The Morse–Sard theorem, generalized Luzin property and level sets of Sobolev functions

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The theorems of A.P. Morse (1939) and A. Sard (1942)

- N^n and M^m (second countable) smooth manifolds with $n > m$
- $f\colon N^n \to M^m$ C^k map
- The **critical set** $C_f = \{x \in N^n : \mathrm{rank} Df(x) < m\}$

If $k \geq n - m + 1$**, then** $\mathcal{H}^m(f(C_f)) = 0$**.**

H. Whitney (1935): False if $k = n - m$.

The theorems of A.Ya. Dubovitskiĭ (1966) and H. Federer (1966)

- N^n and M^m (second countable) smooth manifolds
- $f\colon N^n \to M^m$ C^k map
- The d–**critical set** $C_f^d = \{x \in N^n : \operatorname{rank} Df(x) < d\}$, where $d \in \{1, \ldots, n\}$

Then $\mathcal{H}^\beta(f(C_f^d)) = 0$**, when** $\beta \geq d - 1 + \frac{n-d+1}{k}$**.**

$\Rightarrow$ **If** $n > m = d$**, then** $\mathcal{H}^\beta(f(C_f^m)) = 0$**, when** $\beta \geq m - 1 + \frac{n-m+1}{k}$**.**

Result is sharp on C^k **scale**

Improvements to Hölder spaces

- N^n and M^m (second countable) smooth manifolds
- $f\colon N^n \to M^m$ $\mathrm{C}^{k,\alpha}_{\mathrm{loc}}$ map
- The d–**critical set** $C^d_f = \{x \in N^n : \mathrm{rank} Df(x) < d\}$, where $d \in \{1, \dots, n\}$

Then $\mathcal{H}^\beta(f(C^d_f)) = 0$**, when** $\beta \geq d - 1 + \frac{n-d+1}{k+\alpha}$**.**

Result is sharp on $\mathrm{C}^{k,\alpha}_{\mathrm{loc}}$ **scale**

Y. Yomdin (1983), A. Norton (1986), S.M. Bates (1993), C.G.T. De A. Moreira (2001), ...

An example

- If $f\colon \mathbb{R}^2 \to \mathbb{R}$ is $\mathrm{C}^{1,1}_{\mathrm{loc}}$, then $\mathcal{H}^1(f(C_f)) = 0$.
- False when f is only $\mathrm{C}^{1,\alpha}_{\mathrm{loc}}$ for $\alpha < 1$.

Central Cantor sets: C_λ for $0 < \lambda < 1/2$

- remove from $[0,1]$ the central open interval $I_{1,1}$ of proportion $1-2\lambda$
- remove from two remaining intervals the central open intervals $I_{2,1}$ and $I_{2,2}$ of proportion $1-2\lambda$, and repeat ...
- $$C_\lambda = \bigcap_{i=1}^{\infty} \bigcap_{j=1}^{2^{i-1}} [0,1] \setminus I_{i,j}$$

An example

- Let $\phi\colon \mathbb{R} \to [0,1]$ be C^∞ and
$$\phi(t) = \begin{cases} 0 & t \le 0 \\ 1 & t \ge 1. \end{cases}$$
- Let $\frac{1}{4} < \lambda < \frac{1}{2}$.
- Let $I_{ij}^{\lambda} = (a,b) \subset [0,1]$ and $I_{ij}^{\frac{1}{4}} = (c,d) \subset [0,1]$ be corresponding removed intervals in construction of C_λ and $C_{\frac{1}{4}}$, respectively.
- Put
$$g(t) = c + (d-c)\phi\left(\tfrac{t-a}{b-a}\right), \quad t \in (a,b),$$
and $g(t) = 0$ for $t \le 0$, $g(t) = 1$ for $t \ge 1$.
- Extend g by continuity to C_λ; hereby $g\colon \mathbb{R} \to \mathbb{R}$ is continuous and $g(C_\lambda) = C_{\frac{1}{4}}$.

