A Geometric Description of Large-Scale Atmospheric Flows
(and Navier-Stokes)

Ian Roulstone
University of Surrey

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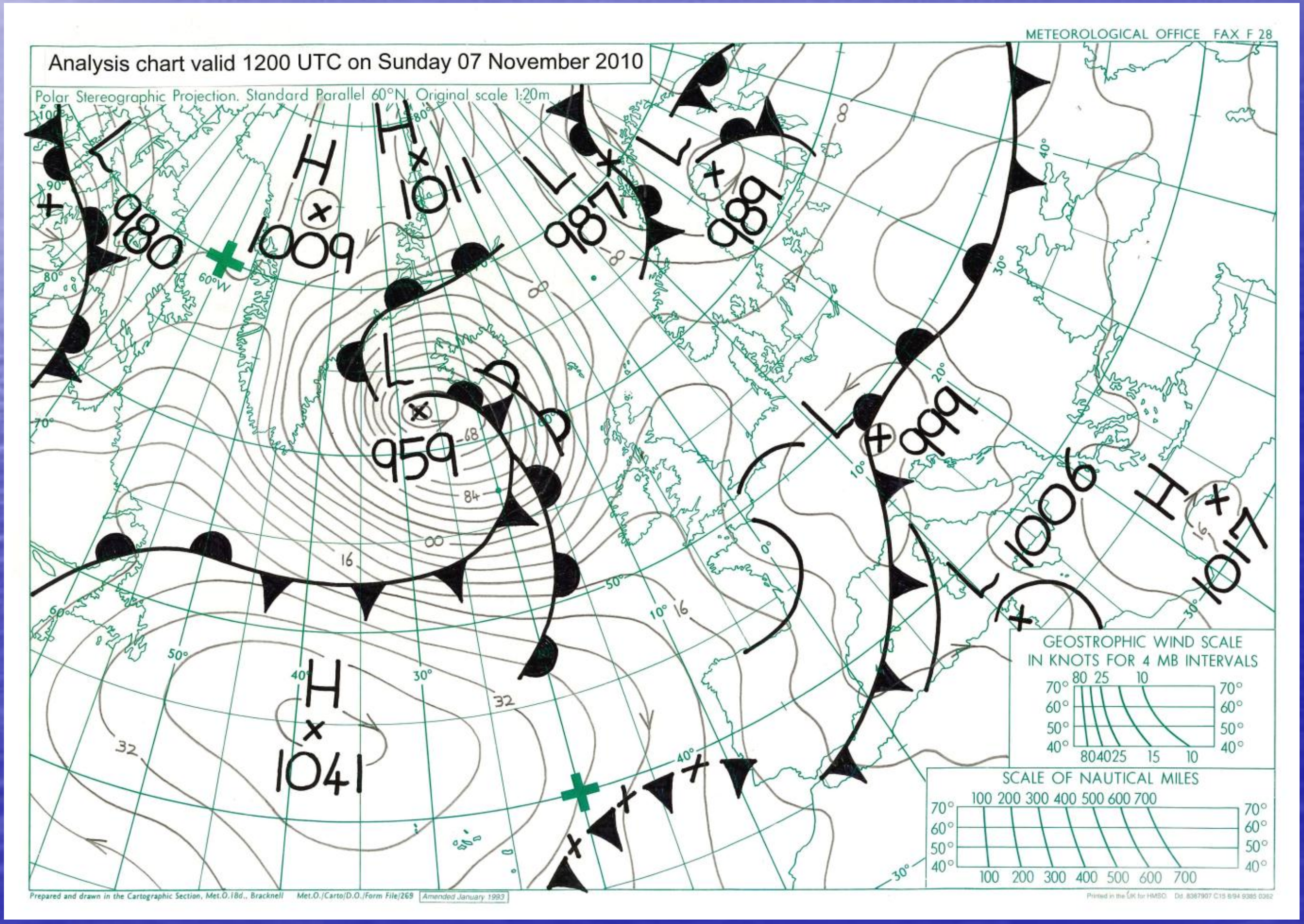
Outline

