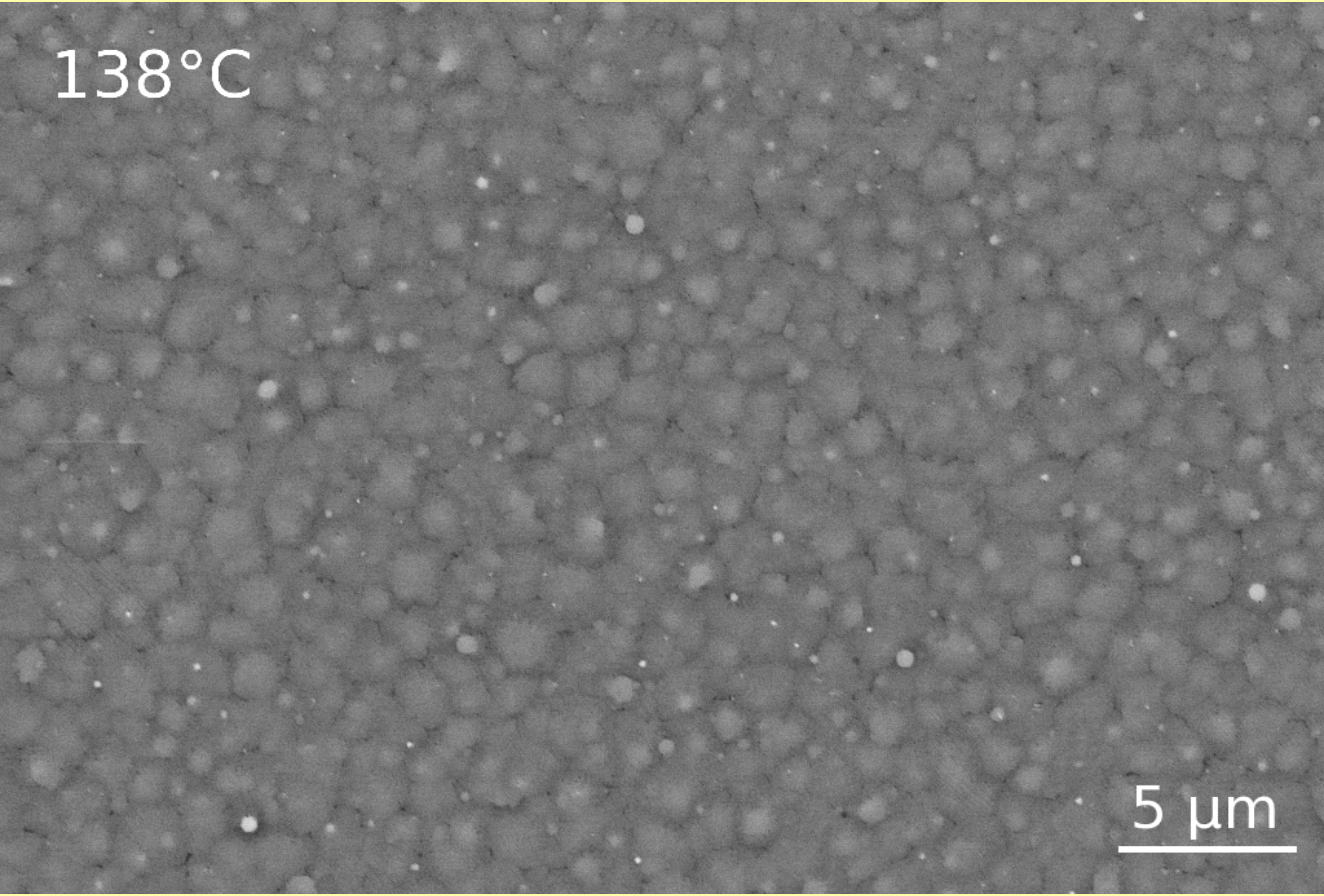
Experimental examples

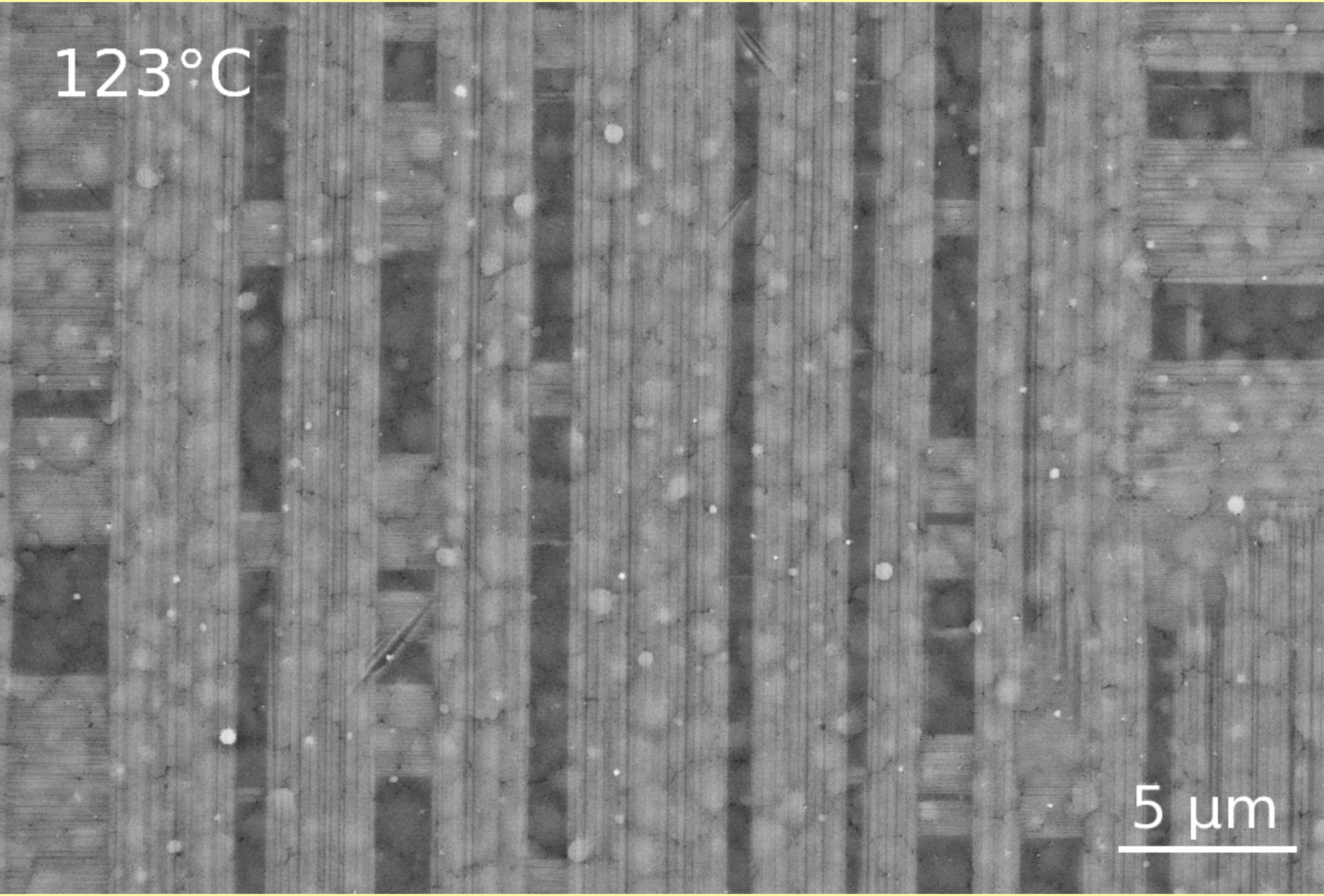
- Video by Robert Niemann PhD Thesis: NiMnGa epitaxial film on a substrate

  http://www.ifw-dresden.de/about-us/people/dr-robert-niemann/

- Acoustic emission & high sensitivity calorimetry studies

  Talk by A.Planes on Wednesday afterlunch


138°C
5 µm


123°C
5 µm

Martensitic phase transition is a sequential process

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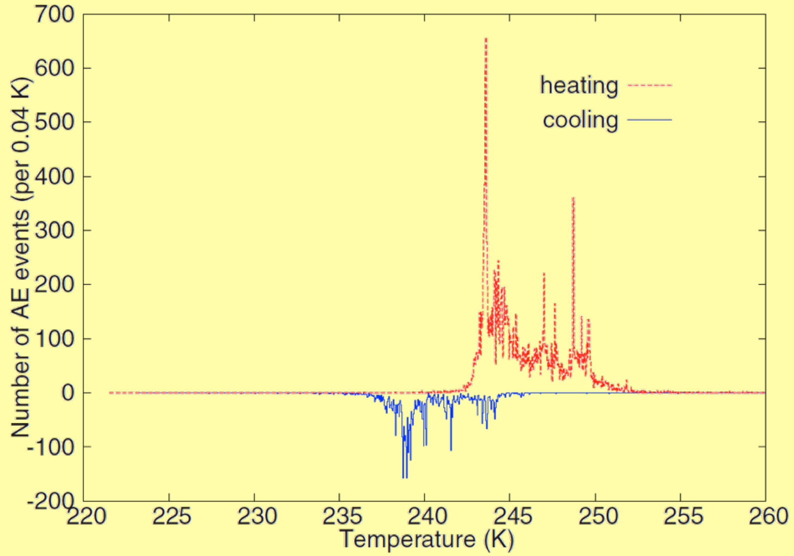
  Talk by Prof. A.Planes on Wednesday afterlunch

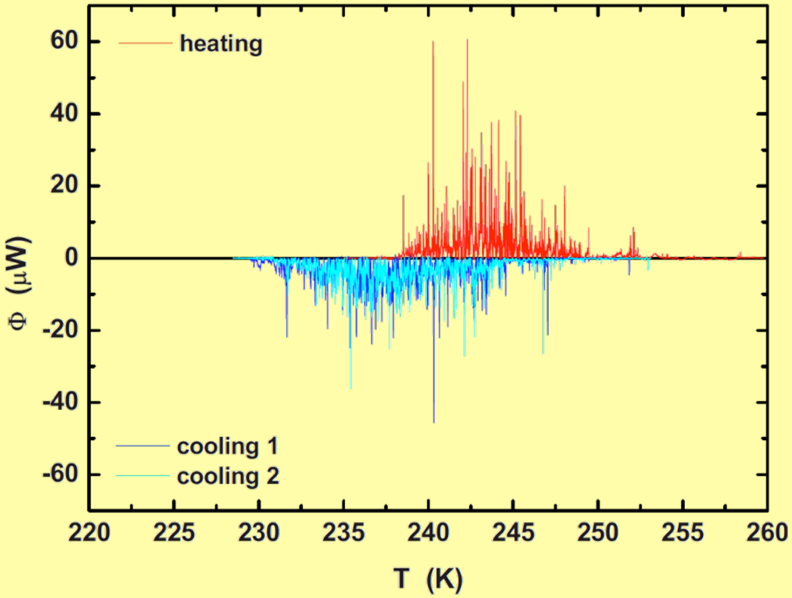
Experimental results from AE and calorimetry studies (1)

Ex: CuZn-Al M.C.Gallardo et al. PRB **81**, 174102 (2010)

Acoustic Emission activty (number of events per temperature interval)



Calorimetry:
Heat power exchange per temperature interval



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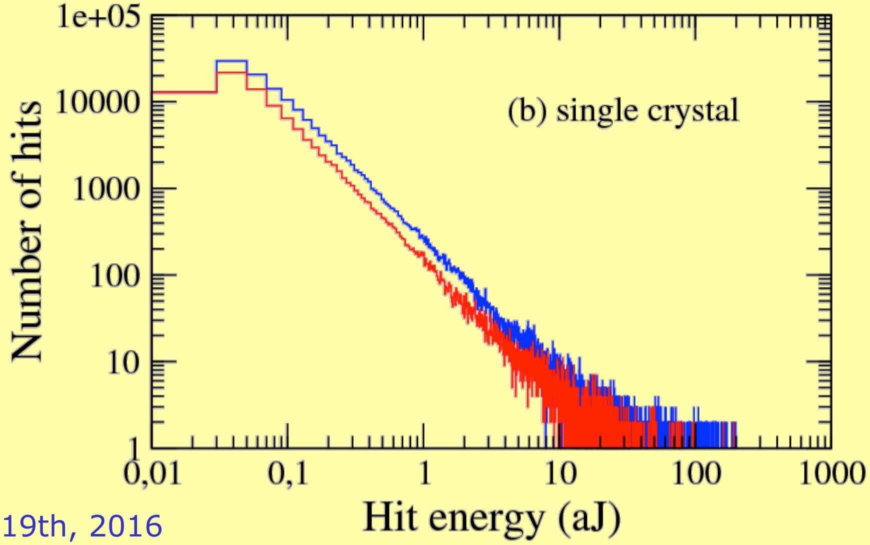
Experimental results from AE and calorimetry studies (2)