An example

- $g \in \mathrm{C}^{1,\alpha}$ for $\alpha = \ln 4/\ln(1/\lambda) - 1$
- $\ln 4/\ln(1/\lambda) - 1 \nearrow 1$ as $\lambda \nearrow 1/2$
- $g'(t) = 0$ for $t \in C_\lambda$ (since $\phi'(0) = \phi'(1) = 0$)
- Put $f(x,y) = g(x) + 2g(y)$, $(x,y) \in \mathbb{R}^2$.

$\Rightarrow$ f is $\mathrm{C}^{1,\alpha}$, $C_f \supset C_\lambda \times C_\lambda$ so

$$\begin{aligned} f(C_f) &\supseteq f(C_\lambda \times C_\lambda) = g(C_\lambda) + 2g(C_\lambda) \\ &= C_{\frac{1}{4}} + 2C_{\frac{1}{4}} = [0,3] \end{aligned}$$

Other constructions, incl. H. Whitney (1935), R. Kaufman (1979), E.L. Grinberg (1985), T. Körner (1988), G. Comte (1996), ...

Summary

- Let $m < n$ be natural numbers.
- Morse–Sard holds for maps $f\colon \mathbb{R}^n \to \mathbb{R}^m$ of class $\mathrm{C}^{n-m,\alpha}_{\mathrm{loc}}$ iff $\alpha = 1$.
- $\mathrm{C}^{n-m,1}_{\mathrm{loc}} = \mathrm{W}^{n-m+1,\infty}_{\mathrm{loc}}$ (precise representatives)

Extension to Sobolev functions

- L. De Pascale (2001): If $f \in \mathrm{W}^{k,p}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^m)$ and $k \geq n - m + 1$, $p > n$, then $\mathcal{H}^m(f(C_f)) = 0$.
- Note that $\mathrm{W}^{k,p}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^m) \subset \mathrm{C}^1(\mathbb{R}^n, \mathbb{R}^m)$ when $k \geq n - m + 1$, $p > n$.

E.M. Landis (1951), B. Bojarski, P. Hajlasz and P. Strzelecki (2005), D. Pavlica and L. Zajíček (2006), A. Figalli (2008), D. Bucur, A. Giacomini and P. Trebeschi (2008), R. Van der Putten (2012), G. Alberti, S. Bianchini and G. Crippa (2013), P. Hajlasz (2014), G. Alberti, M. Csörnyei, E. D'Aniello and B. Kirchheim, ...

The critical set when the function is not differentiable

The precise representative: of $f \in \mathrm{L}_{\mathrm{loc}}(\mathbb{R}^n)$ is

$$f(x) := \limsup_{r \searrow 0} \frac{1}{\mathcal{H}^n(B_r(x))} \int_{B_r(x)} f(y)\,\mathrm{d}y \tag{1}$$

Then $f \colon \mathbb{R}^n \to [-\infty, \infty]$ Borel and (1) is a limit in $\mathbb{R}$ for $\mathcal{H}^n$ almost all x.

Sobolev functions: $f \in \mathrm{W}^{k,p}_{\mathrm{loc}}(\mathbb{R}^n)$. Then

- f is continuous when $p > 1$ & $kp > n$ or when $p = 1$ & $k = n$
- (1) exists as a limit in $\mathbb{R}$ at $\mathcal{H}^t$ almost all x for each $t > n - kp$ when $p > 1$ & $kp < n$ (and for $\mathcal{H}^{n-k}$ almost all x when $p = 1$ & $n > k$)

The critical set when the function is not differentiable

Let $f \in \mathrm{W}^{n,1}(\mathbb{R}^n)$. Then

- $f \in \mathrm{C}_0(\mathbb{R}^n)$
- there exists a Borel set $E \subset \mathbb{R}^n$ with $\mathcal{H}^1(E) = 0$ such that f is Fréchet differentiable at all $x \in \mathbb{R}^n \setminus E$ with differential

$$Df(x) = \lim_{r \searrow 0} (Df)_{x,r}.$$

The critical set: $C_f := \{x \in \mathbb{R}^n \setminus E : \, Df(x) = 0\}$

The generalized Luzin property

Theorem I (Bourgain, Korobkov & K, 2010, 2012)
Let $f \in \mathrm{W}^{n,1}(\mathbb{R}^n)$. Then

$$\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall A \subset \mathbb{R}^n : \quad \mathcal{H}^1_\infty(A) < \delta \Rightarrow \mathcal{H}^1(f(A)) < \varepsilon.$$

Remarks:

- $\mathcal{H}^1(f(E)) = 0$ when $E = \{x : f \text{ not diff. at } x\}$.
- Theorem I also holds for $f \in \mathrm{B}V^n(\mathbb{R}^n)$.

