Image Processing and related PDEs
Lecture 1: Introduction to image processing

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On the menu

- Lecture 1 will be a general introduction, with later lectures focussing on specific image processing problems and methods.
- What kind of problems are tackled in image processing?
- What is an image and how to model it?
- Some necessary mathematical background

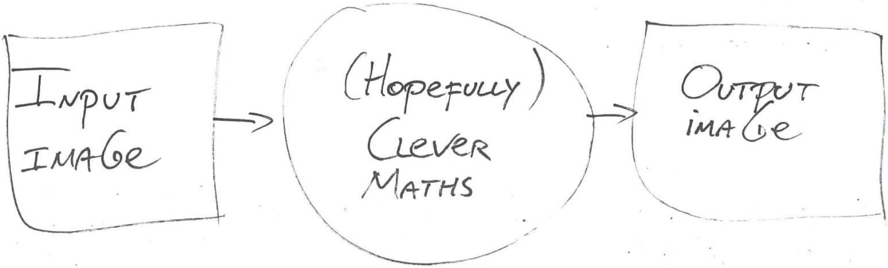
So what's this image processing anyway?

Some not so hard and fast definitions

- Image analysis: get information from images
- Image processing: turn an input image into an output image with desired properties
- Computer vision: gain understanding from images for decision making; have a computer do what humans do, vision wise
- Machine vision: computer vision in an industrial or practical context
- ...

Image processing

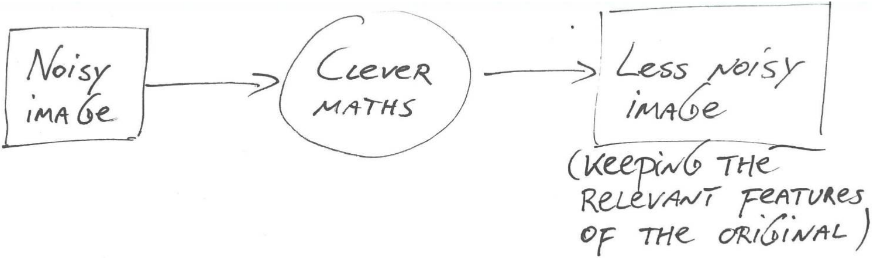
Our focus will mainly be on image processing (but the lines between different 'fields' are blurry...)



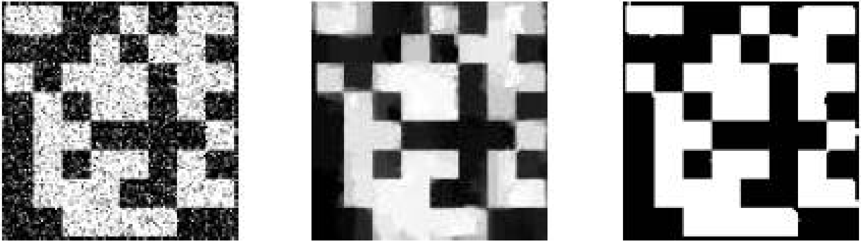
The following slides discuss some standard image processing problems.

Denoising

Image denoising will be addressed in Lecture 2.



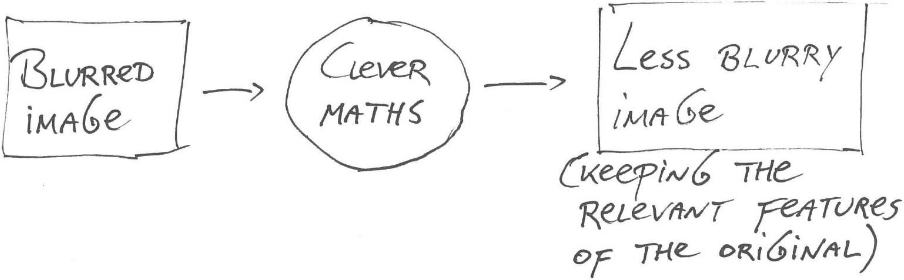
Examples of denoising



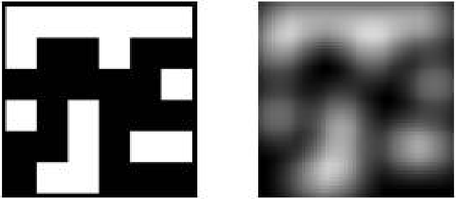
(Choksi, vG, Oberman, 2010)

