Extensions of functions - lecture notes

Krzysztof J. Ciosmak

University of Oxford, Mathematical Institute, Oxford, United Kingdom

Email address: Krzysztof.Ciosmak@maths.ox.ac.uk

Abstract. The aim of the course is to present several results on extensions of functions. Among the most important are Kirszbraun's and Whitney's theorems. They provide powerful technical tools in many problems of analysis. One way to view these theorems is that they show that there exists an interpolation of data with certain properties. In this context they are useful in computer science, e.g. in clustering of data and in dimension reduction.

Contents

Chapter 1. Extensions of Lipschitz maps 5
1. McShane's theorem 5
2. Kirszbraun's theorem 6
3. Kneser–Poulsen conjecture 8
4. Continuity of extensions 9

Chapter 2. Whitney's extension theorem 13
1. Covering theorems 13
2. Partitions of unity 15
3. Whitney's extension theorem 16

Chapter 3. Minimal Lipschtiz extensions to differentiable functions 19
1. Affine jets 19
2. Extension problem 19
3. Proofs 20

Chapter 4. Ball's extension theorem 27
1. Markov type and cotype 27
2. Statement of the theorem 30
3. Examples 36

Chapter 5. Assesment and references 37
1. Reading list 37
2. Assesment 37
Additional references 37

Bibliography 39

Additionnal Topics and References 41

CHAPTER 1

Extensions of Lipschitz maps

The first lecture is devoted to study of extensions of Lipschitz maps. Suppose that we are given two metric spaces (X, d_X) and (Y, d_Y), then we say that a map $f\colon X \to Y$ Lipschitz provided that there exists a finite constant L such that

$$d_Y(f(x_1), f(x_2)) \leq L d_X(x_1, x_2) \text{ for all } x_1, x_2 \in X.$$

An optimal constant is the *Lipschitz constant* of f. If L is the optimal constant then we say that f is L-Lipschitz.

1. McShane's theorem

In this section we shall consider functions defined on an arbitrary metric space (X, d) with values in the real line.

Before we prove the main theorem we shall need simple lemmata.

LEMMA 1.1. *Let $L \geq 0$. Suppose that $\mathcal{F}$ is a family of L-Lipschitz functions $f\colon X \to \mathbb{R}$. Then $\sup\{f \mid f \in \mathcal{F}\}$ and $\inf\{f \mid f \in \mathcal{F}\}$ are L-Lipschitz.*

PROOF. Take any $f \in \mathcal{F}$. Then for any $x, y \in X$ there is

$$f(x) - f(y) \leq L d(x, y).$$

Therefore

$$f(x) \leq L d(x, y) + \sup\{f(y) \mid f \in \mathcal{F}\}.$$

Taking the supremum on the left-hand side proves that $\sup\{f \mid f \in \mathcal{F}\}$ is L-Lipschitz. The proof for $\inf\{f \mid f \in \mathcal{F}\}$ follows analogous lines. $\square$

LEMMA 1.2. *Let $y \in X$. Then the function $d(\cdot, y)\colon X \to \mathbb{R}$ is 1-Lipschitz.*

PROOF. This follows immediately from the triangle inequality:

$$|d(x_1, y) - d(x_2, y)| \leq d(x_1, x_2).$$

$\square$

The following is a theorem of McShane, see [**13**].

THEOREM 1.3 (McShane). *Suppose that $A \subset X$ and that $f\colon A \to \mathbb{R}$ is a Lipschitz function. Then there exists an extension of f, i.e. a function $\tilde{f}\colon X \to \mathbb{R}$ such that $\tilde{f}|_A = f$, with with the Lipschitz constant as that of f.*

Moreover, there exists the smallest and the greatest functions f_1 and f_2 respectively that satisfy these properties. They are given by the formulae

$$f_1(x) = \sup\{f(a) - L d(x, a) \mid a \in A\} \text{ for } x \in X$$

and

$$f_2(x) = \inf\{f(a) + L d(a, x) \mid a \in A\} \text{ for } x \in X.$$

The proof resembles the proof of the Hahn–Banach theorem.

Proof. Let L be the Lipschitz constant of f. We want to find an L-Lipschitz $\tilde{f}$ such that for $x \in X, a \in A$

$$|\tilde{f}(x) - f(a)| \leq Ld(x,a). \tag{1.1}$$

Observe that this already implies that $\tilde{f}$ is an extension of f.