Criticality: Energies and amplitudes of AE events recorded during the full transition are power-law distributed

$$p(E)dE = \frac{1}{Z_\varepsilon} E^{-\varepsilon} dE \qquad p(A)dA = \frac{1}{Z_\alpha} A^{-\alpha} dA$$

Ex: FePd

Bonnot et al. PRB 78, 184103 (2008)



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Results from AE and calorimetry studies (3)

Weak universality Materials transforming to the same structure show the same exponents

Carrillo et al PRL81, 1889 (1998)

Cu-based alloys:

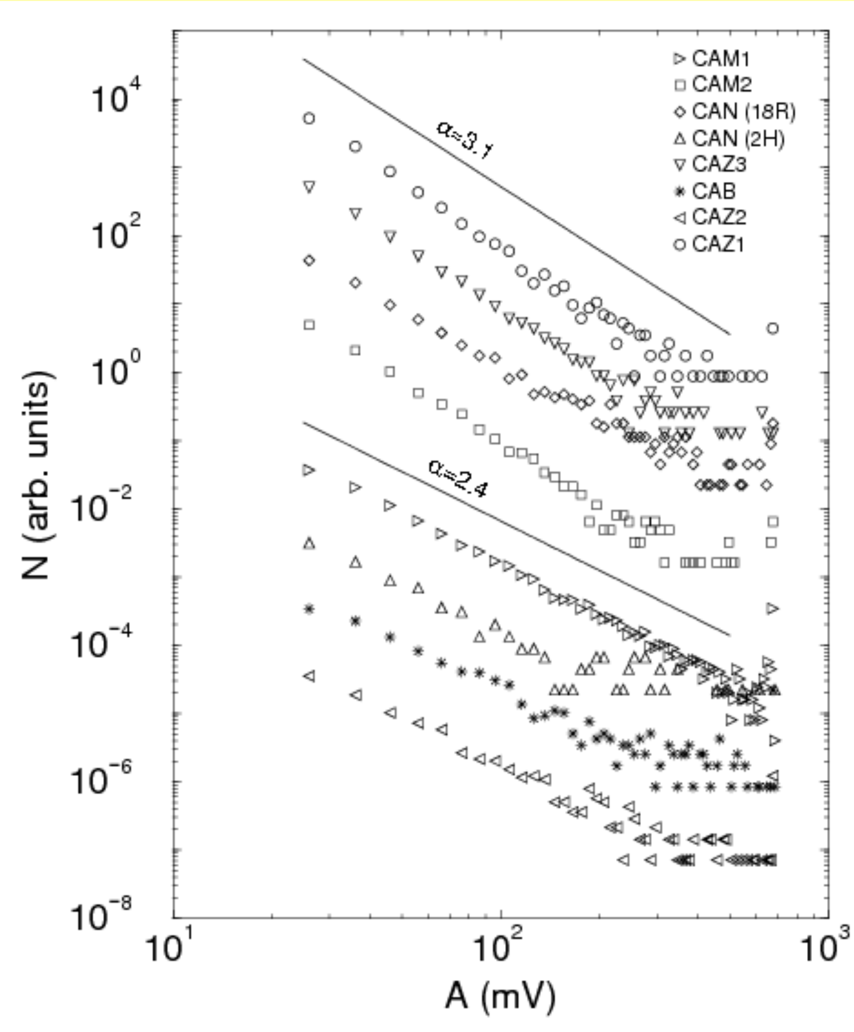
Two families:

Transformation Cubic-18R

(12 variants)

Transformation Cubic-2H

(6 variants)



Results from AE and calorimetry studies

The critical exponents increase with the number of equivalent variants.

When an external field or stress is applied, the number of possible variants is reduced and, correspondingly, the exponents decrease

Mart. phase	Variants	α	ε	z
Monoclininc	12	2.8 – 3	2	2
Orthorrombic	6	2.4-2.6	1.7-1.8	2
Tetragonal	3	2.2-2.4	1.6-1.5	2

$$z = 2$$

$$E \propto A^z$$

$$\alpha = 2\varepsilon - 1$$

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References: models for criticality

Spin-like models with disorder: Zero Temperature RFIM with metastable dynamics

J.P.Sethna, K.Dahmen, et al. PRL **70**, 3347 (1993)

Equation of motion for the strain field

R.Ahluwalia & G.Amanthakrishna, PRL **86**, 4076 (2001)

Phase field models

O.U.Salman, A.Finel, R.Delville & D.Schryvers J.App.Phys. **111**, 103517 (2012)

O.U.Salman, PhD disertation

Connection spin-models – strain field models

F.J.Pérez-Reche, L.Truskinovsky & G.Zanzotto PRL **99**, 75501 (2007)

F.J.Pérez-Reche, L.Truskinovsky & G.Zanzotto PRL **101**, 230601 (2008)

Molecular dynamics

E.K.H.Salje, X.Ding, Z.Zhao, T.Lookman & A.Saxena, PRB 83, 104109 (2011).

Z.Zhao, X.Ding, J.Sun & E.K.H.Salje, Journ. of Phys.: Cond. Matter **26**, 142201 (2014).

Competition between tip speed and nucleation rate

M.Rao & S.Sengupta & H.K.Sahu PRL **75** 2164 (1995)

E.Ben-Naïm & P.L.Krapivsky PRL **76**, 3224 (1996).

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Simple models

Sequential partitioning model (1)

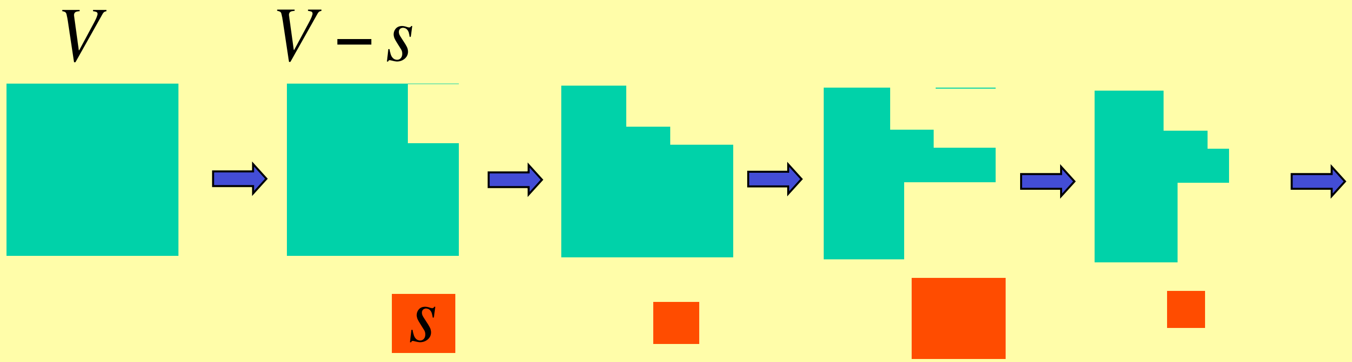
Frontera et al., PRE 52, 5671 (1995)

Minimal model that illustrates the consequences of the sequential character of the athermal phase transitions

Single variant transition (transformed/untransformed)

Excluded volume interaction only (no back transformations)

Scalar model: s size (volume) of an individual transformation event (avalanche)



Question: Is there a power-law distribution of “avalanches”?

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Sequential partitioning model (2)

Scaling hypothesis: probability of choosing a certain fraction *s/V* of the remaining volume *V*

$$\boxed{p(s;V)ds = g\left(\frac{s}{V}\right)\frac{ds}{V}}$$

Let the probability of extracting a fragment of size s in the k-step, from a system with original size V, be: $p_k(s;V)$

Recurrence $p_1(s;V) = p(s;V)$

$$p_2(s,V) = \int_0^{V-s} ds_1 p(s_1,V)p(s,V-s_1)$$

$$p_3(s,V) = \int_0^{V-s} ds_2 \int_0^{V-s_2-s} ds_1 p(s_1,V)p(s_2,V-s_1)p(s,V-s_1-s_2)$$

Doing some algebra one can obtain the recurrence:

$$p_k(s;V) = \int_0^{V-s} p(r;V)p_{k-1}(s;V-r)\,dr$$

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Sequential partitioning model (3)

Expected number of fragments with size between s and s+ds after M extractions

$$n_M(s;V) = \sum_{k=1}^{M} p_k(s;V)$$

$$n_M(s;V) = p(s;V) + \int_0^{V-s} p(r;V)\, n_{M-1}(s;V-r)$$

If the limit $n(s;V) \equiv n_{M\to\infty}(s;V)$ exists, it must satisfy:

$$n(s;V) = p(s;V) + \int_0^{V-s} p(r;V)\, n(s;V-r)$$

And should be "normalizable" $\int_0^{V} s\,n(s;V)\,ds = V$

Sequential partitioning model: solutions

Solution for uniform $g(x) = 1$ $\qquad n(s;V) = 1/s$

Solution for restricted β $g(x) = (\beta+1)(1-x)^{\beta} \quad \beta > -1$

$$n(s;V) = \frac{\beta+1}{s}\left(1-\frac{s}{V}\right)^{\beta} \approx \frac{1}{s} \quad s << V$$

There are also analytical solutions for general β-distributions

$$g \propto x^{\alpha}(1-x)^{\beta}$$

In general there are arguments based on the computations of the momenta of n(s;V) that indicate that

$$n(s;V) \approx s^{-1} \quad s << V$$

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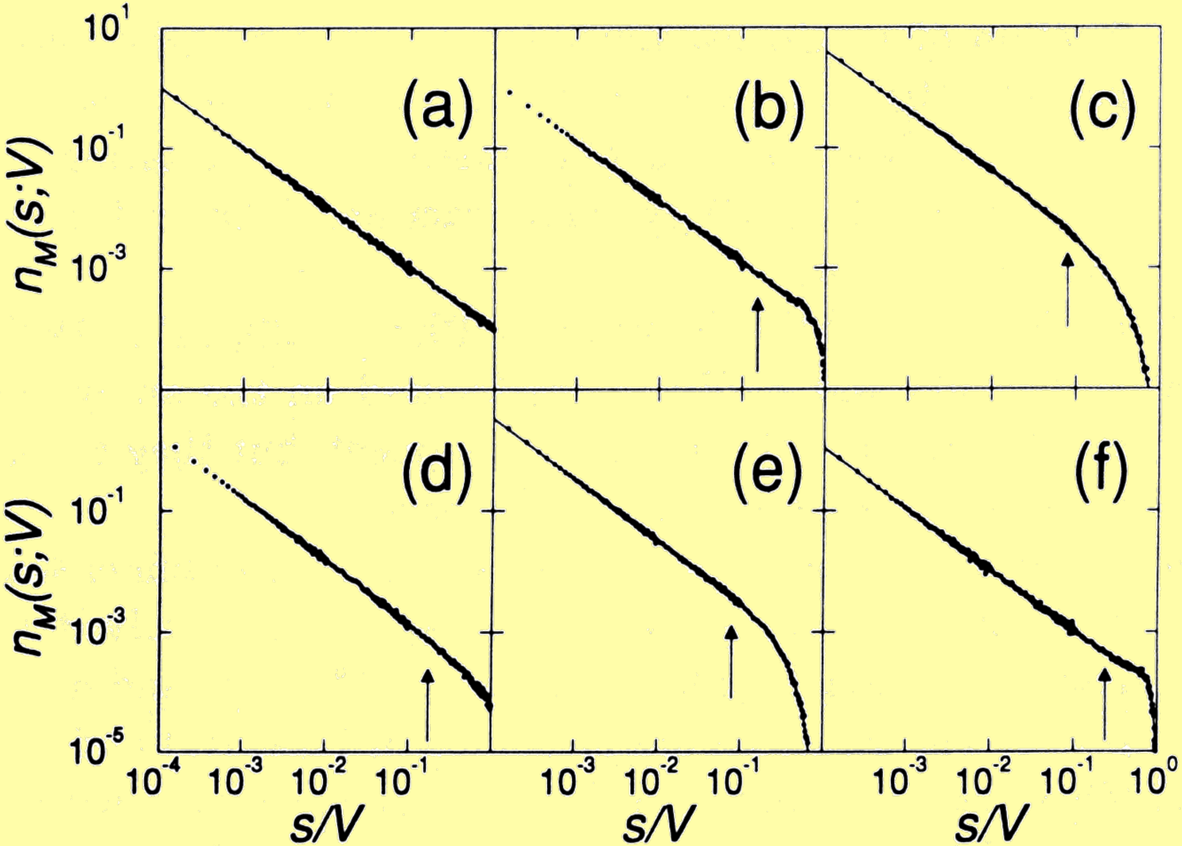
Sequential partitioning model: solutions



a) Uniform
b) Triangular
c) Restricted β-distribution for β=3
d) $g(x)=1/2\sqrt{x}$
e) Beta distribution with α=1, β=5
f) Beta distribution with α=3, β=2

Ball & Planes model (PLKK model?)