Deblurring

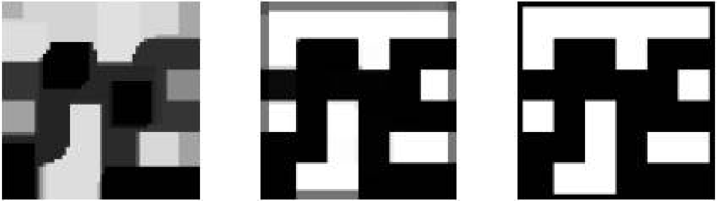
Image deblurring will be addressed in Lecture 3.



Examples of deblurring



(a) Original (b) Blurred



(Choksi, vG, Oberman, 2010)

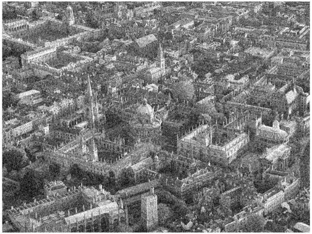
What is noise and what is blur?

There are no precise definitions that cover all cases (that I am aware of), but there are some often recurring characteristics that distinguish the two.

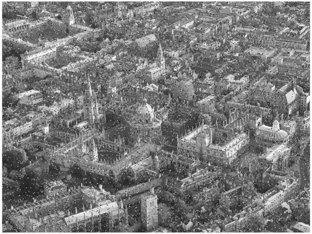
- Noise is often stochastic (at least, it is typically modelled as such) whereas blur is (modelled as being) deterministic.
- Noise adds unwanted high frequencies to the image, whereas blur destroys wanted high frequency information.
- Examples of noise:
  - Gaussian white noise
  - Salt-and-pepper noise
  - Speckle noise
- Examples of blur:
  - Gaussian blur
  - Motion blur
  - Atmospheric blur

Noisy Oxford

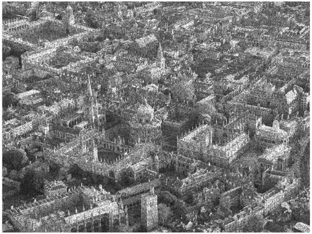
No noise



Gaussian noise



Salt and Pepper noise



Speckle noise

Blurry Oxford (and sign)

No blur

Gaussian blur

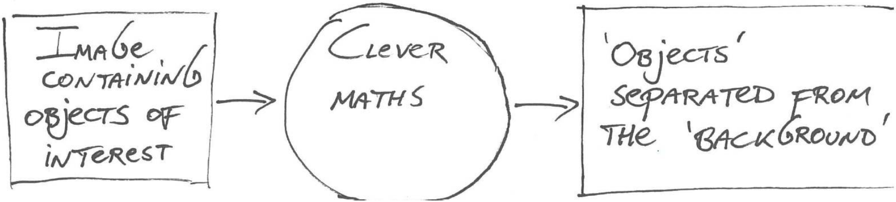
Motion blur

Atmospheric blur[1]

[1] Gilles, Osher, 2012

Segmentation

Image segmentation will be addressed in Lecture 4.