Morse–Sard for Sobolev functions

Theorem II (Bourgain, Korobkov & K, 2010, 2012)
Let $f \in \mathrm{W}^{n,1}(\mathbb{R}^n)$. Denote

$$E = \{x : \ f \text{ not diff. at } x\}$$

and

$$C_f := \{x \in \mathbb{R}^n \setminus E : \ Df(x) = 0\}.$$

Then $\mathcal{H}^1(f(C_f)) = 0$.

Remark: Theorem II also holds for $f \in \mathrm{B}V^n(\mathbb{R}^n)$. When $n = 2$ we used another definition of the critical set.

Typical level sets for Sobolev functions

Theorem III (Bourgain, Korobkov & K, 2010, 2012)
Let $f \in \mathrm{W}^{n,1}(\mathbb{R}^n)$. For $\mathcal{H}^1$ almost all $y \in \mathbb{R}$ the level set $f^{-1}\{y\}$ is a compact $(n-1)$–dimensional C^1 submanifold in $\mathbb{R}^n$.

Remark: f need not be C^1, but is only differentiable $\mathcal{H}^1$ almost everywhere.

Sobolev–Lorentz functions

Lorentz space: $f \in \mathrm{L}^{p,1}(\mathbb{R}^n)$ if

$$\|f\|_{p,1} = \int_0^\infty \mathcal{H}^n\Big(\{x \in \mathbb{R}^n : |f(x)| > t\}\Big)^{\frac{1}{p}}\,\mathrm{d}t < \infty.$$

- $f \in \mathrm{W}^{k,p,1}(\mathbb{R}^n)$ if $f \in \mathrm{W}^{k,p}(\mathbb{R}^n)$ and $|D^k f| \in \mathrm{L}^{p,1}$.

Proposition:
Let $f \in \mathrm{W}^{k,p,1}(\mathbb{R}^n)$ where $k \in \{2, \dots, n-1\}$ and $p = \frac{n}{k}$. Then
• $f \in \mathrm{C}_0(\mathbb{R}^n)$
• There exists a Borel set $E \subset \mathbb{R}^n$ such that $\mathcal{H}^t(E) = 0$ for $t > \frac{n}{k}$, f is (Fréchet–)differentiable at each $x \in \mathbb{R}^n \setminus E$ with differential $Df(x)$ and x is an L^n–Lebesgue point of the weak derivative Df.

Remark: $k = 1$, $t = p = n$ due to Stein (1970),
$k = n$, $t = p = 1$ due to Dorronsoro (1989)

Results for Sobolev–Lorentz mappings

Let $k,\, d \in \{2, \dots, n\}$ and $f \in \mathrm{W}^{k,\frac{n}{k},1}(\mathbb{R}^n, \mathbb{R}^m)$. Put

$$E = \{x :\ f \text{ not diff. at } x,\ \text{or } x \text{ not } \mathrm{L}^n \text{ Lebesgue point of } Df\}$$

and

$$C_f^d = \{x \in \mathbb{R}^n \setminus E :\ \mathrm{rank} Df(x) < d\}.$$

Theorem (Korobkov & K, 2013)

- For $t > \frac{n}{k}$: $\forall \varepsilon > 0\, \exists \delta > 0$, $\mathcal{H}^t_\infty(A) < \delta \Rightarrow \mathcal{H}^t(f(A)) < \varepsilon$
- $\mathcal{H}^\beta(f(C_f^d)) = 0$ where $\beta \geq d - 1 + \frac{n-d+1}{k}$

Corollary (Korobkov & K, 2013)

- $\mathcal{H}^m(f(C_f^m)) = 0$ when $k = n - m + 1$.