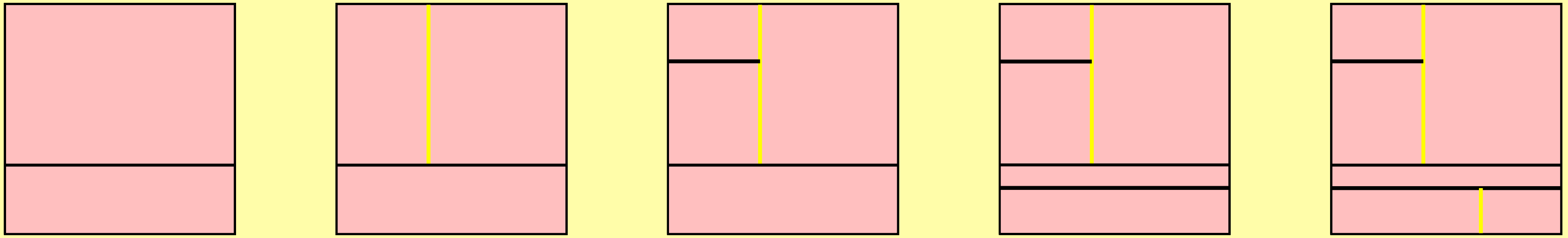
J.M.Ball, P.Cesana & B.Hambly, MATEC WoC 33,02008 (2015).
G.Torrents, X.Illa, E.Vives & A.Planes, submitted to PRE

Elongated martensitic domains (needle like) grow sequentially, nucleating at random sites in untransformed regions and growing linearly

Domains grow along few directions fixed by the symmetries of the problem

Retransformations are not possible (domains cannot cross)

Example: 2d, 2 “variants”, 90°, parallel to the sample boundaries



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Two formulations: continuum and discrete versions

Continuum:
-Needle domains are 1d-lines in a continuum rectangle [0,1]x[0,1]

-The system is never fully transformed

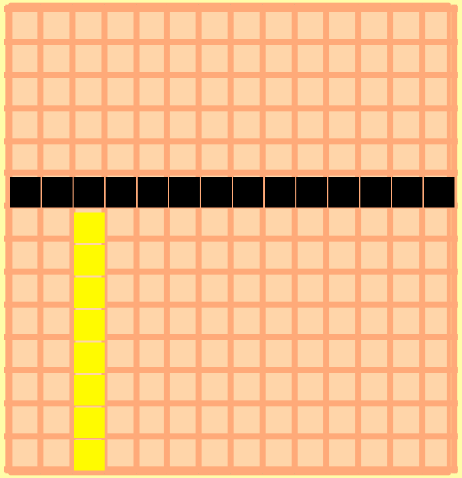
-Question: After many draws, what will be the average number (density) H(ℓ) of horizontal segments with length between ℓ and ℓ +dℓ?

- ℓ will be a continuum variable taking values in (0,1]

Discrete (convenient for simulations)
-Needle domains with width "a", are placed at discrete positions on a LxL grid

-The sample fully transforms.

Question: What is the average number H(ℓ) of variants with a certain size ℓ ?



- ℓ will be a discrete variable taking values 1,2,…L

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Generalization to asymmetric probabilities and sample shape

Analytic treatment will be done in the case in which the probabilities of every variant are not equal:

p_v- probability of drawing a vertical variant
p_h- probability of drawing a horizontal variant

We will not assume $p_v = 1 - p_h$, but we will solve a generic case with:

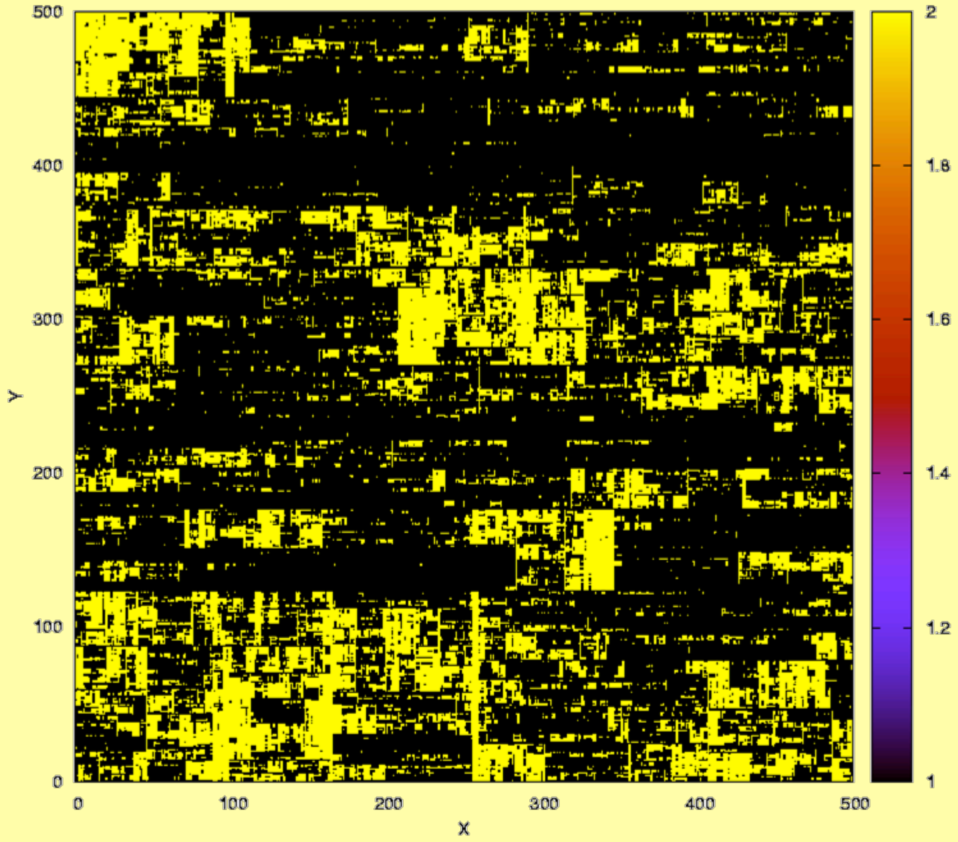
$$p_v + p_h \leq 1$$

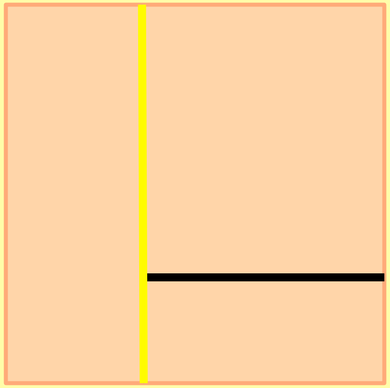
Moreover, for the continumm case, we will assume that the system has dimensions $L_h x L_v$

Numerical simulations

Fortran code that generates final configurations, until complete fill-up. In the initial steps it chooses sites at random on the lattice and in the final steps it keeps a list of empty sites and chooses at random from the list

Example 1: 2 variants: (1,0) & (0,1)



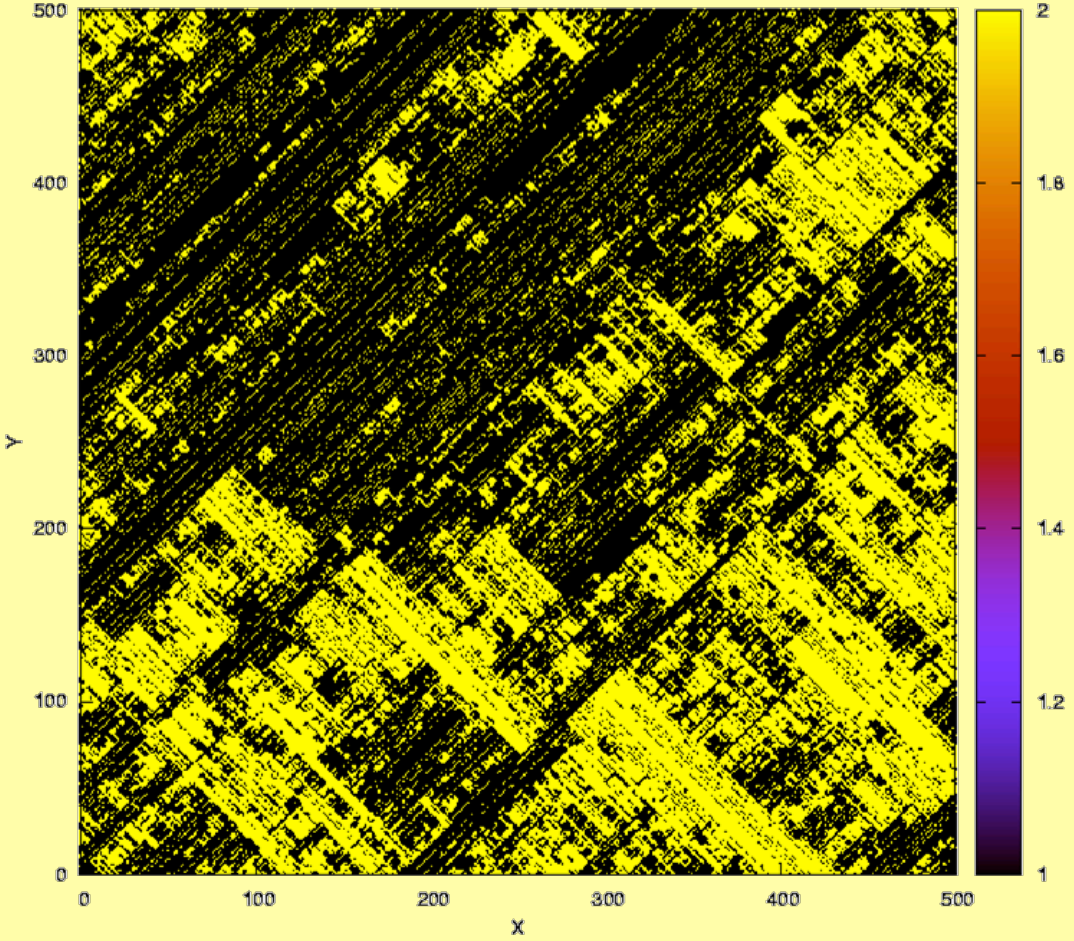


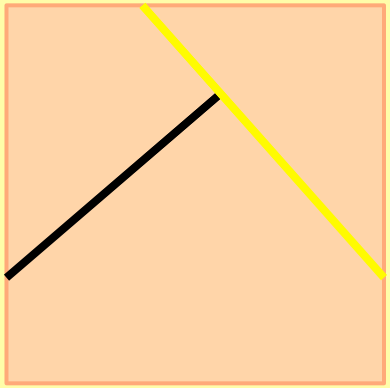
500x500 subset from a 2000x2000 system

Numerical simulations

The code can be easily adapted to different cases: orientation of variants, number of variants, etc…

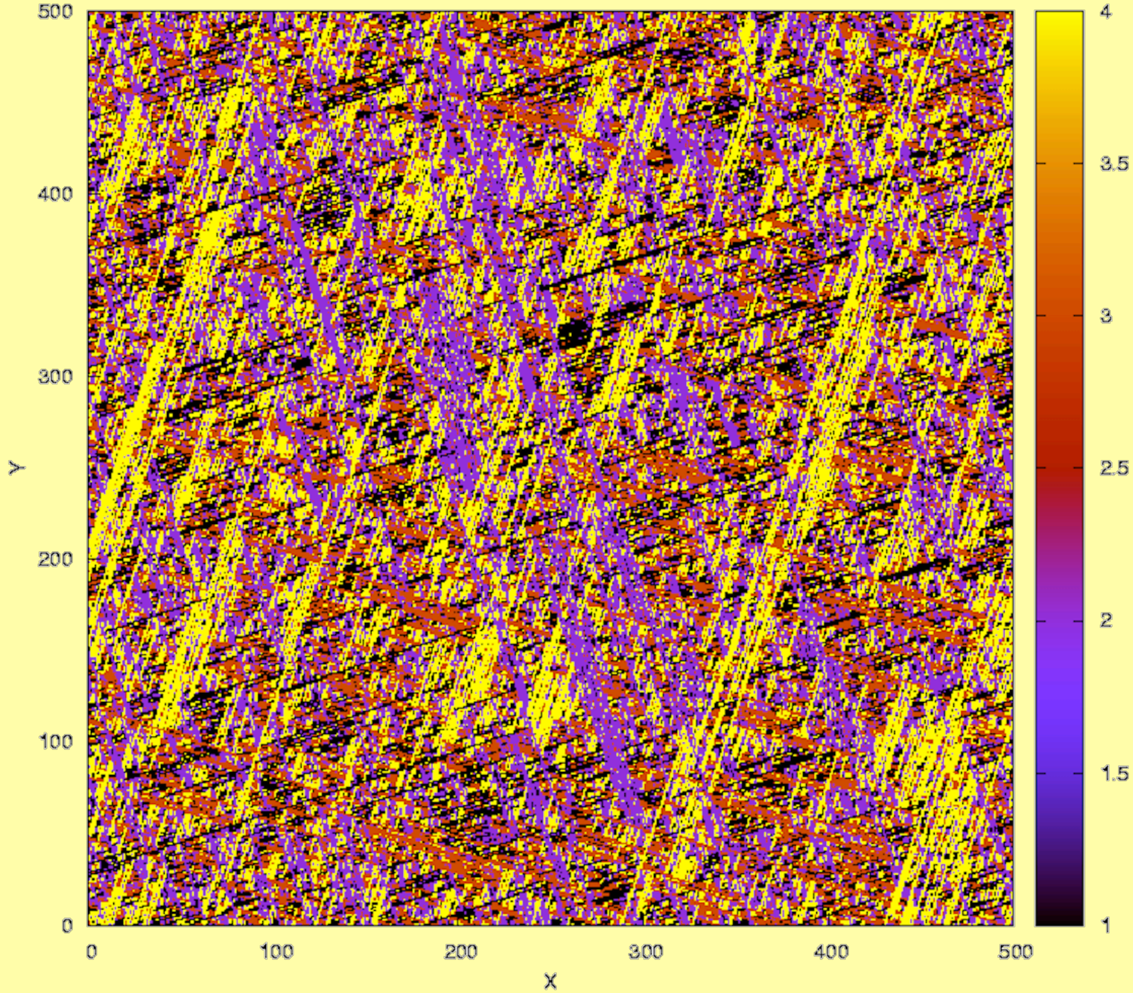
Example: 2 variants: (1,1) & (-1,1)