Now, condition (1.1) is equivalent to that for all $a_1, a_2 \in A$ and $x \in X$ there is

$$f(a_1) - Ld(x,a_1) \leq \tilde{f}(x) \leq f(a_2) + Ld(x,a_2). \tag{1.2}$$

From Lemma 1.2 we infer that functions

$$x \mapsto f(a_1) - Ld(x,a_1) \text{ and } x \mapsto f(a_2) + Ld(a_2,x)$$

are both L-Lipschitz. Lemma 1.1 tells us that both

$$f_1(x) = \sup\{f(a) - Ld(x,a) \mid a \in A\} \text{ and } f_2(x) = \inf\{f(a) + Ld(a,x) \mid a \in A\}$$

are L-Lipschitz. Function f_1 satisfies the left-hand side inequality of (1.2) while f_2 satisfies the right-hand side inequality of (1.2). Also, as f is L-Lipschtiz $f_1 \leq f_2$. Moreover, if $x \in A$, putting $a = x$ in the formulae for f_1, f_2 yields

$$f(a) \leq f_1(a) \leq f_2(a) \leq f(a).$$

Thus we have here equalities. The proof is complete. □

2. Kirszbraun's theorem

Here we consider more delicate question of extending Lipschitz maps with values in a vector space. Suppose that we have map $f\colon A \to \ell^\infty(\mathbb{N})$, on a subset $A \subset X$ of a metric space (X,d) with values in the space $\ell^\infty(\mathbb{N})$ of bounded sequences. Then applying McShane's theorem to each coordinate of f we may extend it to the entire space X, preserving its Lipschitz constant. However, this is not true for maps with values in arbitrary metric spaces.

Another important positive example is provided by the Kirszbraun's theorem, which addresses the problem in the case of Euclidean spaces. We refer the reader to original paper of Kirszbraun [**10**], proofs of Valentine [**18**], of Schoenberg [**17**] and a proof in the discrete setting by Brehm [**6**].

Theorem 2.1 (Kirszbraun). *Suppose that $A \subset \mathbb{R}^n$ and that $f\colon A \to \mathbb{R}^m$ is a Lipschitz map with respect to Euclidean metrics on A and on $\mathbb{R}^m$. Then there exists an extension $\tilde{f}\colon \mathbb{R}^n \to \mathbb{R}^m$ of f with the same Lipschitz constant.*

Proof. Observe first that it is enough to consider a situation when f is 1-Lipschitz. Suppose that $A = \{x_1, \ldots, x_k\}$ is a finite set. Suppose that $x \notin A$ and we want to find a point $\tilde{f}(x)$ such that $\tilde{f}\colon A \cup \{x\} \to \mathbb{R}^n$ is 1-Lipschitz and $\tilde{f}|_A = f$. That is we want $\tilde{f}(x)$ to belong to the set

$$\bigcap_{i=1}^{k} B(f(x_i), \|x - x_i\|).$$

The assumption is that $\|f(x_i) - f(x_j)\| \leq \|x_i - x_j\|$ for all $i, j = 1, \ldots, k$.

Define a function on $\mathbb{R}^m$ by the formula

$$g(y) = \max\Big\{\frac{\|y - f(x_i)\|}{\|x - x_i\|} \mid i = 1, \ldots, k\Big\}.$$

Clearly, it converges to infinity, as y converges to infinity. Therefore there exists $y_0 \in \mathbb{R}^n$ such that $g(y_0) = \min\{g(y) \mid y \in \mathbb{R}^m\}$. Pick indices $i_1, \ldots, i_l$ for which

$$\frac{\|y_0 - f(x_{i_j})\|}{\|x - x_{i_j}\|} = g(y_0) \text{ for } j = 1, \ldots, l. \tag{2.1}$$

We may assume that

$$y_0 \in \operatorname{Conv}\{f(x_{i_j}) \mid j = 1, \ldots, l\}.$$

Indeed, if not, then there exists a separating hyperplane, i.e. a unit vector $v \in \mathbb{R}^m$ such that for all $j = 1, \ldots, l$ there is

$$\langle v, y_0 - f(x_{i_j})\rangle > \epsilon$$

for some $\epsilon > 0$. Set $y_0' = y_0 - \epsilon v$. Let P be the orthogonal projection onto the space perpendicular to v. Then

$$\|y_0' - f(x_{i_j})\|^2 = \|P(y_0 - f(x_{i_j})\|^2 + (\langle v, y_0 - f(x_{i_j})\rangle - \epsilon)^2 < \|y_0 - f(x_{i_j})\|^2.$$

This contradicts the fact that at y_0 the minimum of g is attained. Therefore there exist non-negative real numbers $\lambda_{i_1}, \ldots, \lambda_{i_l}$ that sum up to one such that

$$y_0 = \sum_{j=1}^{l} \lambda_{i_j} f(x_{i_j}).$$

Define a random vector X, with values in $\mathbb{R}^n$ so that

$$\mathbb{P}(X = x_{i_j}) = \lambda_{i_j} \text{ for } j = 1, \ldots, l.$$

Then

$$y_0 = \mathbb{E} f(X)$$

and

$$\|X - x\|^2 = g(y_0)^2 \big\|f(X) - \mathbb{E} f(X)\big\|^2. \tag{2.2}$$

Let X' be an independent copy of X. Then

$$\mathbb{E}\big\|X - \mathbb{E}X\big\|^2 = \frac{1}{2}\mathbb{E}\|X - X'\|^2 \text{ and } \frac{1}{2}\mathbb{E}\|f(X) - f(X')\|^2 = \mathbb{E}\big\|f(X) - \mathbb{E}f(X)\big\|^2.$$

