Lattice-Based Cryptography

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Lattice-Based Cryptography

\[
y = g^x \mod p
\]

\[
m^e \mod N
\]

\[
N = p \cdot q
\]

\[
e(g^a, g^b)
\]

(Images courtesy xkcd.org)
Lattice-Based Cryptography

\[ L \cdot L = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + bd & bc + d^2 \end{pmatrix} \]

Why?

▶ Efficient: linear, embarrassingly parallel operations
▶ Appears to resist quantum attacks, contra \[\text{Shor'97}\]
▶ Security from mild worst-case assumptions
▶ Solutions to 'holy grail' problems in crypto: FHE and related

(Images courtesy xkcd.org)
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\[
\begin{align*}
N &= p \cdot q \\
y &= g^x \mod p \\
m &= e^x \mod N \\
e(g^a, g^b) &\Rightarrow (Images \ courtesy \ xkcd.org)
\end{align*}
\]

Why?

- **Efficient**: linear, embarrassingly parallel operations
- Appears to resist **quantum** attacks, contra [Shor'97]
- Security from mild **worst-case** assumptions

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(Images courtesy xkcd.org)
This Talk

1. Historical and mathematical background
2. Framework for lattice-based encryption/key exchange
3. Cryptanalysis, parameters, and NIST candidates
Part 1:

Background
A Brief History

1978– Rise and fall of ‘knapsack’ cryptosystems
A Brief History

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1996-7 Ajtai’s worst-case/average-case reduction, one-way function & (with Dwork) public-key encryption (very inefficient)

2002 Micciancio’s ring-based one-way function with worst-case hardness (no encryption)

2005 Regev’s LWE: encryption with worst-case hardness (efficient-ish)

2010– Ring/Module-LWE: efficient encryption, worst-case hardness

2015– Practical implementations of (Ring/Module-)LWE encryption
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What’s a Lattice?

- A periodic ‘grid’ in (subgroup of) $\mathbb{Z}^m$. 

---

Hard Lattice Problems

- ‘Find/detect short’ nonzero lattice vectors.
- Decode a point ‘somewhat near to’ the lattice.

Both seem to require $2^{\Omega(m)}$ time (and space).
What’s a Lattice?

- A periodic ‘grid’ in (subgroup of) $\mathbb{Z}^m$.

- Basis $B = \{b_1, \ldots, b_m\}$:

$$\mathcal{L} = \sum_{i=1}^{m} (\mathbb{Z} \cdot b_i)$$
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(Other representations as well...)

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▶ Decode a point ‘somewhat near to’ the lattice.

▶ Both seem to require $2^{\Omega(m)}$ time (and space).
Shortest Vector Problem: $\text{SVP}_\gamma$ and $\text{GapSVP}_\gamma$

Approximation problems with factor $\gamma = \gamma(n)$:

**Search:** given basis $B$, find nonzero $v \in \mathcal{L}$ s.t. $\|v\| \leq \gamma \cdot \lambda_1(\mathcal{L})$. 

Clearly $\text{GapSVP}_\gamma \leq \text{SVP}_\gamma$, but the reverse direction is open!

**Minkowski:**

$$\min_i \|\tilde{b}_i\| \leq \lambda_1(\mathcal{L}) \leq \sqrt{n \cdot \det(\mathcal{L})}^{1/n},$$

but usually very loose.
Shortest Vector Problem: $\text{SVP}_\gamma$ and $\text{GapSVP}_\gamma$

Approximation problems with factor $\gamma = \gamma(n)$:

**Search:** given basis $B$, find nonzero $v \in \mathcal{L}$ s.t. $\|v\| \leq \gamma \cdot \lambda_1(\mathcal{L})$.

**Decision:** given basis $B$ and real $d$, decide whether $\lambda_1(\mathcal{L}) \leq d$ OR $\lambda_1(\mathcal{L}) > \gamma \cdot d$. 
Shortest Vector Problem: $\text{SVP}_\gamma$ and $\text{GapSVP}_\gamma$

Approximation problems with factor $\gamma = \gamma(n)$:

**Search:** given basis $B$, find nonzero $v \in L$ s.t. $\|v\| \leq \gamma \cdot \lambda_1(L)$.

**Decision:** given basis $B$ and real $d$, decide whether

$$\lambda_1(L) \leq d \ \text{OR} \ \lambda_1(L) > \gamma \cdot d.$$ 

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Minkowski: \( \min_i \|\tilde{\mathbf{b}}_i\| \leq \lambda_1(\mathcal{L}) \leq \sqrt{n} \cdot \det(\mathcal{L})^{1/n} \), but usually very loose.
Bounded-Distance Decoding (BDD)

**Search:** given basis $B$, point $t$, and real $d < \lambda_1/2$ s.t. $\text{dist}(t, \mathcal{L}) \leq d$, find the (unique) $v \in \mathcal{L}$ closest to $t$. 
Bounded-Distance Decoding (BDD)

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Bounded-Distance Decoding (BDD)

**Search:** given basis $B$, point $t$, and real $d < \lambda_1/2$ s.t. $\text{dist}(t, L) \leq d$, find the (unique) $v \in L$ closest to $t$.