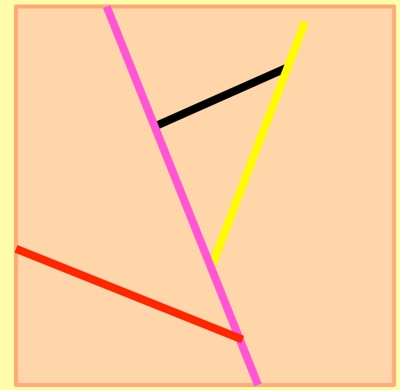




Numerical simulations

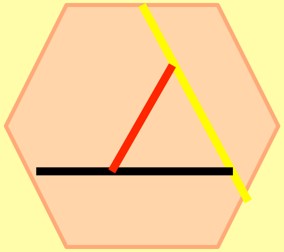
Example: 4 variants (1,2) (-1,2) (2,1) (2,-1)

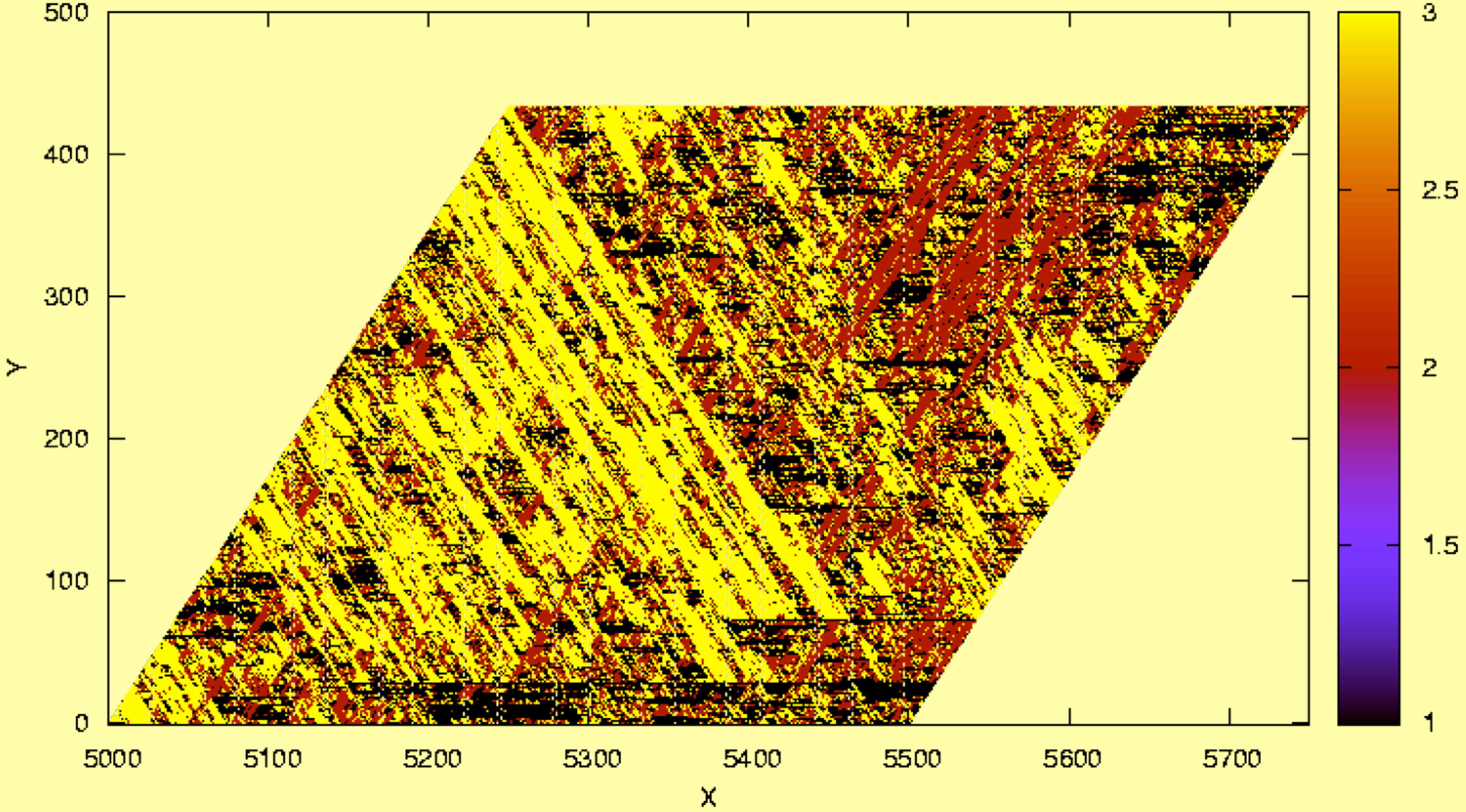




Numerical simulations

Hexagonal lattice: 3 variants (1,0), (0,1) (-1,1)





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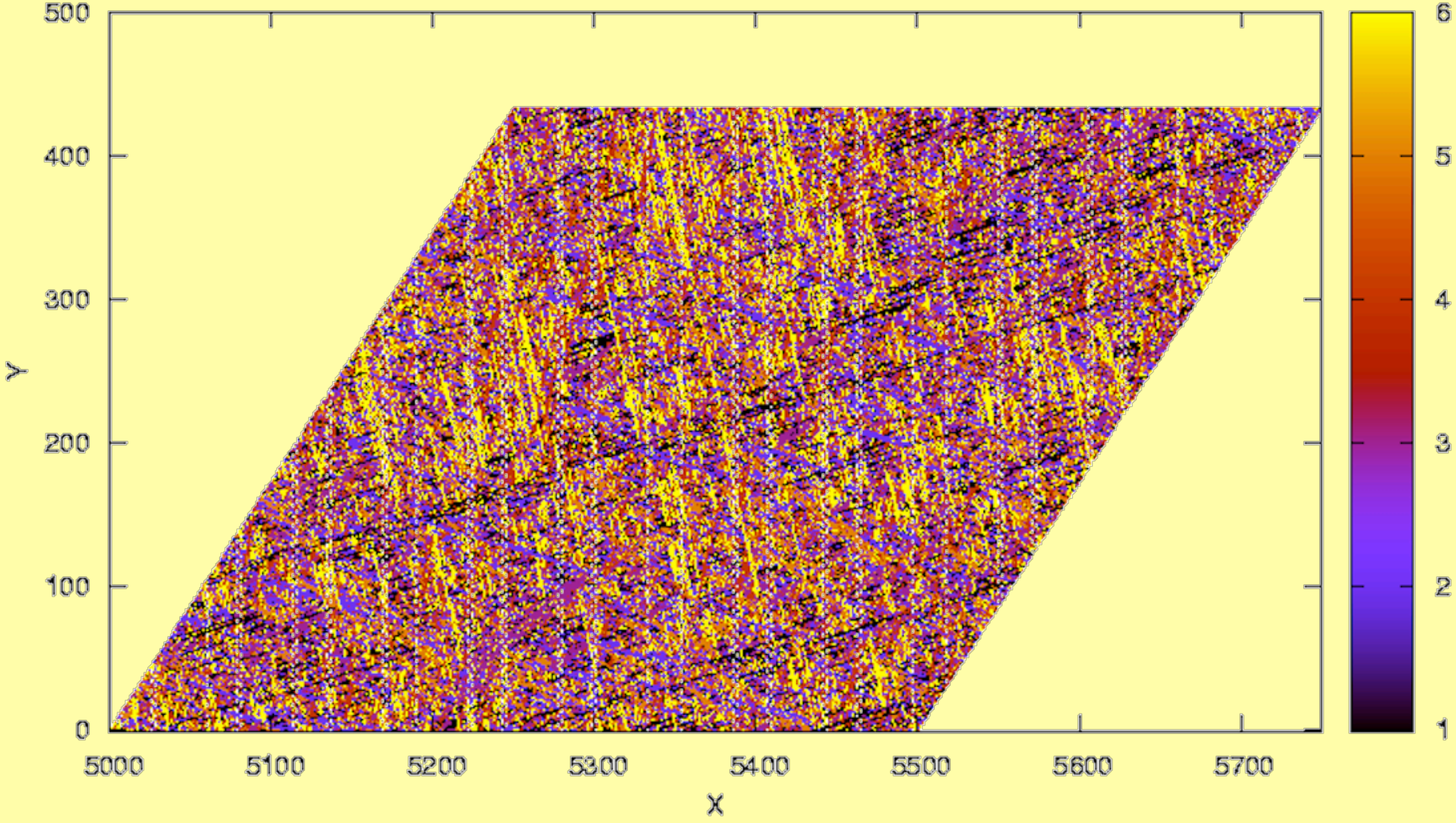
Hexagonal lattice: 6 variants



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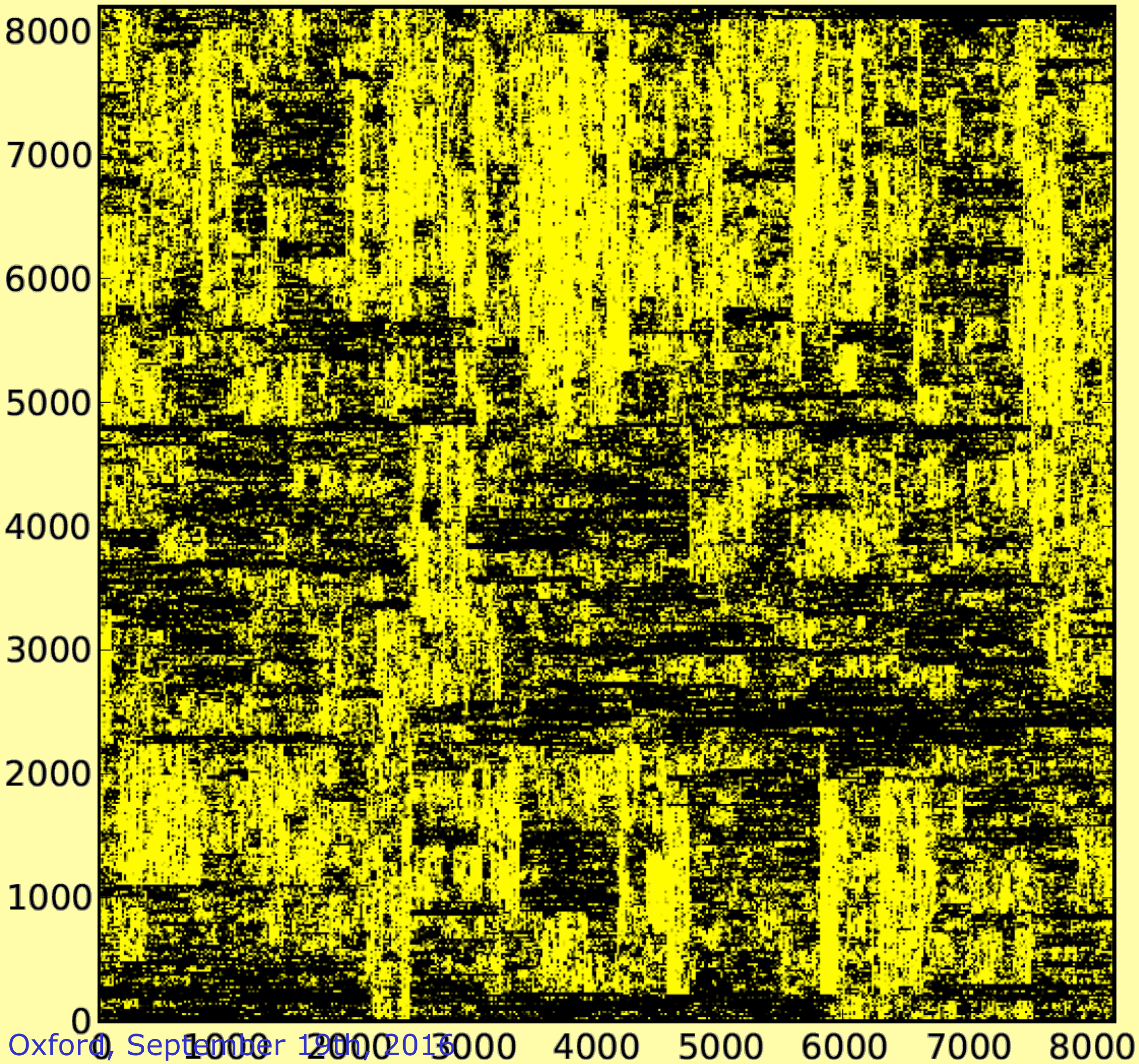
Dipolar and boundary effects

