# On some pointwise properties of functions of bounded variation on metric spaces

Panu Lahti, Aalto University

March 23, 2015

# BV functions and metric measure spaces

- Functions of bounded variation, or BV functions, are a more general class than Sobolev functions, and are a natural class in e.g. many minimization problems
- Various properties of BV functions are well understood in Euclidean spaces and also other settings such as weighted Euclidean spaces, Heisenberg groups etc.
- During the past decade, a theory of BV functions has been developed on general metric measure spaces

# Metric measure space

- Setting: $(X, d, \mu)$ is a complete metric space with metric $d$ and measure $\mu$.
- $\mu$ is a Borel regular outer measure and also *doubling*, meaning that for some constant $c_d \geq 1$ we have

$$0 < \mu(B(x, 2r)) \leq c_d \mu(B(x, r)) < \infty$$

  for all balls $B(x, r)$ in $X$.
- It follows that for every $y \in B(x, R)$ with $r \leq R$, we have

$$\frac{\mu(B(y, r))}{\mu(B(x, R))} \geq C(c_d) \left(\frac{r}{R}\right)^Q$$

  for $Q = Q(c_d) > 1$.

# Curves and upper gradients

A curve $\gamma$ is a rectifiable, continuous mapping from a compact interval $[0, \ell_\gamma]$ into the space $X$, parametrized by arc-length.

**Definition (Heinonen and Koskela, 1998)**

A nonnegative Borel function $g$ on $X$ is an upper gradient of an extended real-valued function $u$ on $X$ if for all curves $\gamma$,

$$|u(x) - u(y)| \leq \int_0^{\ell_\gamma} g(\gamma(s))\, ds,$$

where $x$ and $y$ are the end points of $\gamma$.

# Poincaré inequality

## Definition

The space $X$ supports a $(1,1)$-Poincaré inequality if there exist constants $c_P > 0$ and $\lambda \geq 1$ such that for all locally integrable functions $u$, all upper gradients $g$ of $u$, and all balls $B(x,r)$, we have

$$\fint_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \leq c_P r \fint_{B(x,\lambda r)} g \, d\mu.$$

Here

$$u_{B(x,r)} := \fint_{B(x,r)} u \, d\mu := \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u \, d\mu.$$

# Functions of bounded variation

Let $\Omega \subset X$ be an open set.

## Definition

The *total variation* of $u \in L^1_{\mathrm{loc}}(\Omega)$ is

$$\|Du\|(\Omega)$$
$$:= \inf \left\{ \liminf_{i\to\infty} \int_\Omega g_{u_i}\, d\mu,\ \mathrm{Lip}_{\mathrm{loc}}(\Omega) \ni u_i \to u \text{ in } L^1_{\mathrm{loc}}(\Omega) \right\},$$

where each $g_{u_i}$ is an upper gradient of $u_i$.

We say that a function $u \in L^1(\Omega)$ is in the class $\mathrm{BV}(\Omega)$ if $\|Du\|(\Omega) < \infty$.

# The variation measure

- For an arbitrary set $A \subset X$, we define

$$\|Du\|(A) := \inf\{\|Du\|(\Omega),\, \Omega \text{ open},\, A \subset \Omega\}.$$

- If $u \in \mathrm{BV}(X)$, $\|Du\|$ is a Radon measure on $X$, called the *variation measure* (Miranda 2003).
- For a set $E \subset X$, the *perimeter* of $E$ in $\Omega$ is

$$P(E, \Omega) := \|D\chi_E\|(\Omega).$$

- We also have a BV version of the $(1,1)$-Poincaré inequality.

# Coarea formula

- For any $u \in \mathrm{BV}(\Omega)$, we have

$$\|Du\|(\Omega) = \int_{-\infty}^{\infty} P(\{u > t\}, \Omega)\, dt.$$

- Take $x \in X$, $r > 0$, $u := d(\cdot, x)$, with upper gradient $g \equiv 1$.
- By coarea formula

$$\mu(B(x,2r)) = \int_{B(x,2r)} g\, d\mu \geq \|Du\|(B(x,2r))$$
$$= \int_{-\infty}^{\infty} P(\{u > t\}, B(x,2r))\, dt = \int_{0}^{2r} P(B(x,t), B(x,2r))\, dt$$

- Thus $\exists t \in (r, 2r)$ s.t.

$$P(B(x,t), B(x,2r)) \leq \frac{\mu(B(x,2r))}{r} \leq c_d \frac{\mu(B(x,r))}{r}$$

# Codimension 1 Hausdorff measure

- The codimension 1 Hausdorff measure is defined for $A \subset X$ by

$$\mathcal{H}(A) := \lim_{R \to 0} \mathcal{H}_R(A)$$

with

$$\mathcal{H}_R(A) := \inf \left\{ \sum_{i=1}^{\infty} \frac{\mu(B(x_i, r_i))}{r_i},\ A \subset \bigcup_{i=1}^{\infty} B(x_i, r_i),\ r_i \leq R \right\}.$$