- Semi-geostrophic theory – Legendre duality and Hamiltonian structure
- Higher-order balanced models
- Complex structures
- Kähler geometry
- Complex manifolds and the incompressible Navier-Stokes equations

Semi-geostrophic theory

- Jets and fronts – two length scales



Semi-geostrophic equations: shallow water

$$\frac{\mathrm{D}u_g}{\mathrm{D}t} - fv + g\frac{\partial h}{\partial x} = 0, \quad \frac{\mathrm{D}v_g}{\mathrm{D}t} + fu + g\frac{\partial h}{\partial y} = 0$$

$$\frac{\mathrm{D}h}{\mathrm{D}t} + h\nabla \cdot \boldsymbol{u} = 0,$$

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla$$

$h(x, y, t)$ **is the depth of the fluid**

$$u_g = -\frac{g}{f}\frac{\partial h}{\partial y}, \quad v_g = \frac{g}{f}\frac{\partial h}{\partial x}$$

Geostrophic wind

Conservation laws

- The SG equations conserve energy and potential vorticity, q

$$q = \frac{1}{h}\left(f + \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} + \frac{1}{f}\frac{\partial(u_g, v_g)}{\partial(x, y)} \right)$$

$$\frac{\mathrm{D}q}{\mathrm{D}t} = 0$$

Geostrophic momentum coordinates

$$\boldsymbol{X} \equiv (X, Y) \equiv \left[x + \frac{v_g}{f}, y - \frac{u_g}{f}\right]$$

Equations of motion become

$$\frac{\mathrm{D}\boldsymbol{X}}{\mathrm{D}t} = \mathbf{u}_g \equiv (u_g, v_g)$$

Potential vorticity

$$q = \frac{f}{h} \frac{\partial (X, Y)}{\partial (x, y)}$$

Legendre transformation

Define $\phi = \frac{g}{f^2} h(x, y, t)$ $P = \frac{1}{2} f \left(x^2 + y^2\right) + \phi$

then $\boldsymbol{X} = \nabla_x P$ $R(\boldsymbol{X}) = \boldsymbol{x} \cdot \boldsymbol{X} - P(\boldsymbol{x})$

and $\boldsymbol{x} = \nabla_X R$ Singularities/Fronts

PV and Monge-Ampère equation

$$q(x, y, t) = \frac{g}{f\phi} \frac{\partial (X, Y)}{\partial (x, y)}$$

Geometric model (Cullen et al, 1984)

