Image Processing and related PDEs
Lecture 3: Image deblurring

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Coming up

- Variational non-blind deblurring
- Variational blind deblurring

Examples of blur

No blur

Gaussian blur

Motion blur

Atmospheric blur[1]

[1] Gilles, Osher, 2012

Modelling blur

Chan, Shen §5.1

Linear blur (and additive noise)

$$f = K(u) + n$$

- f is the observed image, u is the clean image, and n is the noise, as before
- K is a linear operator which models the effect of blur
- Very commonly K will be a convolution operator, i.e.

$$(K(u))(x) = (k * u)(x) = \int_{\mathbb{R}^2} k(x-y)u(y)\,dy,$$

  for some appropriate kernel function k (*point spread function* (PSF)).
- Convolution operators are the only linear, shift-invariant operators (Oppenheim, Willsky, §2). K is shift-invariant if it commutes with all shift operators S_a, where $(S_a f)(x) := f(x+a)$.

More blur modelling

- Want $K(1) = 1$. For convolution operators, this means $\int_{\mathbb{R}^2} k(x)\,dx = 1$.
- Typically, most physical blurs are best modelled by a *lowpass* operator.
- On a finite domain Ω we can still use convolution, but it will no longer be shift-invariant (Chan, Shen §5.1, Schönlieb §3.3) :
  - either introduce an extension operator E, such that Eu is defined on all of $\mathbb{R}^2$ (or at least on a neighbourhood of Ω large enough to accommodate the support of k) and consider the new blurring operator $K \circ E$ (while preserving $K \circ E(1) = 1$),
  - or restrict $k(x-y)$ to Ω^2 and normalise, to satisfy $K(1) = 1$:

$$\tilde{k}(x,y) = \frac{k(x-y)}{\int_\Omega k(x-z)\,dz}, \quad (K(u))(x) := \int_\Omega \tilde{k}(x,y)u(y)\,dy.$$

Ill-posed problem

- If $K : L^2(\Omega) \to L^2(\Omega)$ is a convolution operator, the inverse problem is ill-posed, in the sense that solving $f = Ku$, can lead to very different solutions u, for inputs f that are very $(L^2(\Omega))$ similar. (Tikhonov, Arsenin, §1)
- One way of proving that, is by noting first that K is a compact operator (Conway §4) ...
- ... and second that the inverse of an injective, compact operator between Hilbert spaces is not continuous (Bal §8.1).
- For example, if k is a Gaussian, inverting K will correspond to solving the backwards heat equation, as we saw in the previous lecture!
- This problem occurs in other settings as well, not just when K's domain and codomain are $L^2(\Omega)$.
- When noise is present in f, this will be a problem!

Non-blind and blind deblurring

- In non-blind deblurring, the blurring operator K is assumed to be known.
- In (partially or completely) blind deblurring K is not known:
  - Parametric deblurring: assume K belongs to a parametrised family of operators.
  - Blind deblurring: make only minimal mathematical assumptions on K
  - Data driven blind deblurring: learn the (approximate) K from training data (which can be other images that are believed to have the same blur, or known parts of the current image)

Non-blind deblurring

Deblurring, Chan, Shen §5.4

Given $f \in L^2(\Omega)$, $K : L^1(\Omega) \to L^2(\Omega)$ a bounded, injective, linear operator satisfying $K(1) = 1$, and $\lambda > 0$. Minimise

$$u^* \in \operatorname*{arg\,min}_{u \in BV(\Omega))} F(u),$$

where

$$F(u) := \int_\Omega |\nabla u| + \frac{\lambda}{2} \int_\Omega (K(u) - f)^2.$$

If $K(u) := k * u$ for some kernel $k \in L^2(\Omega)$, then by Young's inequality for convolutions, $K(u) \in L^2(\Omega)$ for all $u \in L^1(\Omega)$.

Existence and uniqueness

Chan, Shen §5.4.2

Let f, K, λ, and F be as on the previous slide. Then there is a unique minimiser

$$u^* := \operatorname*{arg\,min}_{u \in BV(\Omega))} F(u),$$

- Uniqueness follows as in the denoising case (functional G in Lecture 2), since the strict convexity of F is not affected by the linear operator K.
- The existence proof follows in almost the same way as in the denoising case, we just need to work a bit harder to obtain strong convergence in $L^1(\Omega)$. ... TBC...

Existence proof, continued

- We obtain uniform bounds on $\int_\Omega |\nabla u_n|$ and $\|K(u_n)\|_{L^2(\Omega)}$ as before.
- The bound on $\int_\Omega |\nabla u_n|$ combined with Poincaré's inequality for $BV(\Omega)$ functions (Guisti §1, Evans, Gariepy §5.6) gives a uniform bound on $\|u_n - \frac{1}{|\Omega|}\int_\Omega u_n\|_{L^1(\Omega)}$.
- By linearity and $K(1) = 1$, we have $\frac{1}{|\Omega|}\int_\Omega u_n = K(u_n) - K\left(u_n - \frac{1}{|\Omega|}\int_\Omega u_n\right)$.
- By continuity of $K : L^1(\Omega) \to L^2(\Omega)$, the right hand side is $L^2(\Omega)$ bounded, thus also $L^1(\Omega)$ bounded, hence so is $\frac{1}{|\Omega|}\int_\Omega u_n$. We deduce that $\|u_n\|_{L^1(\Omega)}$ is bounded and therefore so is $\|u_n\|_{BV(\Omega)}$.
- The rest of the proof follows as before, because K is continuous.