# Absolute continuity of $P(E,\cdot)$ w.r.t. $\mathcal{H}$

- Suppose $K \subset X$ is compact with $\mathcal{H}(K) = 0$.
- For any $\varepsilon > 0$, there exist balls $\cup_{i=1}^n B(x_i, r_i) \supset K$ with

$$\sum_{i=1}^n P(B(x_i, r_i), X) \le c_d \sum_{i=1}^n \frac{\mu(B(x_i, r_i))}{r_i} \le \varepsilon.$$

- Thus for a set of finite perimeter $E$, if we define $E_\varepsilon := E \cup \bigcup_{i=1}^n B(x_i, r_i)$, then

$$\begin{aligned}
&P(E, K) = P(E, X) - P(E, X \setminus K) \\
&\le \liminf_{\varepsilon \to 0} P(E_\varepsilon, X) - P(E, X \setminus K) \\
&= \liminf_{\varepsilon \to 0} P(E_\varepsilon, X \setminus K) - P(E, X \setminus K) \\
&\le \liminf_{\varepsilon \to 0} \left( P(E, X \setminus K) + \sum_{i=1}^n P(B(x_i, r_i), X) - P(E, X \setminus K) \right) \\
&= 0.
\end{aligned}$$

# Structure of sets of finite perimeter

For $E \subset X$, define the measure theoretic boundary $\partial^* E$ as the set of points $x \in X$ where

$$\limsup_{r\to 0} \frac{\mu(B(x,r)\cap E)}{\mu(B(x,r))} > 0 \quad \text{and} \quad \limsup_{r\to 0} \frac{\mu(B(x,r)\setminus E)}{\mu(B(x,r))} > 0.$$

**Theorem (Ambrosio et al., 2003)**

*For a set of finite perimeter $E \subset X$, there are constants $C \geq 1, \gamma \in (0, 1/2]$, only depending on doubling and Poincaré constants, such that*

$$\frac{1}{C}\mathcal{H}(\partial^* E \cap A) \leq P(E, A) \leq C\mathcal{H}(\partial^* E \cap A),$$

*and for $\mathcal{H}$-a.e. $x \in \partial^* E$,*

$$\gamma \leq \liminf_{r\to 0} \frac{\mu(B(x,r)\cap E)}{\mu(B(x,r))} \leq \limsup_{r\to 0} \frac{\mu(B(x,r)\cap E)}{\mu(B(x,r))} \leq 1-\gamma.$$

# Approximate limits of BV functions

- For $u \in \mathrm{BV}(X)$, define the approximate lower and upper limits of $u$ as

$$u^\wedge(x) := \sup\left\{t \in \overline{\mathbb{R}} : \lim_{r\to 0} \frac{\mu(\{u < t\} \cap B(x,r))}{\mu(B(x,r))} = 0\right\},$$

$$u^\vee(x) := \inf\left\{t \in \overline{\mathbb{R}} : \lim_{r\to 0} \frac{\mu(\{u > t\} \cap B(x,r))}{\mu(B(x,r))} = 0\right\}.$$

- Define the jump set $S_u$ as the set where $u^\wedge(x) < u^\vee(x)$.

**Theorem (Kinnunen et al., 2013)**

*For $u \in \mathrm{BV}(X)$ and $\mathcal{H}$-a.e. $x \in X \setminus S_u$, we have*

$$\lim_{r\to 0} \fint_{B(x,r)} |u - u^\vee(x)|^{Q/(Q-1)}\, d\mu = 0.$$

# Behavior in jump set

In the Euclidean setting, for $\mathcal{H}$-a.e. $x \in S_u$ we have

$$\lim_{r \to 0} \fint_{B(x,r) \cap H_\nu^+(x)} |u - u^\vee(x)|^{n/(n-1)} \, d\mathcal{L}^n = 0$$

and

$$\lim_{r \to 0} \fint_{B(x,r) \cap H_\nu^-(x)} |u - u^\wedge(x)|^{n/(n-1)} \, d\mathcal{L}^n = 0$$

for half-spaces $H_\nu^-(x)$, $H_\nu^+(x)$ with normal $\nu \in \mathbb{S}^{n-1}$.

**Theorem (L., 2014)**

*For $u \in \mathrm{BV}(X)$ and $\mathcal{H}$-a.e. $x \in S_u$, there exist $t_l, t_u \in (u^\wedge(x), u^\vee(x))$ such that*

$$\lim_{r \to 0} \fint_{B(x,r) \cap \{u > t_u\}} |u - u^\vee(x)|^{Q/(Q-1)} \, d\mu = 0$$

*and*

$$\lim_{r \to 0} \fint_{B(x,r) \cap \{u < t_l\}} |u - u^\wedge(x)|^{Q/(Q-1)} \, d\mu = 0.$$