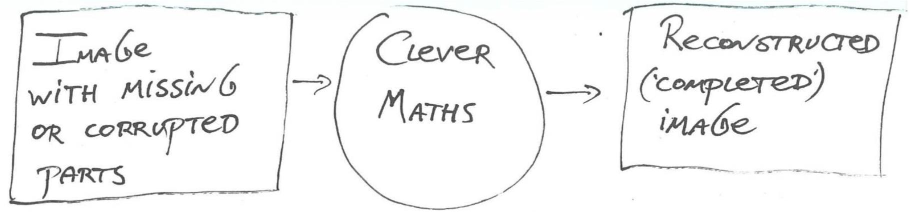


Examples of segmentation

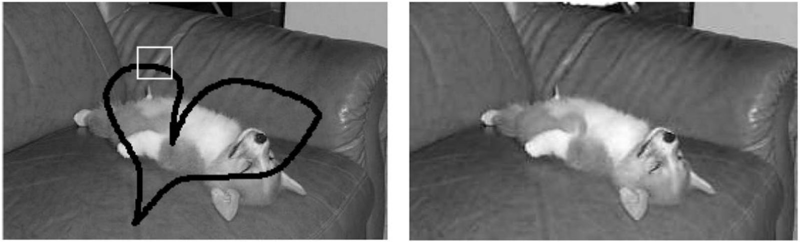
(Calatroni, vG, Schönlieb, Rowland, Flenner, 2016)

Inpainting

We will not discuss image inpainting in these lectures.



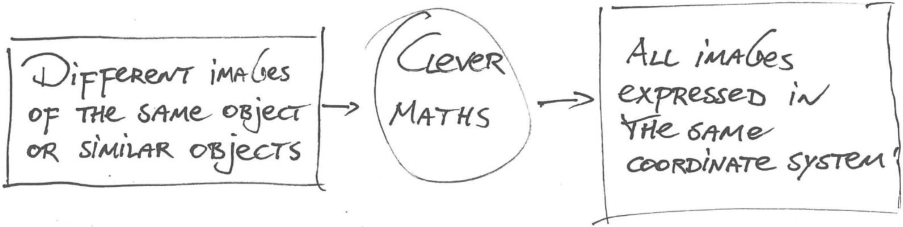
Example of inpainting



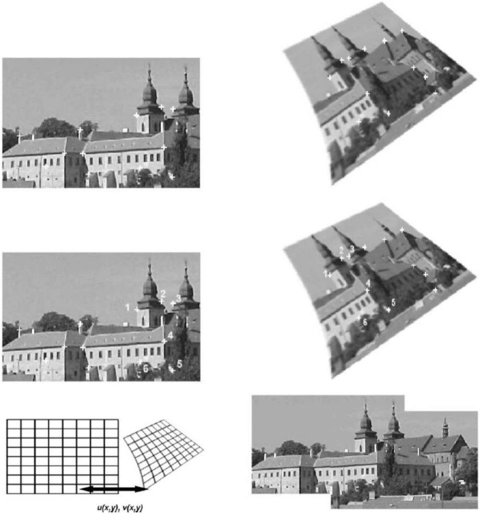
(Schönlieb, Bertozzi, 2011)

Registration

We will not discuss image registration in these lectures.



Example of registration



(Zitová, Flusser, 2003)

What is an image?

Let us restrict ourselves to digital (raster) images:

- $M \times N$ rectangular grid of pixels
- Each pixel is assigned a vector in $\left([0, 255] \cap \mathbb{N}\right)^n$
  - For a greyscale image $n = 1$.
  - For a colour image (typically) $n = 3$ (red, green, blue)
  - Hyperspectral images can have $n \approx 200$

  We will focus on greyscale images in these lectures. We can think of them as $M \times N$ matrices.
- We will (most of the time) consider a continuum description for images. Why?
  - Resolution independent
  - It opens the doors to using the well-developed theoretical machinery from the partial differential equations (PDEs) and variational methods world.

Continuum model for an image

We will focus on deterministic (as opposed to stochastic) descriptions.