**Decision:** given basis $B$, point $t$, and real $d$, decide whether

$$\text{dist}(t, L) \leq d \quad \text{OR} \quad > \gamma \cdot d.$$
A Central Hard Problem: Learning With Errors [Regev’05]

- Parameters: dimension $n$, modulus $q$, error distribution $\chi$
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- Parameters: dimension $n$, modulus $q$, error distribution $\chi$
- **Search**: find secret $s \in \mathbb{Z}_q^n$ given many ‘noisy inner products’

\[
\begin{align*}
    a_1 &\leftarrow \mathbb{Z}_q^n , \quad b_1 \approx \langle s , a_1 \rangle \mod q \\
    a_2 &\leftarrow \mathbb{Z}_q^n , \quad b_2 \approx \langle s , a_2 \rangle \mod q \\
    &\vdots \\
    a_m &\leftarrow \mathbb{Z}_q^n , \quad b_m \approx \langle s , a_m \rangle \mod q
\end{align*}
\]
A Central Hard Problem: Learning With Errors [Regev’05]

- **Parameters:** dimension $n$, modulus $q$, error distribution $\chi$
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\[
\begin{align*}
\mathbf{a}_1 &\leftarrow \mathbb{Z}_q^n, & \mathbf{b}_1 &= \langle s, \mathbf{a}_1 \rangle + e_1 \in \mathbb{Z}_q \\
\mathbf{a}_2 &\leftarrow \mathbb{Z}_q^n, & \mathbf{b}_2 &= \langle s, \mathbf{a}_2 \rangle + e_2 \in \mathbb{Z}_q \\
\vdots & & \\
\mathbf{a}_m &\leftarrow \mathbb{Z}_q^n, & \mathbf{b}_m &= \langle s, \mathbf{a}_m \rangle + e_m \in \mathbb{Z}_q
\end{align*}
\]

- e.g. width $\sqrt{n} \ll q$, ‘rate’ $\alpha$

LWE is Hard \((n/\alpha)\) -approx worst case GapSVP etc.

- Classical reductions for alt. problems & params [Peikert’09, BLPRS’13]
A Central Hard Problem: Learning With Errors [Regev’05]

- **Parameters:** dimension $n$, modulus $q$, error distribution $\chi$
- **Search:** find secret $s \in \mathbb{Z}_q^n$ given many ‘noisy inner products’

\[
\begin{pmatrix}
\cdots & A & \cdots \\
\hline
m
\end{pmatrix}, \quad \begin{pmatrix}
\cdots & b^t & \cdots \\
\hline
\end{pmatrix} = s^t A + e^t
\]

- e.g. width $\sqrt{n} \ll q$, ‘rate’ $\alpha$

\[\text{LWE is Hard} \begin{pmatrix} n/\alpha \\
\end{pmatrix}-\text{approx worst case}\]
\[\text{GapSVP etc.} \leq \begin{pmatrix} \text{quantum [R’05]} \end{pmatrix}\]
\[\text{search-LWE} \leq \begin{pmatrix} \text{[BFKL’93, R’05, . . .]} \end{pmatrix}\]
\[\text{decision-LWE} \leq \begin{pmatrix} \text{crypto} \end{pmatrix}\]

Classical reductions for alt. problems & params
\[\text{[Peikert’09, BLPRS’13]}\]
A Central Hard Problem: Learning With Errors [Regev’05]

- **Parameters:** dimension $n$, modulus $q$, error distribution $\chi$

- **Search:** find secret $s \in \mathbb{Z}_q^n$ given many ‘noisy inner products’

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\begin{pmatrix}
\cdots & b^t & \cdots \\
\end{pmatrix} = s^tA + e^t
\]

- **Example:** width $\sqrt{n} \ll q$, ‘rate’ $\alpha$

- **Decision:** distinguish $(A, b)$ from uniform $(A, b)$
A Central Hard Problem: Learning With Errors [Regev’05]

- Parameters: dimension $n$, modulus $q$, error distribution $\chi$
- **Search**: find $\text{secret } s \in \mathbb{Z}_q^n$ given many ‘noisy inner products’