# Proof I

- Denote $E_t := \{u > t\}$, $t \in \mathbb{R}$. By the coarea formula, there exists a countable dense set $T \subset \mathbb{R}$ such that for every $t \in T$, $P(E_t, X) < \infty$.
- Define $N$ as the set of points $x \in X$ where for some $t \in T$, we have $x \in \partial^* E_t$ but the condition

  $$\gamma \le \liminf_{r\to 0} \frac{\mu(B(x,r)\cap E_t)}{\mu(B(x,r))} \le \limsup_{r\to 0} \frac{\mu(B(x,r)\cap E_t)}{\mu(B(x,r))} \le 1-\gamma$$

  fails.
- Define similarly $\widetilde{N}$ as the set where for some $s, t \in T$, $x \in \partial^*(E_s \setminus E_t)$ but the above condition fails for $E_s \setminus E_t$.
- Then $\mathcal{H}(N \cup \widetilde{N}) = 0$.

# Proof II

- Now take $x \in S_u \setminus (N \cup \widetilde{N})$ and consider a sequence of numbers $t_1 < t_2 < \ldots \in (u^\wedge(x), u^\vee(x)) \cap T$. Note that $x \in \partial^* E_{t_i}$ for all $i = 1, 2, \ldots$.
- Assume that

$$\limsup_{r \to 0} \frac{\mu(B(x,r) \cap E_{t_i} \setminus E_{t_{i+1}})}{\mu(B(x,r))} > 0$$

for all $i = 1, 2, \ldots$.

- But then $x \in \partial^*(E_{t_i} \setminus E_{t_{i+1}})$ for all $i$, and since $x \notin \widetilde{N}$, we have in fact

$$\liminf_{r \to 0} \frac{\mu(B(x,r) \cap E_{t_i} \setminus E_{t_{i+1}})}{\mu(B(x,r))} \geq \gamma$$

# Proof III

- Thus there exists $t_u \in (u^\wedge(x), u^\vee(x))$ such that

$$\limsup_{r \to 0} \frac{\mu(B(x,r) \cap E_{t_u} \setminus E_t)}{\mu(B(x,r))} = 0$$

  for all $t \in (t_u, u^\vee(x))$.

- Hence for any $\varepsilon > 0$, the level sets $\{u^\vee(x) - \varepsilon < u < u^\vee(x) + \varepsilon\}$ have density 1 in $E_{t_u}$ at $x$.
- Technical calculations involving the Sobolev-Poincaré inequality and the coarea formula then give the result.

# Proof IV

$$
\begin{aligned}
&\limsup_{r\to 0} \fint_{B(x,r)\cap\{u>t_u\}} |u-u^\vee(x)|^{Q/(Q-1)}\,d\mu \\
&\le \limsup_{r\to 0} \frac{1}{\mu(B(x,r)\cap E_{t_u})} \int_{B(x,r)\cap\{u>M\}} |u-u^\vee(x)|^{Q/(Q-1)}\,d\mu + \dots \\
&\le \limsup_{r\to 0} \frac{1}{\mu(B(x,r)\cap E_{t_u})} \int_{B(x,r)} (u-M)_+^{Q/(Q-1)}\,d\mu + \dots \\
&\le C \limsup_{r\to 0} \left( r\frac{\|D(u-M)_+\|(B(x,r))}{\mu(B(x,r))} \right)^{Q/(Q-1)} + \dots
\end{aligned}
$$

By the BV coarea formula, $\|D(u-M)_+\|$ converges to zero as $M\to\infty$.

# The locality condition

- In $\mathbb{R}^n$, Alberti's rank one theorem states that for $u = (u_1, u_2) \in [\mathrm{BV}(\mathbb{R}^n)]^2$,

$$\frac{dDu_1}{d|Du|}(x) \quad \| \quad \frac{dDu_2}{d|Du|}(x)$$

  for $|Du|^s$-a.e. $x \in \mathbb{R}^n$.

- We say that the space $X$ satisfies a *locality condition* if given any sets of finite perimeter $E_1 \subset E_2 \subset X$, we have for $\mathcal{H}$-a.e. $x \in \partial^* E_1 \cap \partial^* E_2$ that

$$\lim_{r \to 0} \frac{\mu(B(x,r) \cap E_2 \setminus E_1)}{\mu(B(x,r))} = 0.$$

- If this condition holds, then in the above theorem we can choose $t_l, t_u$ freely from the interval $(u^\wedge(x), u^\vee(x))$.

# The locality condition II

- Take the space

$$X := \{(x, y) \in \mathbb{R}^2 : x = 0 \text{ or } y = 0\},$$

  equipped with metric inherited from $\mathbb{R}^2$ and the 1-dimensional Hausdorff measure.
- Then the locality condition does not hold for $E_1 := \{x > 0\}$ and $E_2 := \{x > 0\} \cup \{y > 0\}$.
- Similarly we see that we cannot always pick $t_l, t_u$ freely from the interval $(u^\wedge(x), u^\vee(x))$.
- If we assume the locality condition, what kind of analogue of Alberti's rank one theorem could we prove in the metric setting?