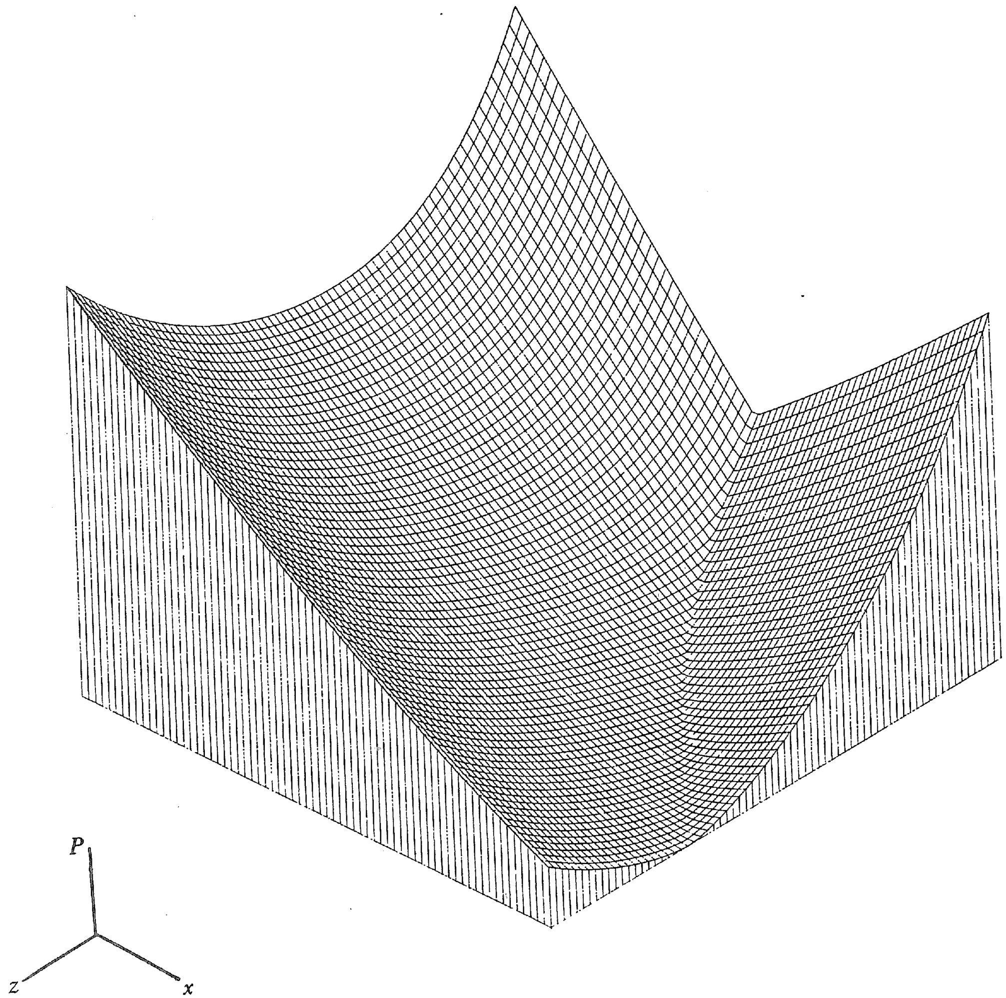


FIGURE 12. Example in §16 of $P[x, z]$ for $q = 1$.

Hamiltonian structure

Define

$$q^{-1} \equiv \rho(\boldsymbol{X}) = \frac{f\phi}{g}\frac{\partial(x,y)}{\partial(X,Y)}$$
$$= \frac{f\phi}{g}(R_{XX}R_{YY} - R_{XY}^2)$$

$$R(X,Y,t) = \Phi - \tfrac{1}{2}(X^2+Y^2)$$

$$\phi = \Phi(X,Y,t) - \tfrac{1}{2}(\Phi_X^2 + \Phi_Y^2)$$

Then

$$\frac{\partial\rho}{\partial t} + \dot{X}\frac{\partial\rho}{\partial X} + \dot{Y}\frac{\partial\rho}{\partial Y} = 0$$

$$\dot{X} = f\frac{\partial\Phi}{\partial Y}, \quad \dot{Y} = -f\frac{\partial\Phi}{\partial X}$$

Higher-order balanced models

There exists a family (Salmon 1985) of balanced models that conserve a PV of the form

$$q = \frac{1}{h}\left(f + \left(\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}\right) + \frac{(1-c^2)}{f}\frac{\partial(u_g, v_g)}{\partial(x, y)}\right)$$

McIntyre and Roulstone (1996)

$$q = \frac{g}{f\phi}\frac{\partial(\mathcal{X}, \overline{\mathcal{Y}})}{\partial(x, y)}$$

$$\mathcal{X} = x + icQ + P, \quad \overline{\mathcal{Y}} = y - icP + Q$$

$$(P = \phi_x, Q = \phi_y)$$

Complex structure

Introduce a symplectic structure

$$(\{x, y; P, Q\}, T^*\mathbb{R}^2, \Omega) \qquad \Omega = \mathrm{d}x \wedge \mathrm{d}P + \mathrm{d}y \wedge \mathrm{d}Q$$

and a two-form

$$\omega = A\mathrm{d}P \wedge \mathrm{d}y + B(\mathrm{d}x \wedge \mathrm{d}P - \mathrm{d}y \wedge \mathrm{d}Q) + C\mathrm{d}x \wedge \mathrm{d}Q + D\mathrm{d}P \wedge \mathrm{d}Q + E\mathrm{d}x \wedge \mathrm{d}y$$