- If we wanted to be very general, we could describe images as distributions: an image is determined by its response to all smooth, compactly supported test functions ('sensors').
- However, for our purposes it will suffice to describe (greyscale) images by functions $u : \Omega \to \mathbb{R}$.
- The codomain would be $\mathbb{R}^n$ for colour images ($n = 3$) and hyperspectral images (n large).
- Usually $\Omega = (0, a) \times (0, b) \subset \mathbb{R}^2$, representing the geometry of the image, but sometimes other choices are mathematically expedient, e.g. $\Omega = \mathbb{R}^2$.
- The precise mathematical formulation of the problem at hand will determine the required regularity for the function u. Especially the Banach spaces $L^1(\Omega)$, $L^2(\Omega)$, $BV(\Omega)$, and $SBV(\Omega)$ are important for us. We will introduce the latter two when we need them.

Three steps of image processing

1. Modelling: how to represent the image (what is a suitable Banach space) and the process (clever maths) in mathematical terms?
2. Mathematical analysis: what properties does the mathematical system formulated in point (1) have? Think of: existence and uniqueness of solutions, stability and regularity properties, geometric properties, mathematical measures of how close the solution is to what the image processing objective requires, ...
3. Computation: how can we implement the clever maths (which in our case usually lives in a continuum setting)? Which numerical techniques can be used for simulation?

By restricting ourselves to greyscale images described by deterministic functions in Banach spaces, we have already made quite a few choices for the modelling step. We will mainly focus on point (2), the mathematical analysis, in particular the existence and uniqueness of solutions.

The computational aspects are also very important. After all, we want to be able to actually process images. However, in these lectures we will not spend much, if any, time on point (3). This is just a matter of me choosing a particular focus for this limited series of lectures, not the importance of the subject.

What is to come?

- Lecture 2: Image denoising:
  - Heat equation
  - $W^{1,2}$ regulariser + L^2 fidelity variational method
  - Rudin-Osher-Fatemi (ROF) variational method
- Lecture 3: Image deblurring:
  - Variational non-blind deblurring
  - Variational blind deblurring
- Lecture 4: Image segmentation
  - Mumford-Shah variational segmentation
  - Chan-Vese variational segmentation
  - Graph based variational segmentation

Mathematical background

By far the most important (mathematical) skill you will need to follow these lectures is the one of asking questions when something is unclear. But it will also help if you refresh your memory (if needed) on measure theory (at least to the point where you are familiar with L^p spaces) and functional analysis (weak and strong convergence, compactness, compact Sobolev embeddings (Rellich-Kondrachov), lower semicontinuity).

Bibliography: some good mathematical image analysis references

1. G. Aubert, P. Kornprobst, *Mathematical Problems in Image Processing — Partial Differential Equations and the Calculus of Variations*, Springer, 2006
2. T.F. Chan, J. Shen, *Image Processing and Analysis*, SIAM, 2005
3. C.-B. Schönlieb, *Image Processing — Variational and PDE Methods*, online lecture notes, 2013/14, `http://www.damtp.cam.ac.uk/user/cbs31/Teaching_files/PDEImageLectureNotesLent2014.pdf`

Bibliography: papers referenced in this lecture

- L. Calatroni, Y. van Gennip, C.-B. Schönlieb, H.M. Rowland, A. Flenner, *Graph Clustering, Variational Image Segmentation Methods and Hough Transform Scale Detection for Object Measurement in Images*, J. Math. Imaging Vis, 2016, Springer Online First DOI 10.1007/s10851-016-0678-0
- R. Choksi, Y. van Gennip, A. Oberman, *Anisotropic Total Variation Regularized L^1-Approximation and Denoising/Deblurring of 2D Bar Codes*, Inverse Problems and Imaging, 5(3), 591–617, 2010
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- C.-B. Schönlieb, A.L. Bertozzi, *Unconditionally stable schemes for higher order inpainting*, Commun. Math. Sci 9.2, 413–457, 2011
- B. Zitová, J. Flusser, *Image registration methods: a survey*, Image and Vision Computing 21, 977–1000, 2003