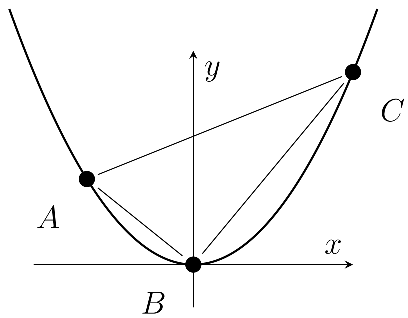
This is an interview question I used in December 2019 for Maths applicants to Oxford.

First part

I've got three points A, B, and C, which all lie on the parabola $y = x^2$. In fact, B is the origin; $B = (0, 0)$. A and C are two other points on the curve; A is somewhere on the left, and C is somewhere on the right.



I'd like triangle ABC to be an equilateral triangle. Where should I put A and C?

Solution and discussion on the next page.

The answer is: put A at $(-\sqrt{3}, 3)$ and C at $(\sqrt{3}, 3)$.

Most people started by saying calling the coordinates of A and C something like (a, a^2) and (c, c^2) respectively, where $a < 0$ and $c > 0$. Then A and C should have the same y-coordinate so that the sides AB and BC have the same length, so $a^2 = c^2$ and so $a = -c$. Now we've got a choice; we could think about angles or side lengths. If you had a go at this question, you could pause now to think about how you could have done it another way.

Method 1: Angles

We want angle ABC to be $60°$, and the picture will be symmetric when it's reflected in the y-axis, so the line BC must make an angle of $60°$ with the x-axis. That means that $\tan 60° = c^2/c$, so $c = \sqrt{3}$.

Method 2: Side Lengths

By Pythagoras, the length of BC is $\sqrt{c^2 + c^4}$, and we want this to be equal to the length of AC, which is $2c$. This gives us an equation to solve;

$$2c = \sqrt{c^2 + c^4}$$

which has solutions $c = 0$ (not much of a triangle) or $c = \sqrt{3}$.

Either way, we've found out where to put C, and then A is symmetrically opposite.

Moving to the next part

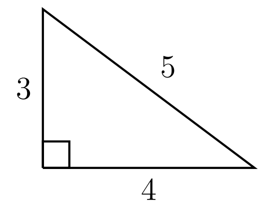
Once we get to this stage, I would ask an open-ended question- what other sorts of triangles do you think we can make by moving A and C? I'm not looking for a precise answer here, I just want to get the interviewee to think about the general set-up again (rather than the special equilateral case we considered above).

Second part on the next page

Second part

Once again, A, B, and C are points on the curve $y = x^2$ with B at the origin. This time, I'd like the triangle ABC to be similar to a 3-4-5 triangle. Where could I put A and C?

(There might be several different solutions this time- I just want one of them!)



Solution and discussion on the next page.

Comments or corrections to `james.munro@maths.ox.ac.uk`

One solution is to put A at $\left(-\left(\frac{3}{4}\right)^{1/3}, \left(\frac{3}{4}\right)^{2/3}\right)$ and C at $\left(\left(\frac{4}{3}\right)^{1/3}, \left(\frac{4}{3}\right)^{2/3}\right)$. There are other solutions.

I think the best starting point is to put the right angle at B and see if we can find a solution like that. Then the lines AB and BC must be perpendicular. Writing (a, a^2) and (c, c^2) for the coordinates of A and C again, the gradients of AB and BC are a and c respectively. The condition for two lines to be perpendicular is that their gradients multiply to minus one, so $ac = -1$.

Now think about the ratio of side lengths; let's look for a solution with the side BC longer by a factor of 4/3 than the side AB. Using Pythagoras, we want

$$\frac{4}{3}\sqrt{a^2 + a^4} = \sqrt{c^2 + c^4}$$

Now use $ac = -1$ to eliminate a and solve for c;

$$\frac{4}{3}\sqrt{\frac{1}{c^2} + \frac{1}{c^4}} = \sqrt{c^2 + c^4}$$

so

$$\frac{4}{3}\sqrt{\frac{c^2 + 1}{c^4}} = c\sqrt{1 + c^2}$$

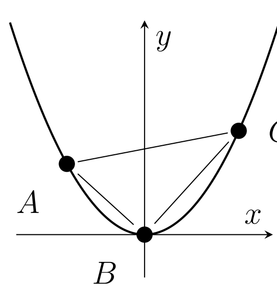
and so $c = 0$ or

$$c^3 = \frac{4}{3}$$

and so

$$c = \left(\frac{4}{3}\right)^{1/3}.$$

This gives the coordinates of C and then, using $a = -1/c$, the coordinates of A.



It's probably worth thinking about other solutions and other methods, but I'm going to leave that to you!