Square sample, (1,0)&(0,1) $p_h=p_v=1/2$



The distribution of variants is not homogeneous, but influenced by the system boundaries

There exist and effective correlation between domains belonging to the same variant

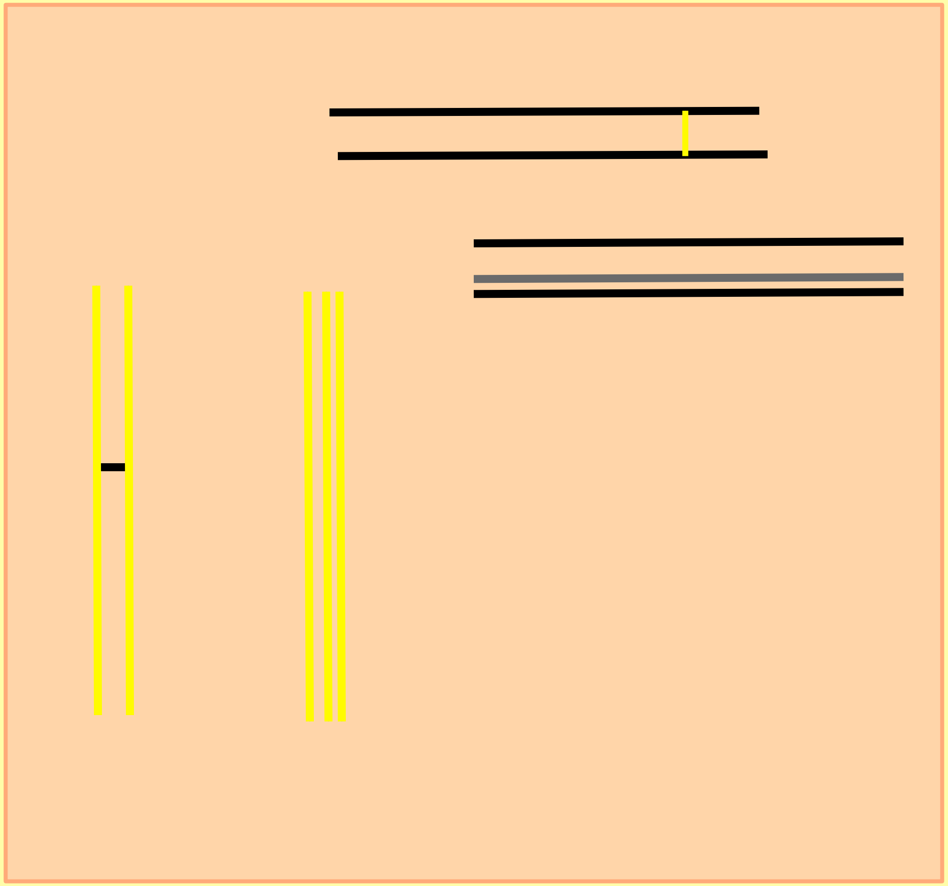
It can be interpreted as a dipolar/ferroelastic effective interactions

8192x8192 system

Dipolar and boundary effects: why?

Between two vertical domains (yellow)that are close one to the other it is more probable to find points associated to vertical variants (yellow) than horizontal variants (black). Why?

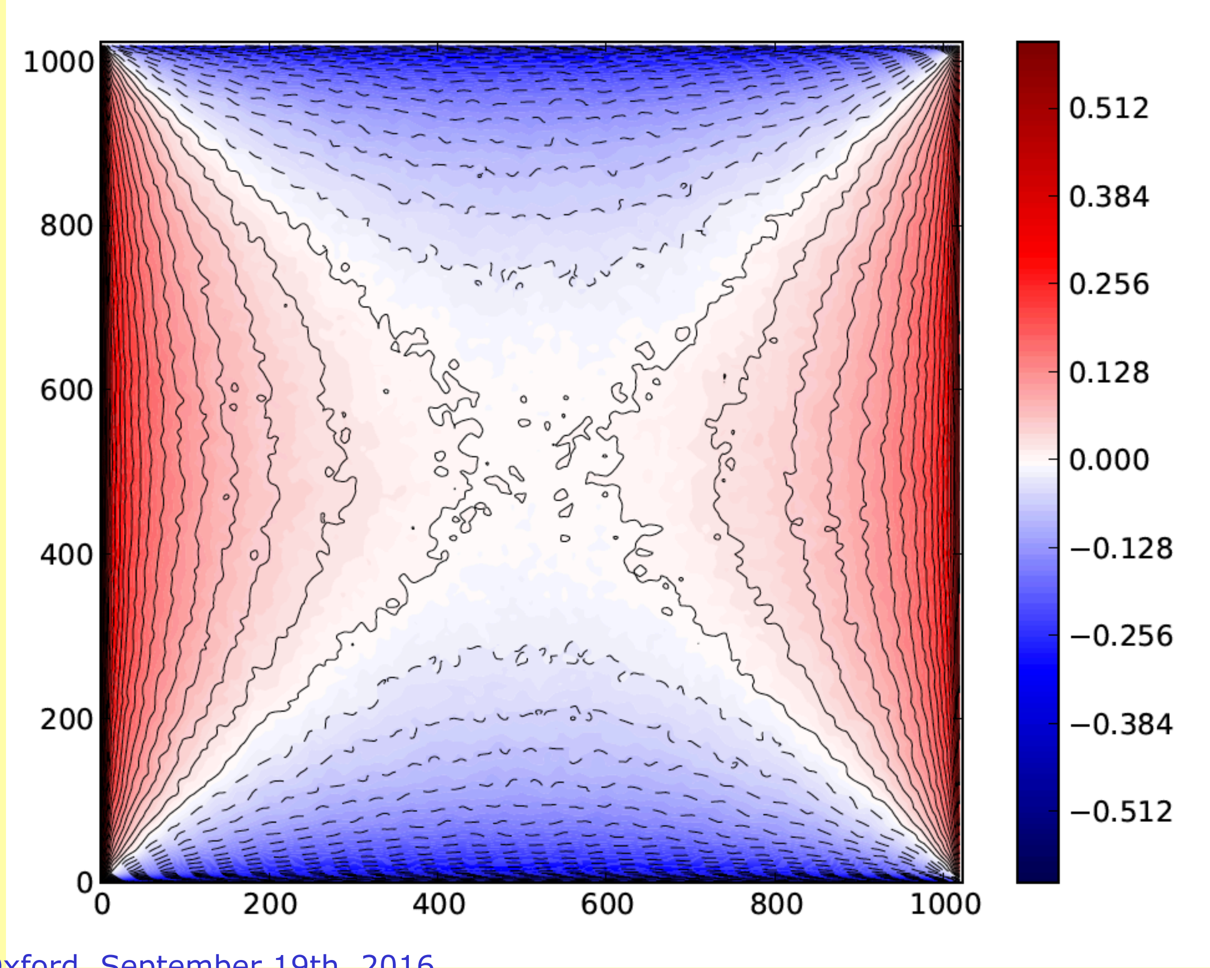
Despite $p_h = p_v$, the vertical lines will be much longer than the horizontal ones, thus creating an effective atractive dipolar/ferroelastic interaction.



Boundary effects: order parameter

Coarse-grain the lattice in small cells (8x8) and and measure:

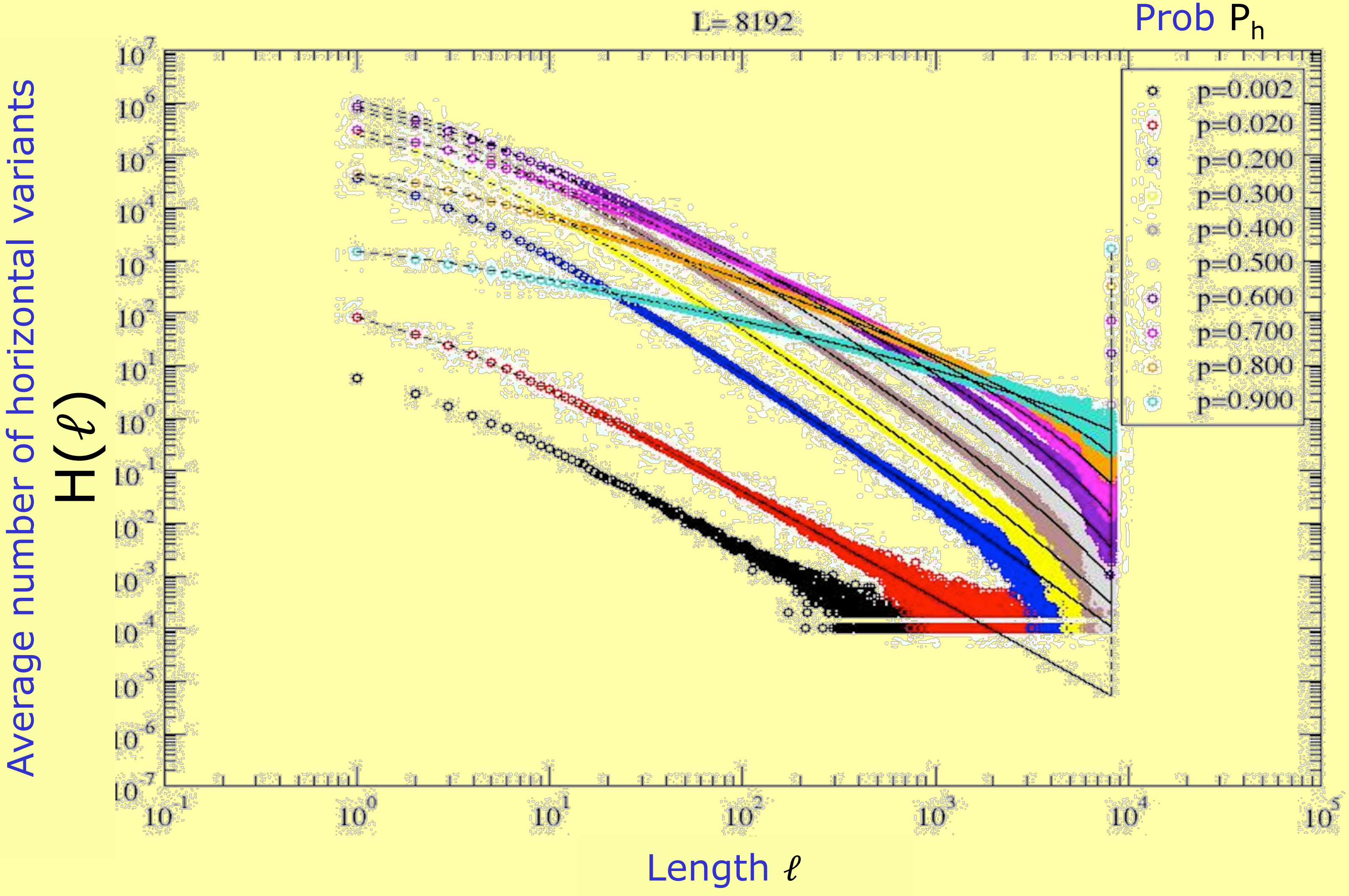
$$\frac{n_{vertical\ sites} - n_{horizontal\ sites}}{n_{sites\ in\ the\ cell}}$$



1024x1024 cells
8192x8192 system

Numerical simulation of distributions of lengths

Histograms corresponding to the full LxL system (irrespective of the spatial position)



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Analytic solution of the continuum case

Let us consider the continuum case: a square [0,1]x[0,1].

Let p_h and p_v be the probabilities of drawing a horizontal and a vertical line respectively

Let H(ℓ) be the average number (density) of horizontal lines (a.n.h.l.) with length within (ℓ,ℓ+dℓ)

We can assume that: $$H(\ell) = A\delta(\ell - 1) + h(\ell)$$

where h(ℓ) is continuum and non-zero on the interval ℓ= (0,1)

After drawing the first line, the system splits into two subsystems equivalent to the first one, but eventually re-scaled. Let us consider the first draw, H(ℓ) will have two contributions:

$$H(\ell) = p_h \times H^{[h]}(\ell) + p_v \times H^{[v]}(\ell)$$

where $H^{[h,v]}(\ell)$ are conditional probabilities.