Typical level sets for Sobolev-Lorentz maps

Theorem (Korobkov & K, 2013)
Let $2 \leq m \leq n$, $k = n - m + 1$, and $f \in \mathrm{W}^{k,\frac{n}{k},1}(\mathbb{R}^n, \mathbb{R}^m)$.
Then for $\mathcal{H}^m$ almost all $y \in \mathbb{R}^m$ the level set $f^{-1}\{y\}$ is a compact C^1 $(n-m)$–dimensional submanifold of $\mathbb{R}^n$

Notes.
(1) $\frac{n}{n-m+1} < m$ iff $m < n$ ($\Rightarrow \mathcal{H}^m(f(E)) = 0$)
(2) $\mathcal{H}^m(f(C_f)) = 0$

Comments on proof for Theorem III

Luzin–type theorem: (BKK 2012)
Let $f \in \mathrm{W}^{n,1}(\mathbb{R}^n)$. Then $\forall \varepsilon > 0$ $\exists$ open $U \subset \mathbb{R}^n$ with $\mathcal{H}^1_\infty(U) < \varepsilon$ and $g \in \mathrm{C}^1(\mathbb{R}^n)$ so

$$f = g \text{ and } Dg = Df \text{ on } \mathbb{R}^n \setminus U.$$

Fix $\varepsilon > 0$. By Theorems I & II, $\mathcal{H}^1(f(E \cup C_f) = 0$ so $\exists$ open set $V_1 \supset f(E \cup C_f)$ with $\mathcal{H}^1(V_1) < \varepsilon/2$. For given $\delta > 0$ by Luzin $\exists$ U, g so $\mathcal{H}^1_\infty(U) < \delta$ and

$$f = g \text{ and } Dg = Df \text{ on } \mathbb{R}^n \setminus U.$$

Theorem I allows to take δ so $\mathcal{H}^1(f(U)) < \varepsilon/2$. Take open set $V_2 \supset f(U)$ with $\mathcal{H}^1(V_2) < \varepsilon/2$. Put $V = V_1 \cup V_2$ and record:

$$f(E) \subset V, \quad f|_{f^{-1}(\mathbb{R}\setminus V)} = g|_{f^{-1}(\mathbb{R}\setminus V)}$$

$$Df|_{f^{-1}(\mathbb{R}\setminus V)} = Dg|_{f^{-1}(\mathbb{R}\setminus V)} \quad \mathcal{H}^1(V) < \varepsilon.$$

Comments on proof for Theorem III

Fix $y \in f(\mathbb{R}^n) \setminus V$, $y \neq 0$.
Then

(i) $f^{-1}\{y\}$ compact ($\Leftarrow f \in \mathrm{C}_0(\mathbb{R}^n)$)

(ii) $f^{-1}\{y\} \subseteq g^{-1}\{y\}$

(iii) $Df = Dg \neq 0$ on $f^{-1}\{y\}$

(iv) f diff. at each $x \in f^{-1}\{y\}$ with differential $Df(x)$ and Df has Lebesgue point at x

Claim: $\forall x_0 \in f^{-1}\{y\} \, \exists r > 0$ so
$f^{-1}\{y\} \cap B_r(x_0) = g^{-1}\{y\} \cap B_r(x_0)$
($\Rightarrow$ conclusion)

Comments on proof for Theorem III

Suppose not.
Then $\exists x_0 \in f^{-1}\{y\}$ and $x_j \in g^{-1}\{y\} \setminus f^{-1}\{y\}$ with $x_j \to x_0$.
Put

$$I_x = \left\{ x + \frac{Dg(x_0)}{|Dg(x_0)|} t : \, |t| < r \right\}.$$

Since $g|_{I_x}$ strictly monotone:

$$I_x \cap g^{-1}\{y\} = \{x\} \quad \forall x \in g^{-1}\{y\} \cap B_r(x_0)$$

when $r > 0$ sufficiently small. By (ii) we get for large j:

$$I_{x_j} \cap f^{-1}\{y\} = \emptyset$$

So either $f > y$ or $f < y$ on I_{x_j}. WLOG $f > y$ on I_{x_j}, so by cont. $f \geq y = f(x_0)$ on I_{x_0}. But f diff. at x_0 so $Df(x_0) = 0$ contradicting (iii) and (iv).

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