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\quad , \\
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- e.g. width $\sqrt{n} \ll q$, ‘rate’ $\alpha$
- **Decision**: distinguish $(A, b)$ from uniform $(A, b)$

**LWE is Hard**

$(n/\alpha)$-approx worst case \[ \text{GapSVP etc.} \leq \text{search-LWE} \leq \text{decision-LWE} \leq \text{crypto} \]

(quantum [R’05]) \[ [\text{BFKL’93, R’05, \ldots}] \]
A Central Hard Problem: Learning With Errors [Regev’05]

- **Parameters:** dimension $n$, modulus $q$, error distribution $\chi$
- **Search:** find secret $s \in \mathbb{Z}^n_q$ given many ‘noisy inner products’

\[
\begin{pmatrix}
\cdots & A & \cdots \\
\end{pmatrix}
m,
\begin{pmatrix}
\cdots & b^t & \cdots \\
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= s^t A + e^t
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- **Decision:** distinguish $(A, b)$ from uniform $(A, b)$

LWE is Hard

- $(n/\alpha)$-approx worst case GapSVP etc. $\leq$ search-LWE $\leq$ decision-LWE $\leq$ crypto

  (quantum [R’05]) $\uparrow$ [BFKL’93, R’05, ...]

- **Classical** reductions for alt. problems & params [Peikert’09, BLPRS’13]
LWE as a Lattice Problem

\[
\begin{pmatrix}
\cdots & A & \cdots \\
\end{pmatrix}
\in \mathbb{Z}_q^{n \times m} \quad , \quad b^t = s^t A + e^t \quad \text{OR} \quad b \leftarrow \mathbb{Z}_q^m
\]
LWE as a Lattice Problem

\[
\begin{pmatrix}
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\end{pmatrix} \in \mathbb{Z}_q^{n \times m}, \quad b^t = s^t A + e^t \quad \text{OR} \quad b \leftarrow \mathbb{Z}_q^m
\]

Lattice interpretation:

\[
\mathcal{L}(A) = \{ z^t \equiv s^t A \mod q \}
\]

Finding \( s, e \): BDD on \( \mathcal{L}(A) \).

Distinguishing \( b \) from \( b \): decision-BDD.
LWE as a Lattice Problem

\[
\begin{pmatrix}
\ldots & A & \ldots
\end{pmatrix} \in \mathbb{Z}_q^{n \times m}, \quad b^t = s^t A + e^t \quad \text{OR} \quad b \leftarrow \mathbb{Z}_q^m
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\[m\]

- Lattice interpretation:
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Finding \(s, e\): BDD on \(\mathcal{L}(A)\).

Distinguishing \(b\) from \(\hat{b}\): decision-BDD.

- WLOG, ‘normal form’ short \(s \leftarrow \chi^n\) with entries from error distribution [ACPS’09]
Learning With Rounding [BanerjeePeikertRosen’12]

- Generate errors deterministically by rounding \( \mathbb{Z}_q \) to a “sparser” subset (e.g., a subgroup).
Learning With Rounding [BanerjeePeikertRosen'12]

- Generate errors deterministically by rounding \( \mathbb{Z}_q \) to a “sparser” subset (e.g., a subgroup).

Let \( p < q \) and define \( \lfloor x \rfloor_p = \lfloor (p/q) \cdot x \rfloor \mod p \).
Learning With Rounding [BanerjeePeikertRosen'12]

- Generate errors deterministically by rounding $\mathbb{Z}_q$ to a “sparser” subset (e.g., a subgroup).

Let $p < q$ and define $[x]_p = \lfloor (p/q) \cdot x \rfloor \mod p$.

- **Decision-LWR problem**: for secret $s \in \mathbb{Z}_q^n$, distinguish $m$ pairs

  $a_i \leftarrow \mathbb{Z}_q^n$, $\lfloor \langle s, a_i \rangle \rfloor_p \in \mathbb{Z}_q^n \times \mathbb{Z}_p$ from uniform.
Generate errors deterministically by rounding $\mathbb{Z}_q$ to a “sparser” subset (e.g., a subgroup).

Let $p < q$ and define $\lfloor x \rfloor_p = \lfloor (p/q) \cdot x \rfloor \mod p$.

Decision-LWR problem: for secret $s \in \mathbb{Z}_q^n$, distinguish $m$ pairs

$$a_i \leftarrow \mathbb{Z}_q^n, \ \lfloor \langle s, a_i \rangle \rfloor_p \in \mathbb{Z}_q^n \times \mathbb{Z}_p$$

from uniform.