Let us analyze the two problems separately:

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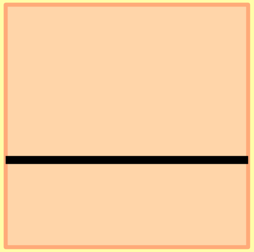
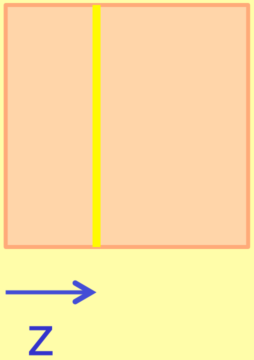
Analytical solution of the continuum case

a.n.h.l. given the first line is horizontal



$$H^{[h]}(\ell) = 1\ \delta(\ell - 1) + 2\left[A\delta(\ell - 1) + h(\ell)\right]$$

a.n.h.l. given the first line is vertical



$$H^{[v]}(\ell) = 2\int_0^1 \Theta(z-\ell)\,dz\left[A\frac{1}{z}\delta\left(\frac{\ell}{z}-1\right)+\frac{1}{z}h\left(\frac{\ell}{z}\right)\right] =$$

… changing variables ($t = \ell/z,\ dt = -\ \ell\ dz/z^2$)

$$= 2\int_\ell^1 dt\left[A\frac{1}{t}\delta\left(t-1\right)+\frac{1}{t}h\left(t\right)\right]$$

Putting things together:

$$A\delta(\ell-1)+h(\ell) = p_h\left[(1+2A)\delta(\ell-1)+2h(\ell)\right]+p_v\left[2A+2\int_\ell^1 h(t)\frac{dt}{t}\right]$$

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Analytical solution of the continuum case

By comparing the terms that multiply the δ-function $A = \dfrac{p_h}{1-2p_h}$

and for the continuum part: $h(\ell)(1-2p_h) = \dfrac{2p_h p_v}{1-2p_h} + 2p_v \int_\ell^1 h(t)\dfrac{dt}{t}$

Diferentiating: $\dfrac{dh(\ell)}{d\ell} = -\dfrac{2p_v}{1-2p_h}\dfrac{h(\ell)}{\ell} \rightarrow \dfrac{dh}{h} = -\dfrac{2p_v}{1-2p_h}\dfrac{d\ell}{\ell}$

Solving: $h(\ell) = \dfrac{2p_v p_h}{(1-2p_h)^2}\,\ell^{-\frac{2p_v}{1-2p_h}}$

The continuum part is a power-law with an exponent that depends on p_h and p_v , but it only makes sense for $p_h<1/2$.

For $p_h \geq 1/2$, the assumption of infinitely thin domains makes non-sense: one can generate too many horizontal lines. We need a cutoff !

For $p_h \rightarrow 0$ the size distribution “of the very minoritary variant” is a power-law with an exponent -2 (if $p_h+p_v=1$)

Analytical solution of the discrete case (Genís Torrents)

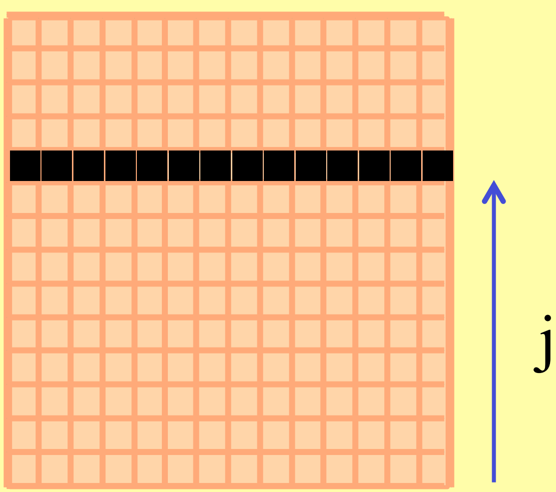
In this case, we want to compute the average number of horizontal lines in a system with size $L_h x L_v$

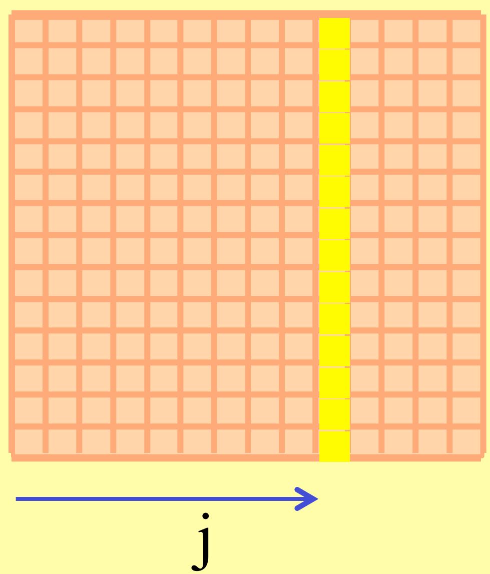
$$H(\ell; L_h, L_v)$$

where the variable ℓ is now discrete and ranges from 1 to L_h

Following the same arguments as before, we can write:

$$H(\ell; L_h, L_v) = p_h \delta(\ell - L_h) + \frac{2p_h}{L_v} \sum_{j=1}^{L_v - 1} H(\ell; L_h, j) + \frac{2p_v}{L_h} \sum_{j=\ell}^{L_h - 1} H(\ell; j, L_v)$$

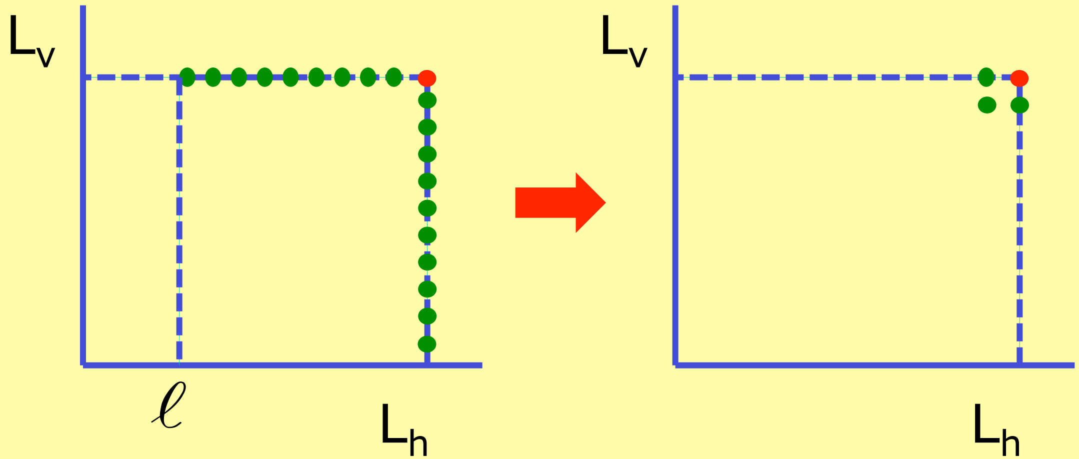




Construction of recurrence

$$H(\ell;L_h,L_v)=p_h\delta(\ell-L_h)+\frac{2p_h}{L_v}\sum_{j=1}^{L_v-1}H(\ell;L_h,j)+\frac{2p_v}{L_h}\sum_{j=\ell}^{L_h-1}H(\ell;j,L_v)$$

In a L_h x L_v diagram, this equation involves summing many terms:



For $\ell<L_h-1$

$$H(\ell;L_h,L_v)=\frac{2p_h+L_v-1}{L_v}H(\ell;L_h,L_v-1)+\frac{2p_v+L_h-1}{L_h}H(\ell;L_h-1,L_v)$$

$$-\frac{(2p_v+L_h-1)(2p_h+L_v-1)-2p_h2p_v}{L_hL_v}H(\ell;L_h-1,L_v-1)$$

General recurrence and simple solutions

In general, it is possible to write en equation as a sum of 4 terms that equals to a inhomogeneous term independent of H

$$H(\ell;L_h,L_v)-\frac{2p_h+L_v-1}{L_v}H(\ell;L_h,L_v-1)-\frac{2p_v+L_h-1}{L_h}H(\ell;L_h-1,L_v)$$
$$+\frac{(2p_v+L_h-1)(2p_h+L_v-1)-2p_h2p_v}{L_hL_v}H(\ell;L_h-1,L_v-1)=\frac{p_h}{L_v}\left(\delta(\ell-L_h)-\frac{L_h-1}{L_h}\delta(\ell-L_h+1)\right)$$

(Note that this equation is linear in H and that mixes problems corresponding to different sizes)

Let us start by finding some simple solutions, corresponding to the cases $\ell=L_h$ and $\ell=L_h-1$

For $L_v=1$, the only way to get a horizontal line of size $\ell=L_h$ is to be lucky and choose first to draw horizontal:

$$H(L_h;L_h,L_v=1)=p_h$$

Then, applying the above recurrence one gets $H(L_h-1;L_h,L_v=1)=\dfrac{2p_vp_h}{L_h}$

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Solution for the cases $\ell = L_h$ & $\ell = L_h - 1$ (1)

It is also simple to extend these two cases $\ell = L_h$ and $\ell = L_h - 1$ to any generic L_v,
By rewriting the recurrence, imposing $\ell = L_h$

$$H(L_h; L_h, L_v) - \frac{2p_h + L_v - 1}{L_v} H(L_h; L_h, L_v - 1) = \frac{p_h}{L_v}$$

Which can be rewritten as en homogeneous recurrence for a [H-const] term

$$\left[H(L_h; L_h, L_v) - \frac{p_h}{1-2p_h}\right] = \frac{2p_h + L_v - 1}{L_v}\left[H(L_h; L_h, L_v - 1) - \frac{p_h}{1-2p_h}\right]$$

The solution is:

$$H(L_h; L_h, L_v) = \frac{p_h}{1-2p_h}\left(1 - \frac{2p_h}{L_v!}\frac{\Gamma(2p_h + L_v)}{\Gamma(2p_h + 1)}\right)$$

This term should be compared with the term multiplying the δ-function in the continuum case

$$H(\ell) = \frac{p_H}{1-2p_H}\delta(\ell - 1) + h(\ell)$$

It corresponds to the limit $L_v \to \infty$. The parenthesis corresponds to the finite size correction.

Solution for the cases $\ell = L_h$ & $\ell = L_h - 1$ (2)

Similarly one can also solve the case $\ell = L_h - 1$.

After some algebra one gets

$$H\left(L_h - 1; L_h, L_v\right) = \frac{2 p_h p_v}{L_h (1-2p_h)^2} \left\{ 1 + \frac{\Gamma\left(2p_h + L_v\right)}{L_v!\,\Gamma\left(2p_h+1\right)} 2p_h \left[\left(2-2p_h\right) + \left(1-2p_h\right) \sum_{k=1}^{L_v - 1} \frac{1}{k+2p_h} \right] \right\}$$

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Construction of a general solution

These two solutions give some clues in order to find the general solution.
In order to do it, it is convenient to construct the objects:

$$A^{L_h,L_v}_{k;\ell_h,\ell_v} \equiv A^{L_h,L_v}_{k;\ell_h,\ell_v}(2p_h,2p_v)$$

defined from the functions

$$A^{L_h,L_v}_{k;\ell_h,\ell_v}(x_h,x_v) = \sum_{j=0}^{\infty} \frac{x_h^j x_v^{j+k}}{j!(j+k)!} \frac{d^{2j+k}\Delta^{L_h,L_v}_{\ell_h,\ell_v}}{dx_h^j dx_v^{j+k}}$$

where

$$\Delta^{L_h,L_v}_{\ell_h,\ell_v}(x_h,x_v) \equiv \frac{\ell_h!\Gamma(x_h+L_h)}{L_h!\Gamma(x_h+\ell_h)} \frac{\ell_v!\Gamma(x_v+L_v)}{L_v!\Gamma(x_v+\ell_v)}$$

It must be understood that k is an integer, and when k is negative, only terms with positive or zero factorial in the divisor contribute to the sum.
Note also that the Δ's are polynomials with finite degree,

$$\Delta^{L_h,L_v}_{\ell_h,\ell_v}(x_h,x_v) \propto (x_h)^{L_h-\ell_h}(x_v)^{L_v-\ell_v} + \ldots.$$

so that the sum j=0,..,∞ will have a finite number of terms.