On the graph of φ $\quad P = \phi_x, Q = \phi_y \quad \omega|_\phi = 0$

$$A\phi_{xx} + 2B\phi_{xy} + C\phi_{yy} + D(\phi_{xx}\phi_{yy} - \phi_{xy}^2) + E = 0$$

Define the Pfaffian $\omega \wedge \omega = \mathrm{pf}(\omega)\Omega \wedge \Omega$

$$\mathrm{pf}(\omega) = AC - B^2 - DE$$

$\mathrm{pf}(\omega) > 0$ then ω is *elliptic*, and

$$I_{\mu\nu} = \frac{1}{\sqrt{\mathrm{pf}(\omega)}}\Omega_{\mu\sigma}\omega_{\sigma\nu}$$

$$I_\omega = \frac{1}{\sqrt{\mathrm{pf}(\omega)}}\begin{pmatrix} B & -A & 0 & -D \\ C & -B & D & 0 \\ 0 & E & B & C \\ -E & 0 & -A & -B \end{pmatrix}$$

is an *almost-complex* structure $I_\omega{}^2 = -Id$

$T^*\mathbb{R}^2, \omega, I_\omega$ is an almost-Kähler manifold

(Delahaies & Roulstone, Proc. R. Soc. Lond. 2010)

Incompressible Navier-Stokes

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = \nu \nabla^2 \boldsymbol{u}$$

$$\nabla \cdot \boldsymbol{u} = 0$$

Apply div v = 0

$$-\nabla^2 p = u_{i,j} u_{j,i}$$

2d: Stream function

$$\boldsymbol{u} = \boldsymbol{k} \times \nabla \psi$$

$$\nabla^2 p = -2(\psi_{xy}^2 - \psi_{xx}\psi_{yy})$$

Complex structure

Poisson eqn

$$\omega \equiv \nabla^2 p \, \mathrm{d}x \wedge \mathrm{d}y - 2\mathrm{d}u \wedge \mathrm{d}v$$

Components

$$\omega_{\mu\nu} = \begin{pmatrix} 0 & \nabla^2 p & 0 & 0 \\ -\nabla^2 p & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{pmatrix}$$

Complex structure

$$I_{\mu\nu} = \frac{1}{\sqrt{\mathrm{pf}(\omega)}} \Omega_{\mu\sigma} \omega_{\sigma\nu}$$

$$I_\omega \equiv I_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{\alpha} \\ 0 & 0 & \frac{1}{\alpha} & 0 \\ 0 & -\alpha & 0 & 0 \\ \alpha & 0 & 0 & 0 \end{pmatrix}$$

$$\nabla^2 p \;=\; 2\alpha^2$$

$$I_\omega^2 = -I \text{ if } \nabla^2 p > 0$$

Vorticity and Rate of Strain (Weiss Criterion)

$$\mathcal{Q} = \frac{1}{2}(W_{ij}W_{ij} - S_{ij}S_{ij}) = \frac{1}{4}(\zeta^2 - 2S_{ij}S_{ij})$$

$$S_{ij} = \tfrac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

$$W_{ij} = \tfrac{1}{2}\left(u_{i,j} - u_{j,i}\right)$$

$$\zeta = \nabla \times \boldsymbol{u}$$

$$\mathcal{Q} = -\tfrac{1}{2}u_{i,j}u_{j,i} \qquad -\nabla^2 p = u_{i,j}u_{j,i}$$

$\mathcal{Q}>0$ implies almost-complex structure (ellipticity)