LWE conceals low-order bits of $\langle s, a_i \rangle$ by adding small random error. LWR just discards those bits instead.
Learning With Rounding [BanerjeePeikertRosen’12]

- Generate errors deterministically by rounding $\mathbb{Z}_q$ to a “sparser” subset (e.g., a subgroup).
  
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  LWE conceals low-order bits of $\langle s, a_i \rangle$ by adding small random error. LWR just discards those bits instead.

- [BPR’12,AKPW’13] proves that $\text{LWE} \leq \text{LWR}$ for $q \geq p \cdot \text{poly}(m)$ . . .
  
  . . . but LWR appears hard for more aggressive parameters.
  
  How aggressive? Not well understood.
LWE/LWR are (Extremely) Versatile

What kinds of crypto can we do with LWE/LWR?

- Key Exchange
- Public Key Encryption
- Oblivious Transfer
- Chosen Ciphertext-Secure Encryption (w/o random oracles)
- Symmetric Crypto: (Constrained & Key-Homomorphic) PRFs
- Identity-Based Encryption (w/o RO)
- Hierarchical ID-Based Encryption (w/o RO)
- NIZK for any NP language
- Fully Homomorphic Encryption
- Attribute-Based & Predicate Encryption for arbitrary policies
- And much, much more...
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✓ Oblivious Transfer
✓ Chosen Ciphertext-Secure Encryption (w/o random oracles)
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✔ Oblivious Transfer
✔ Chosen Ciphertext-Secure Encryption (w/o random oracles)
✔ Symmetric Crypto: (Constrained & Key-Homomorphic) PRFs

✔✔ Identity-Based Encryption (w/o RO)
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✔✔ NIZK for any NP language

!!! Fully Homomorphic Encryption
!!! Attribute-Based & Predicate Encryption for arbitrary policies

and much, much more…
Part 2:
Framework for Lattice-Based Encryption
LWE-Based Encryption/Key Ex [Regev’05, PVW’08, LPS’10, LP’11, ...]

\[ A \leftarrow \mathbb{Z}_{q}^{n \times n} \]

(can be shared and/or expanded from a seed)

\[ U \approx RA \]

(public key)

short \( R \leftarrow \chi^{k \times n} \)
LWE-Based Encryption/Key Ex [Regev’05, PVW’08, LPS’10, LP’11, ...]

\[
\begin{align*}
A & \leftarrow \mathbb{Z}_q^{n \times n} \\
U & \approx RA \\
V & \approx AS \\
S & \leftarrow \chi^{n \times \ell} \\
M & \in \mathbb{Z}_p^{k \times \ell}
\end{align*}
\]

(can be shared and/or expanded from a seed)

(public key)

(ciphertext ‘preamble’)

(msg \(M\))
LWE-Based Encryption/Key Ex [Regev’05, PVW’08, LPS’10, LP’11, . . . ]

short \( R \leftarrow \mathcal{X}^{k \times n} \)

\( A \leftarrow \mathbb{Z}_q^{n \times n} \) (can be shared and/or expanded from a seed)

\( U \approx RA \) (public key)

\( V \approx AS \) (ciphertext ‘preamble’)

\( R \approx RAV \)

\( C \approx US + \frac{q}{p} \cdot M \) (ciphertext ‘payload’)

\( RV \approx RAS \)

\( US \approx RAS \in \mathbb{Z}_q^{k \times \ell} \)

short \( S \leftarrow \mathcal{X}^{n \times \ell} \)

msg \( M \in \mathbb{Z}_p^{k \times \ell} \)
LWE-Based Encryption/Key Ex \[\text{[Regev'05, PVW'08, LPS'10, LP'11, ...]}\]

\[
\begin{align*}
A & \leftarrow \mathbb{Z}_q^{n \times n} \\
U & \approx RA \\
V & \approx AS \\
\mathsf{msg} \; M & \in \mathbb{Z}_p^{k \times \ell} \\
\mathsf{C} & \approx US + \frac{q}{p} \cdot M \\
\mathsf{US} & \approx RAS \in \mathbb{Z}_q^{k \times \ell}
\end{align*}
\]

(short \(R \leftarrow \chi^{k \times n}\))

\[
\begin{align*}
\mathsf{RV} & \approx RAS \\
\mathsf{S} & \leftarrow \chi^{n \times \ell} \\
(A, U, V, C)
\end{align*}
\]

(can be shared and/or expanded from a seed)

(public key)

(ciphertext ‘preamble’)

(ciphertext ‘payload’)
LWE-Based Encryption/Key Ex [Regev’05, PVW’08, LPS’10, LP’11, . . . ]

