This is an interview question I used in December 2019 for Maths applicants to Oxford.

**First part**

I’ve got three points $A$, $B$, and $C$, which all lie on the parabola $y = x^2$. In fact, $B$ is the origin; $B = (0,0)$. $A$ and $C$ are two other points on the curve; $A$ is somewhere on the left, and $C$ is somewhere on the right.

I’d like triangle $ABC$ to be an equilateral triangle. Where should I put $A$ and $C$?

*Solution and discussion on the next page.*
The answer is: put $A$ at $(-\sqrt{3}, 3)$ and $C$ at $(\sqrt{3}, 3)$.

Most people started by saying calling the coordinates of $A$ and $C$ something like $(a, a^2)$ and $(c, c^2)$ respectively, where $a < 0$ and $c > 0$. Then $A$ and $C$ should have the same $y$-coordinate so that the sides $AB$ and $BC$ have the same length, so $a^2 = c^2$ and so $a = -c$. Now we’ve got a choice; we could think about angles or side lengths. If you had a go at this question, you could pause now to think about how you could have done it another way.

**Method 1: Angles**

We want angle $ABC$ to be $60^\circ$, and the picture will be symmetric when it’s reflected in the $y$-axis, so the line $BC$ must make an angle of $60^\circ$ with the $x$-axis. That means that $\tan 60^\circ = c^2/c$, so $c = \sqrt{3}$.

**Method 2: Side Lengths**

By Pythagoras, the length of $BC$ is $\sqrt{c^2 + c^4}$, and we want this to be equal to the length of $AC$, which is $2c$. This gives us an equation to solve;

$$2c = \sqrt{c^2 + c^4}$$

which has solutions $c = 0$ (not much of a triangle) or $c = \sqrt{3}$.

Either way, we’ve found out where to put $C$, and then $A$ is symmetrically opposite.

**Moving to the next part**

Once we get to this stage, I would ask an open-ended question- what other sorts of triangles do you think we can make by moving $A$ and $C$? I’m not looking for a precise answer here, I just want to get the interviewee to think about the general set-up again (rather than the special equilateral case we considered above).

*Second part on the next page*
Second part

Once again, $A$, $B$, and $C$ are points on the curve $y = x^2$ with $B$ at the origin. This time, I’d like the triangle $ABC$ to be similar to a 3-4-5 triangle. Where could I put $A$ and $C$?

(There might be several different solutions this time- I just want one of them!)

Solution and discussion on the next page.

Comments or corrections to james.munro@maths.ox.ac.uk
One solution is to put $A$ at $\left(-\left(\frac{3}{4}\right)^{1/3},\frac{3}{4}\right)^{2/3}$ and $C$ at $\left(\left(\frac{4}{3}\right)^{1/3},\frac{4}{3}\right)^{2/3}$. There are other solutions.

I think the best starting point is to put the right angle at $B$ and see if we can find a solution like that. Then the lines $AB$ and $BC$ must be perpendicular. Writing $(a, a^2)$ and $(c, c^2)$ for the coordinates of $A$ and $C$ again, the gradients of $AB$ and $BC$ are $a$ and $c$ respectively. The condition for two lines to be perpendicular is that their gradients multiply to minus one, so $ac = -1$.

Now think about the ratio of side lengths; let’s look for a solution with the side $BC$ longer by a factor of $4/3$ than the side $AB$. Using Pythagoras, we want

$$\frac{4}{3}\sqrt{a^2 + a^4} = \sqrt{c^2 + c^4}$$

Now use $ac = -1$ to eliminate $a$ and solve for $c$;

$$\frac{4}{3}\sqrt{\frac{1}{c^2} + \frac{1}{c^4}} = \sqrt{1 + c^2}$$

so

$$\frac{4}{3}\sqrt{\frac{c^2 + 1}{c^4}} = c\sqrt{1 + c^2}$$

and so $c = 0$ or

$$c^3 = \frac{4}{3}$$

and so

$$c = \left(\frac{4}{3}\right)^{1/3}.$$  

This gives the coordinates of $C$ and then, using $a = -1/c$, the coordinates of $A$.

It’s probably worth thinking about other solutions and other methods, but I’m going to leave that to you!

Comments or corrections to james.munro@maths.ox.ac.uk