A Bayesian aside

The variational methods can also be interpreted in a Bayesian maximum a posteriori (MAP) estimation way (Chan, Shen, §5.4.1):

- Bayes' formula: $P(u|f) = \frac{P(f|u)P(u)}{P(f)}$
- Taking logarithms, $L(X) := \log(P(X))$, gives

$$L(u|f) = L(f|u) + L(u) - L(f).$$

- Maximising $P(u|f)$ is equivalent to maximising $L(u|f)$.
- Since $L(f)$ is independent of u, we maximise $L(f|u) + L(u)$.
- If the noise is additive white Gaussian noise (mean 0, standard variation σ), we use $P(u|f) = e^{-p(u)}$, with $p(u) = \frac{1}{2\sigma^2}\int_\Omega (K(u) - f)^2$.
  (We forget about normalisation constants, which will only give additional additive constants in $L(u|f)$.)
- For the prior we choose $P(u) = e^{-q(u)}$, with $q(u) = \int_\Omega |\nabla u|$.
- Hence we *minimise* $p(u) + q(u)$.
- For other noise models, see e.g. Schönlieb §3.2.

Examples

Show example(s) on Image Processing On Line:
`http://www.ipol.im/`

Blind deblurring

Deblurring, Chan, Shen §5.5

Given $f \in L^2(\Omega)$ and $\lambda > 0$, minimise

$$(u^*, K) \in \underset{u \in BV(\Omega)) \times \mathcal{K}}{\arg\min} G(u, K),$$

where $$G(u,K) := \int_\Omega |\nabla u| + \phi(K) + \frac{\lambda}{2} \int_\Omega (K(u) - f)^2.$$

- $\mathcal{K}$ is the set of admissible blur operators. In its most general form we could let it consist of all linear, bounded, injective operators $K : L^1(\Omega) \to L^2(\Omega)$ which satisfy $K(1) = 1$...
- ... but typically we want to restrict $\mathcal{K}$ further, either for mathematical or modelling reasons, e.g.
  - restrict K to be convolution operators with certain regularity ($W^{1,2}$ or BV) imposed on the kernel, or
  - restrict K to a particular class of operators (e.g. convolutions with Gaussian kernels), with only a finite number of free parameters.

...TBC...

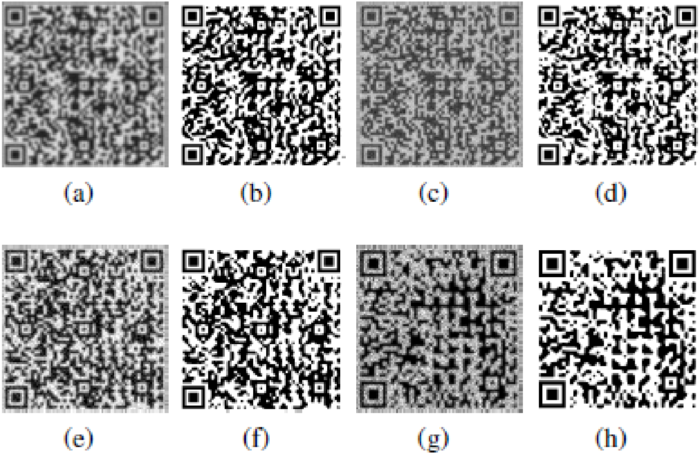
Blind deblurring, continued

- $\phi : \mathcal{K} \to \mathbb{R}$ is a regulariser on the kernel. Its choice will again be driven by a combination of mathematical and modelling concerns. Typical choices are

$$\phi(K) = \int_{\text{supp}(k)} |\nabla k|^2, \quad \text{or} \quad \int_{\text{supp}(k)} |\nabla k|,$$

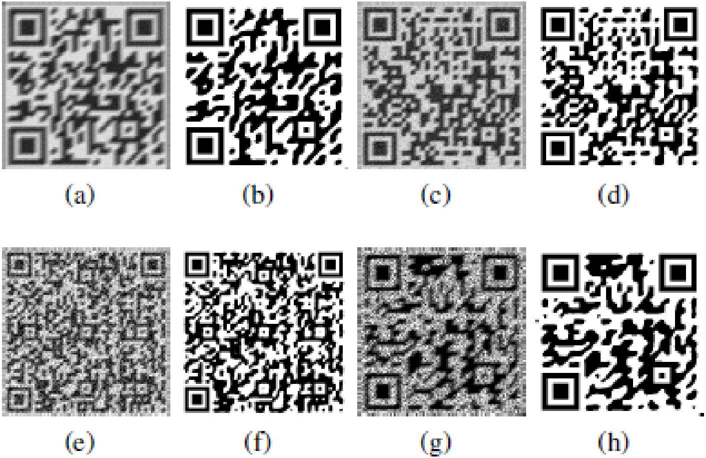
  where K is a convolution operator with kernel function k.
- Chan, Shen §5.5 has an extensive discussion about various existence and uniqueness results. Uniqueness does typically not hold in full generality, as various simultaneous rescalings/translations/phase shifts of u and K can leave the energy unchanged.
- Implementations usually apply an alternating minimisation method, which iterates over minimisations of u and K, respectively, while keeping the other variable fixed.

Using training data to estimate the PSF: noise + Gaussian blur



(vG, Athavale, Gilles, Choksi, 2015)

Using training data to estimate the PSF: noise + motion blur



(vG, Athavale, Gilles, Choksi, 2015)

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