Properties of A's

Symmetry

$$A_{k;\ell_h,\ell_v}^{L_h,L_v}(x_h,x_v) = A_{-k;\ell_v,\ell_h}^{L_h,L_v}(x_h,x_v)$$

Sum completion

$$\sum_{k=-\infty}^{\infty} a^k A_{k;\ell_h,\ell_v}^{L_h,L_v}(x_h,x_v) = \Delta_{\ell_h,\ell_v}^{L_h,L_v}\left(x_h + \frac{x_h}{a}, x_v + a x_v\right)$$

Recurrence in L

$$A_{k;\ell_h,\ell_v}^{L_h,L_v}(x_h,x_v) = \frac{\left[x_h x_v - (x_h + L_h - 1)(x_v + L_v - 1)\right]}{L_v L_h} A_{k;\ell_h,\ell_v}^{L_h-1,L_v-1}(x_h,x_v) +$$

$$+ \frac{(x_h + L_h - 1)}{L_h} A_{k;\ell_h,\ell_v}^{L_h-1,L_v}(x_h,x_v) + \frac{(x_v + L_v - 1)}{L_v} A_{k;\ell_h,\ell_v}^{L_h,L_v-1}(x_h,x_v)$$

That corresponds exactly to the homogeneous part of the recurrence

General solution

The general solution is built by a linear compination of $A_{k;\ell_h,\ell_v}^{L_h,L_v}(2p_h,2p_v)$ with different k, ℓ_h and ℓ_v that vanishes when $L_v=0$ and that matches the appropriate boundary conditions at $\ell=L_h-1$.

$$H\left(\ell;L_h,L_v\right)=$$

$$=\frac{2p_hp_v}{(\ell+1)(1-2p_h)^2}\left\{\sum_{j=0}^{\infty}\left(\frac{1}{2p_h}-1\right)^j A_{j;\ell+1,1}^{L_h,L_v}-2p_h\left[(2-2p_h)A_{0;\ell+1,1}^{L_h,L_v}+(1-2p_h)A_{1;\ell+1,1}^{L_h,L_v}\right]\right\}$$

when $\ell<L_h$

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Comparison with the continuum limit when p_H is small

Using the completion property one can make more explicit the divergence cancellations and get:

$$H(\ell < L_h; L_h, L_v) = \frac{2 p_h p_v \left[\Delta_{\ell,1}^{L_h, L_v}\left(\frac{2 p_v}{1-2p_h}, 1 \right) + \vartheta(p_h) \right]}{(1-2p_h)^2 (\ell+1)}$$

Using Stirling approximation

$$H(\ell < L_h; L_h, L_v) = \frac{2 p_h p_v}{L_h (1-2p_h)^2} \left(\frac{\ell_H}{L_H} \right)^{-\frac{2 p_v}{1-2 p_h}}$$

That can be directly compared with the continuum case when p_H is small,taking into accout that now ℓ = 1,2, .. L_h and before $0<\ell\leqslant 1$

$$h(\ell) = \frac{2 p_V p_H}{(1-2p_H)^2} \ell^{-\frac{2 p_v}{1-2 p_H}}$$

Numerical algorithm for computing the exact solution

Usually one is interested in the solution for a system with a certain size $L_h x L_v$
The use of the general recurrence is tedious because one has to solve all the problems with sizes smaller than $L_h x L_v$

There is a tricky way for avoiding lots of sums. The idea is to use another recurrence for the A's in which instead of decreasing the index L's we have en increase of the index ℓ 's

$$A_{k;\ell_h,\ell_v}^{L_h,L_v}(2p_v,2p_h) = \frac{\left[2p_h 2p_v - (2p_v+\ell_h)(2p_h+\ell_v)\right]}{(\ell_h+1)(\ell_v+1)} A_{k;\ell_h+1,\ell_v+1}^{L_h,L_v}(2p_v,2p_h) +$$

$$+\frac{(2p_v+\ell_h)}{\ell_h+1} A_{k;\ell_h+1,\ell_v}^{L_h,L_v}(2p_v,2p_h) + \frac{(2p_h+\ell_v)}{\ell_v+1} A_{k;\ell_h,\ell_v+1}^{L_h,L_v}(2p_v,2p_h)$$

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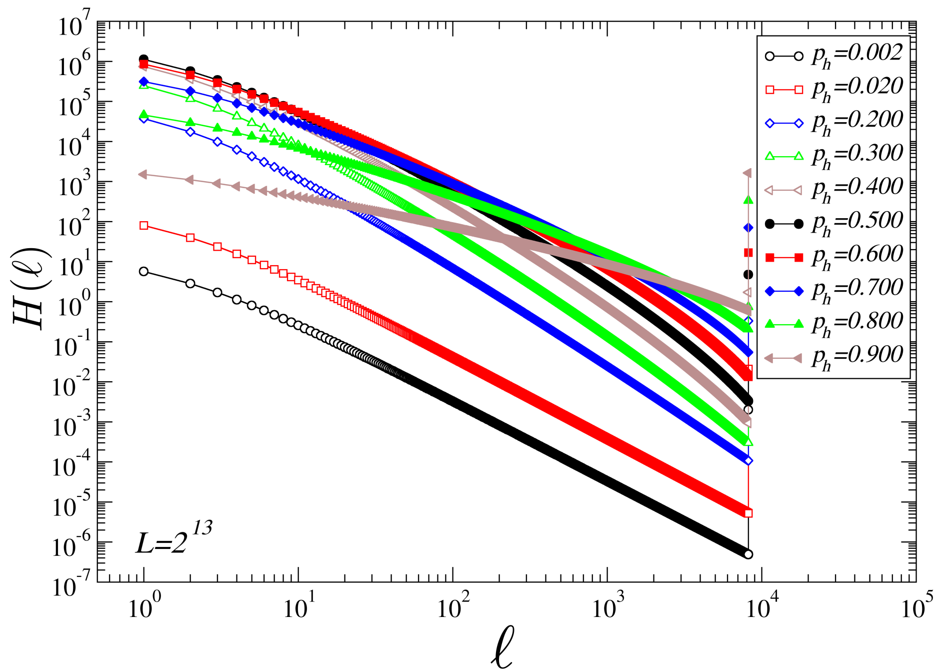
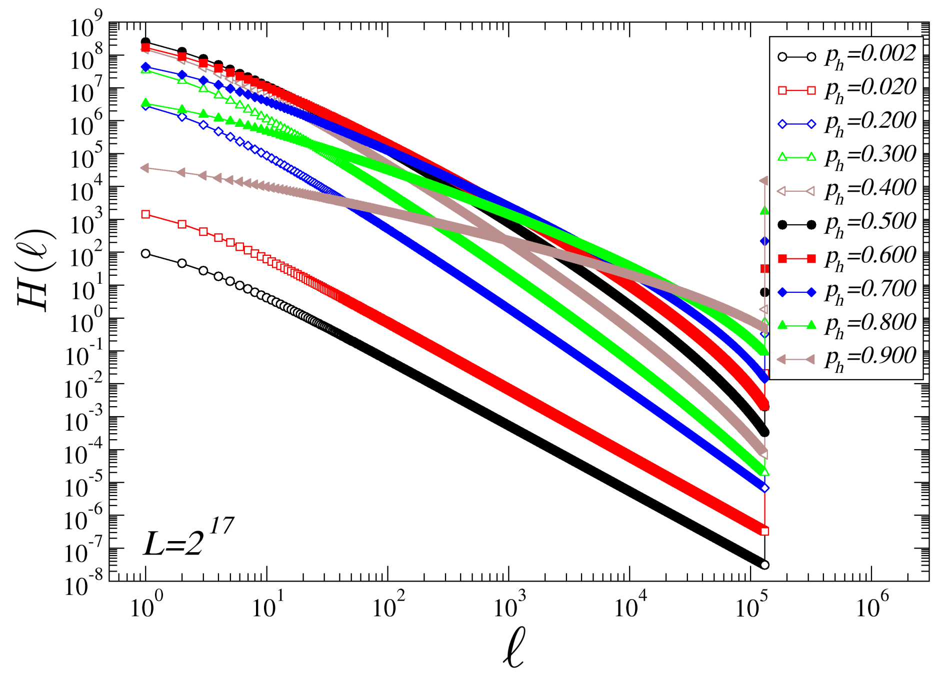
Graphical results