A ← \mathbb{Z}_q^{n \times n}  
\quad (\text{can be shared and/or expanded from a seed})

U \approx RA  
\quad (\text{public key})

V \approx AS  
\quad (\text{ciphertext ‘preamble’})

C \approx US + \frac{q}{p} \cdot M  
\quad (\text{ciphertext ‘payload’})

US \approx RAS \in \mathbb{Z}_q^{k \times \ell}

RV \approx RAS

short R \leftarrow \chi^{k \times n}

msg M \in \mathbb{Z}_p^{k \times \ell}

short S \leftarrow \chi^{n \times \ell}

by decision-LWE
LWE-Based Encryption/Key Ex [Regev’05, PVW’08, LPS’10, LP’11, …]

\[ \text{short } R \leftarrow \chi^{k \times n} \]

\[ A \leftarrow \mathbb{Z}_{q}^{n \times n} \]

\[ U \approx RA \] (public key)

\[ V \approx AS \] (ciphertext ‘preamble’)

\[ \text{short } S \leftarrow \chi^{n \times \ell} \]

\[ \text{msg } M \in \mathbb{Z}_{p}^{k \times \ell} \]

\[ RV \approx RAS \]

\[ C \approx US + \frac{q}{p} \cdot M \] (ciphertext ‘payload’)

\[ US \approx RAS \in \mathbb{Z}_{q}^{k \times \ell} \]

\[ \text{(can be shared and/or expanded from a seed)} \]

\[ \text{by decision-LWE} \]

\[ (A, U, V, C) \]
Design Considerations

1. System as shown is only CPA secure. Good for ephemeral key-ex, but needs a Fujisaki–Okamoto-like transform for CCA-secure KEM. An active area of research; mostly orthogonal to other design aspects.

2. Share $A$ across many public keys? May allow (expensive) preprocessing, making it easier to break many public keys at once.

3. Use random errors, or deterministic rounding? Rounding makes keys/ciphertexts smaller; security is less understood.

4. How large can/should errors be? ⋆ All else being equal, larger $|\varepsilon|/q = \Rightarrow$ more security. ⋆ But need entries of $\text{RE} - E' S + E''$ to have magnitudes $< q^2 p$, with high probability. So $q > p |\varepsilon|^2$.

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$$a_i \leftarrow R_q, \; b_i \approx s \cdot a_i \in R_q.$$
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  Also (weaker) worst-case hardness theorems based on ideal lattices.
Ring-LWE public keys \((a, b)\) satisfy the \textit{inhomogeneous} relation

\[ a \cdot s - b \approx 0 \in R_q. \]
NTRU [HoffsteinPipherSilverman'96,…]

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- Encryption is similar: choose short \(t\) and send \(c \approx t \cdot a + \frac{q}{p} \cdot m \in R_q\).
  (Just one ring element!)

  Decryption:
  \[
  c \cdot s \approx t \cdot a \cdot s + \frac{q}{p} \cdot m \cdot s \approx \frac{q}{p} \cdot m \cdot s,
  \]
  from which we can recover \(m\).
Part 3:

Cryptanalysis, Parameters, and NIST Candidates
Lattice Attacks

- Standard approach: given $[A \mid b = As + e]$, find the (unique mod $\pm$) ‘unusually short’ vector $(s, e, 1)$ in the lattice

$$\mathcal{L} = \{x : [A \mid -I \mid -b] \cdot x = 0\}.$$
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Core-SVP Methodology

Use Block Korkin-Zolotarev (BKZ) with large enough block size \(b\) to succeed. Conservatively lower-bound the cost by a single exact-SVP computations in dimension \(b\). (BKZ actually makes several SVP calls.)
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Exploit Ring Structure?

- To date, we have only trivial \(O(d)\)-factor speedups for Ring/Module-LWE over \(d\)-dimensional rings. (NTRU? Stay tuned...)

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Combinatorial/Algebraic Attacks

**Arora-Ge’11**

- Solves LWE in $\approx n^{S\omega}$ time given $\approx n^S$ pairs, where $S = |\text{Support}(\chi)|$ is the number of possible integer error values.

- This suggests a potential risk of very small (rounding) errors, e.g., $\{0, \pm 1\}$ as in NTRU, NTRU Prime, LAC, ThreeBears—although they provide few pairs.

- FrodoKEM, Kyber, NewHope, SABER use relatively larger errors, at the cost of larger keys/ciphertexts. (Indeed, FrodoKEM’s error distributions even conform to a nontrivial worst-case/average-case reduction.)
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▶ By contrast, BDD problems like (Ring-/Module-)LWE plant a unique shortest vector, which [KirchnerFouque’16] explicitly recommend.
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