3d Incompressible Flows

PROCEEDINGS OF THE ROYAL SOCIETY A

FirstCite® e-publishing

Proc. R. Soc. A
doi:10.1098/rspa.2008.0483
Published online

A geometric interpretation of coherent structures in Navier–Stokes flows

By I. Roulstone[1,*], B. Banos[2], J. D. Gibbon[3] and V. N. Roubtsov[4,5]

$$\Delta p = -u_{i,j}u_{j,i} = \frac{1}{2}\zeta^2 - \mathrm{Tr}\ S^2$$

- J.D. Gibbon (Physica D 2008 – *Euler, 250 years on*): "The elliptic equation for the pressure is by no means fully understood and *locally* holds the key to the formation of vortical structures through the sign of the Laplacian of pressure. In this relation, which is often thought of as a constraint, may lie a deeper knowledge of the geometry of both the Euler and Navier-Stokes equations…The fact that vortex structures are dynamically favoured may be explained by inherent geometrical properties of the Euler equations but little is known about these features."

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Geometry of 3-forms (Hitchin)

$$\Delta p = -u_{i,j}u_{j,i} = \frac{1}{2}\zeta^2 - \mathrm{Tr}\ S^2$$

$$\varpi = \Delta p\,\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 - 2(\mathrm{d}u_1 \wedge \mathrm{d}u_2 \wedge \mathrm{d}x_3 + \mathrm{d}u_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}u_3 + \mathrm{d}x_1 \wedge \mathrm{d}u_2 \wedge \mathrm{d}u_3)$$

Lychagin-Roubtsov (LR) metric

$$q_\varpi(w, w) \equiv -\frac{1}{4} \perp^2(\iota(w)\varpi \wedge \iota(w)\varpi)$$

$$\perp \varpi = \iota\left(\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial u}\right)\varpi$$

Metric and Pfaffian

$$q_{\varpi} = -2\begin{pmatrix} \frac{1}{2}\Delta p I & 0 \\ 0 & I \end{pmatrix}$$

Construct a linear operator, K_{ω}, using LR metric and symplectic structure

$$K_{\varpi} = -2\begin{pmatrix} 0 & I \\ -\frac{1}{2}\Delta p I & 0 \end{pmatrix}$$

The "pfaffian"

$$\lambda(\varpi) = \frac{1}{6}\mathrm{Tr}\, K_{\varpi}^{2} = -2\Delta p$$

Complex structure

In particular, when $\lambda(\varpi)<0$, the tensor

$$J_{\varpi} = \frac{1}{\sqrt{-\lambda(\varpi)}} K_{\varpi}$$

is an almost-complex structure and the real three-form ϖ is the real part of the complex form

$$\varpi^{\mathrm{c}} = \varpi + \mathrm{i}\hat{\varpi},$$

$$\varpi = (\mu_1 + \mathrm{i}\nu_1) \wedge (\mu_2 + \mathrm{i}\nu_2) \wedge (\mu_3 + \mathrm{i}\nu_3) + (\mu_1 - \mathrm{i}\nu_1) \wedge (\mu_2 - \mathrm{i}\nu_2) \wedge (\mu_3 - \mathrm{i}\nu_3)$$

$$\equiv \alpha + \bar{\alpha}$$

where $\mu_i = (\Delta p/2)^{1/3}\, \mathrm{d}x_i$ and $\nu_i = (\Delta p/2)^{-(1/6)}\, \mathrm{d}u_i$, when $\Delta p > 0$

Summary

- Vorticity-dominated incompressible Euler flows in 2D are associated with almost-Kähler structure – a geometric version of the "Weiss criterion", much studied in turbulence
- Using the geometry of 3-forms in six dimensions, we are able to generalize this criterion to 3D incompressible flows

- These ideas originate in models are large-scale atmospheric flows, in which rotation dominates and an elliptic pde relates the flow velocity to the pressure field
- Roubtsov and Roulstone (1997, 2001) showed how hyper-Kähler structures provide a geometric foundation for understanding Legendre duality (singularity theory), Hamiltonian structure and Monge-Ampère equations, in semi-geostrophic theory