Solutions for systems with two different sizes LxL and different p_h

$2^{13}x2^{13}=8192x8192\approx67x10^6$

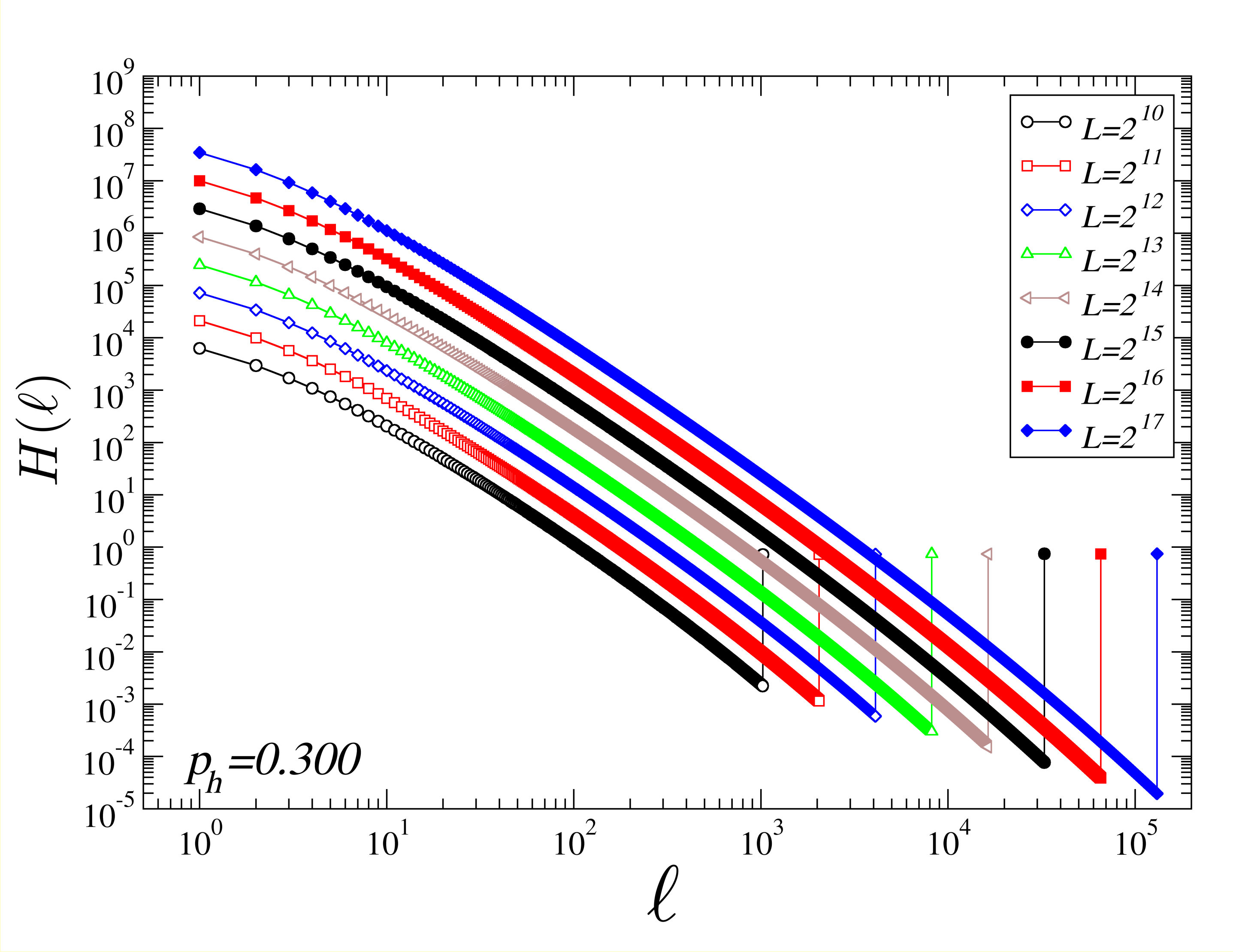


$2^{17}x2^{17}=131072 \times 131072=17x10^9$



Graphical results

Solutions for a fixed $p_h = 0.3 < 0.5$ and different system sizes



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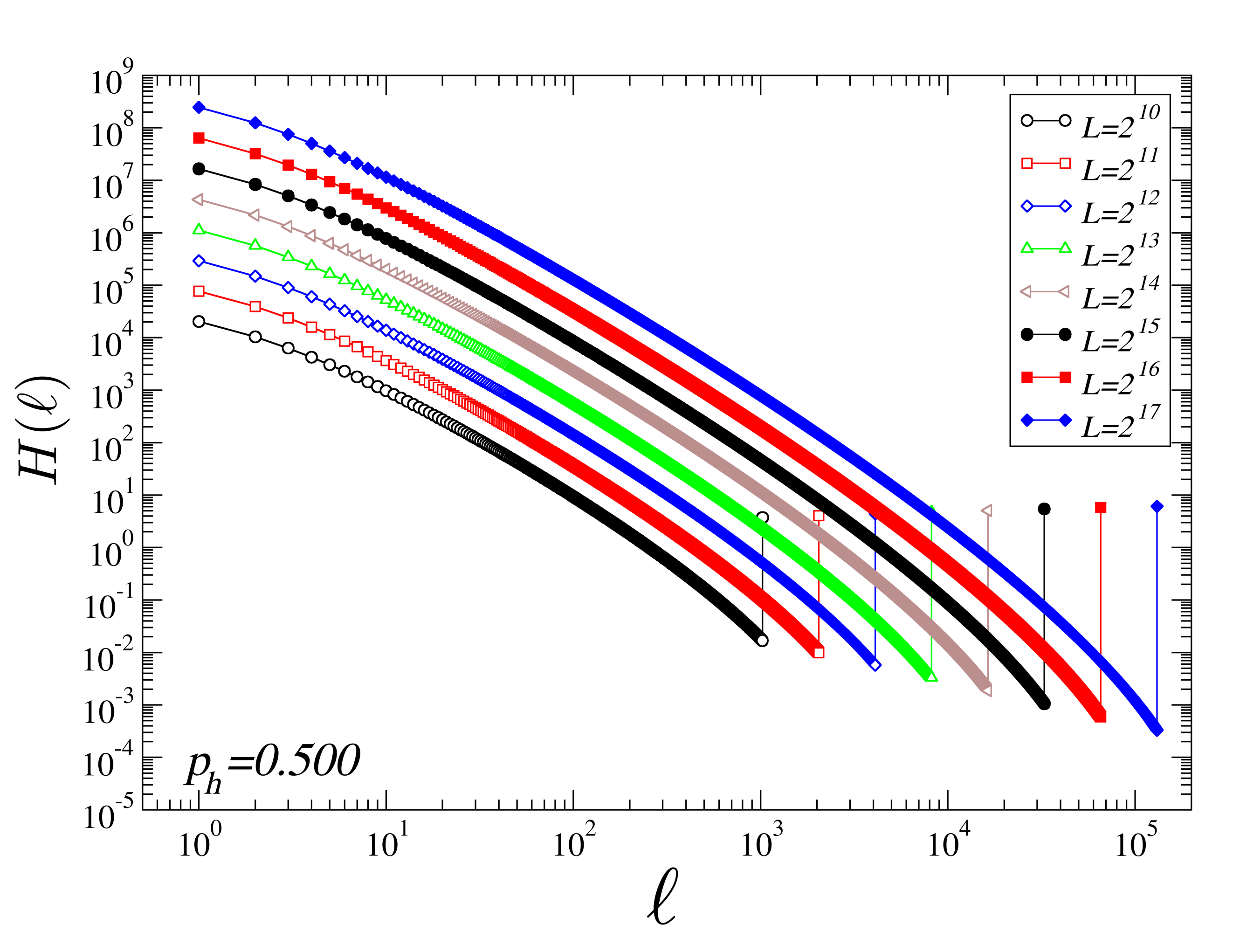
Graphical results

Solution for a fixed $p_h = 0.5$ and different system sizes



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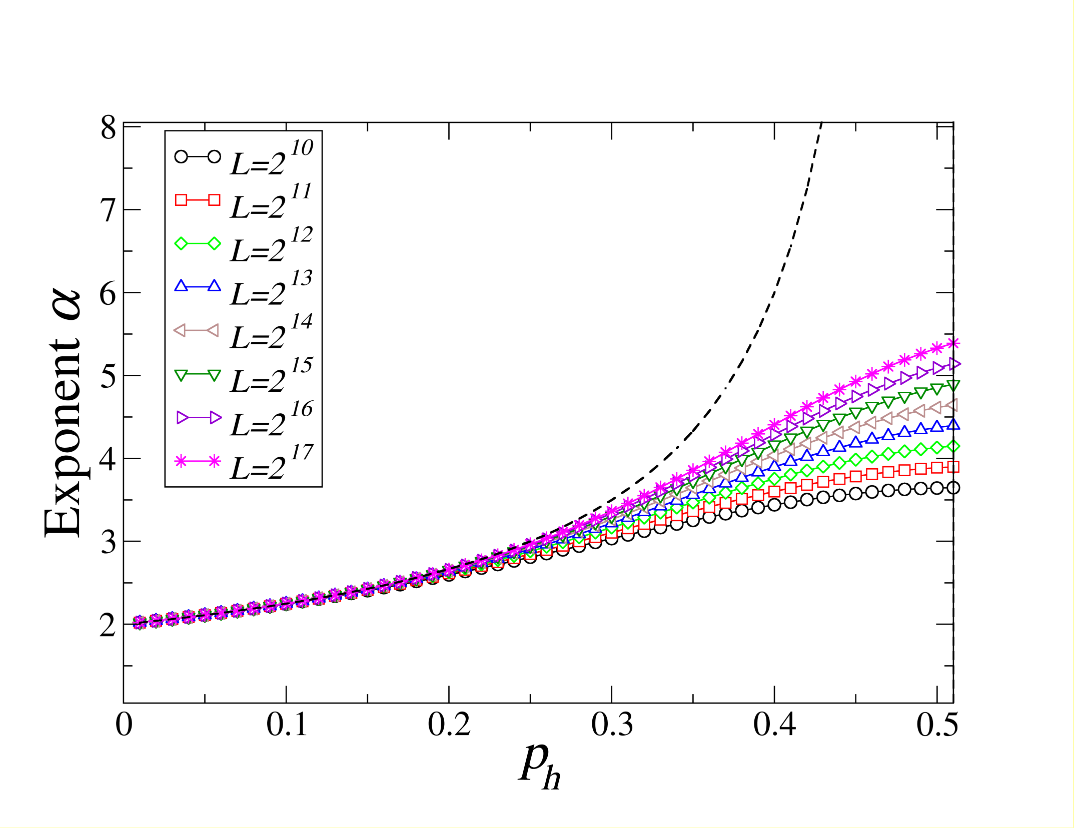
Estimation of the effective power-law exponent

From the values $H(\ell = L_H - 1; L_H, L_V)$ and $H(\ell = L_H - 2; L_H, L_V)$ one can estimate an effective exponent by approaching the logarithmic derivative as:

$$\alpha = -\left.\frac{d\log H}{d\log \ell}\right|_{\ell=L-1} \approx (L-1)\left[\frac{H(\ell = L-2)}{H(\ell = L-1)} - 1\right]$$

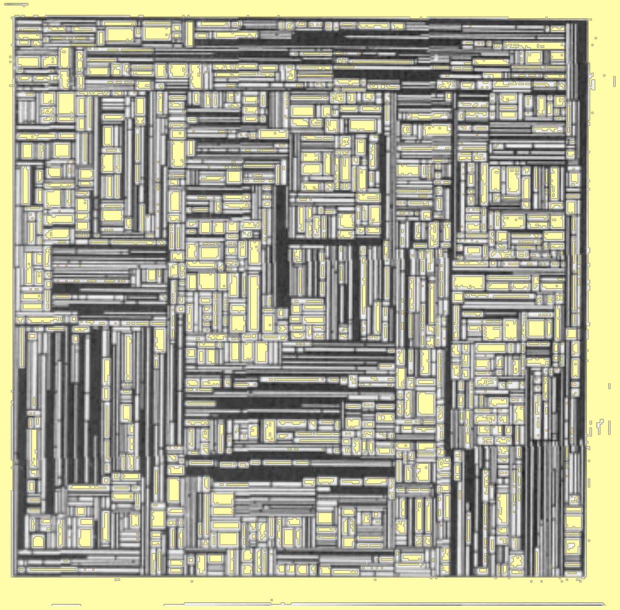


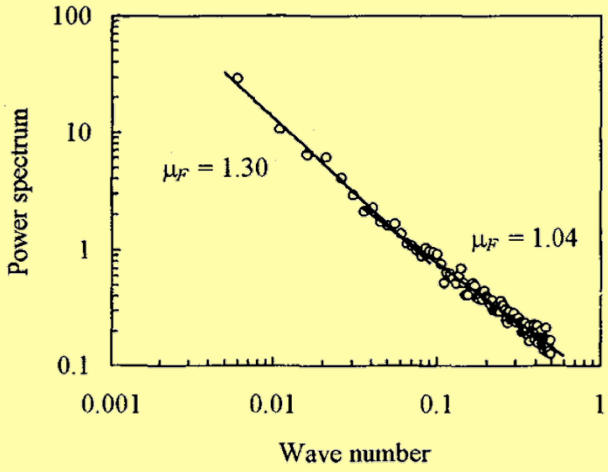
$L_V = L_H = L$
$p_V = 1 - p_h$

PLKK model

A.Y.Pasko, A.A.Likhachev, Y.N.Koval, V.I.Kolomystev, J.PHYS. IV France **7** Colloque C5,435 (1997).

In the present paper we also propose and analyze a simple fractal model for microstructure formation during martensitic transformation. We follow an idea that the fractality and scaling behavior observed in SMA microstructure can be explained as a direct result of impossibility for each new martensitic crystal appeared at any stage of transformation to penetrate the other variant crystals formed before. Therefore the model algorithm is based on the sequence of thin crystals of two possible orientations growing subsequently from randomly distributed "embryo" points up to the intersection with the system of early formed crystals. The resulting microstructure containing about 3000 crystals is shown in Fig. 1(b). Correspondent Fourier spectrum, correlation function and power spectrum radial distribution are presented in Fig. 2(b), 3(b) and 4(b) respectively.





They conclude that the power-law scaling in the large scale region is related to the impossibility of the different variants to penetrate each other

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Conclusions

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Simple geometrical models including only the excluded volume interaction explain the power-law behaviour of the distribution of avalanche sizes

Moreover, if the model includes the acicular geometry and the existence of symmetrically equivalent variants, the power-law exponents depend on the probability of the variants (thus on the number of variants in case of equal probability)

More technical conclusion:

It is unusual to find a solution of a non-trivial model that can be computed analytically for any finite size L and that exhibits a "critical" (power-law) behaviour in the $L \to \infty$ limit.

We are able to sum all the terms of the "partition function" and discover how the length scale a becomes irrelevant

Future: more than 2 variants in 2D
3 variants (planar) in 3D: cubic-tetragonal transformation
RG approach

Announcement

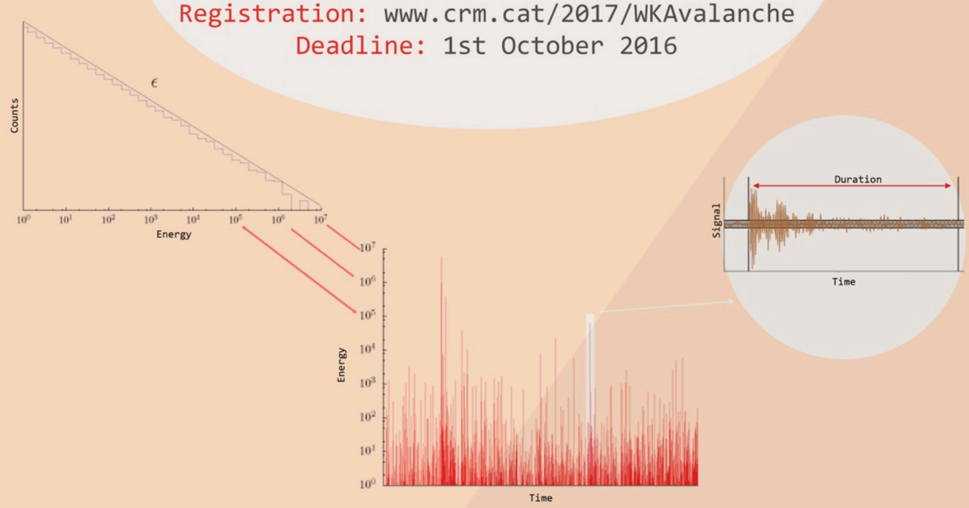
UNIVERSITAT DE BARCELONA

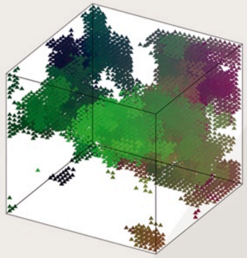
CRM CENTRE DE RECERCA MATEMÀTICA

9th to 13th of January, 2017 at CRM

WORKSHOP ON AVALANCHE PROCESSES IN CONDENSED MATTER PHYSICS AND BEYOND

Registration: www.crm.cat/2017/WKAvalanche
Deadline: 1st October 2016





PLENARY SPEAKERS

Mikko Alava (Aalto University)
Jack Cowan (University of Chicago)
David E. Newman (University of Alaska)
Didier Sornette (ETH Zürich)
Gilles Tarjus (Université Pierre et M. Curie)

ORGANIZING COMMITTEE

Álvaro Corral (Centre de Recerca Matemàtica)
Eduard Vives (Universitat de Barcelona)

INVITED SPEAKERS

Xavier Balandraud
John Beggs
Yehuda Ben-Zion
Karin Dahmen
Joern Davidsen
Gianfranco Durin
Ferenc Kun
Ian Main
Joaquin Marro
Kirsten Martens
Nicholas Moloney
Robert Niemann
F.J. Perez-Reche
John B. Rundle
Ekhard Salje
Konrad Samwer
Stephane Santucci
Avadh Saxena
Doron Shilo
Jordi Soriano
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