Taking the expectation of (2.2) yields

$$g(y_0)^2 \mathbb{E}\big\|f(X) - \mathbb{E}f(X)\big\|^2 = \mathbb{E}\|X - x\|^2 \geq \mathbb{E}\big\|X - \mathbb{E}X\big\|^2.$$

Therefore, as f is 1-Lipschitz,

$$\frac{1}{2}\mathbb{E}\|f(X) - f(X')\|^2 \leq \frac{1}{2}\mathbb{E}\|X - X'\|^2 \leq g(y_0)^2 \frac{1}{2}\mathbb{E}\|f(X) - f(X')\|^2.$$

Hence $g(y_0) \leq 1$, unless $f(X)$ is constant, so that $y_0 = f(X)$. In this case $g(y_0) = 0$.

We have proven that for any $x \in \mathbb{R}^n$ and any finite set A the interesection of closed balls

$$\bigcap_{y \in A} B(f(y), \|x - y\|) \neq \emptyset.$$

By compactness, such intersection is also non-empty for any infinite set A.

We partially order subsets of $\mathbb{R}^n$ containing A and admitting a 1-Lipschitz extension of f by inclusion. By the Kuratowski–Zorn lemma, there exists a maximal element Z of this ordering. If $Z \neq \mathbb{R}^n$, then we could have extended the 1-Lipschitz map to an element $x \notin Z$, contradicting the choice of Z. This completes the proof. $\square$

The Kirszbraun theorem holds true also for Hilbert spaces, with the same proof.

3. Kneser–Poulsen conjecture

The Kirszbraun theorem admits also another formulation.

THEOREM 3.1. *Suppose that* $x_1, \ldots, x_k \in \mathbb{R}^n$ *and let* $r_1, \ldots, r_k \geq 0$. *Suppose that*

$$\bigcap_{i=1}^{k} B(x_i, r_i) \neq \emptyset.$$

Then for any points $y_1, \ldots, y_k \in \mathbb{R}^m$ *such that* $\|y_i - y_j\| \leq \|x_i - x_j\|$ *for all* $i, j = 1, \ldots, k$ *there is*

$$\bigcap_{i=1}^{k} B(y_i, r_i) \neq \emptyset.$$

The above theorem says that if we shrink the centres of the balls, with non-empty intersection, then also the intersection of balls with new centres will be non-empty.

PROOF. Let

$$x \in \bigcap_{i=1}^{k} B(x_i, r_i).$$

Then $\|x - x_i\| \leq r_i$ for each $i = 1, \ldots, k$. By the Kirszbraun theorem there exists

$$y \in \bigcap_{i=1}^{k} B(y_i, \|x - x_i\|) \subset \bigcap_{i=1}^{k} B(y_i, r_i).$$

. □

The Kneser–Poulsen conjecture, see e.g. [**4**], is a form of quantification of the above theorem. That is, it says that non only will the shrinked balls have non-empty intersection, but also that the volume of the intersection will be estimated from below by the volume of the intersection of original balls.

CONJECTURE 3.2 (Kneser–Poulsen). *Suppose that* $x_1, \ldots, x_k \in \mathbb{R}^n$ *and let* $r_1, \ldots, r_k \geq 0$. *Then for any points* $y_1, \ldots, y_k \in \mathbb{R}^m$ *such that* $\|y_i - y_j\| \leq \|x_i - x_j\|$ *for all* $i, j = 1, \ldots, k$ *there is*

$$\lambda\Big(\bigcap_{i=1}^{k} B(x_i, r_i)\Big) \leq \lambda\Big(\bigcap_{i=1}^{k} B(y_i, r_i)\Big).$$

Moreover

$$\lambda\Big(\bigcup_{i=1}^{k} B(x_i, r_i)\Big) \geq \lambda\Big(\bigcup_{i=1}^{k} B(y_i, r_i)\Big).$$

Here λ *stands for the Lebesgue measure.*

In [**4**], the conjecture has been answered in the affirmative for $n = m = 2$.

If $n = m$ and $k \leq n + 1$, then in [**9**] the conjecture has been proven for the volumes of intesections.

We include the references [**11**], [**16**] for the original statement of the conjecture.

4. Continuity of extensions

In this section we shall be concerned with the following question. Let (X, d_X) and (Y, d_Y) be metric spaces and let $A \subset X$. Suppose that we are given 1-Lipschitz maps $v\colon X \to Y$ and $u\colon A \to Y$. Does there exist a 1-Lipschitz extension $\tilde{u}$ of u such that the uniform distance of $\tilde{u}$ to v is equal to the uniform distance from u to v on A?

For a thorough discussion of this and related questions we refer the reader to [**7**].

Let $A \subset B \subset \mathbb{R}^n$, $n \in \mathbb{N}$. We shall prove that given any 1-Lipschitz maps $v\colon \mathbb{R}^n \to \mathbb{R}^m$, for $m \in \mathbb{N}$, and $u\colon A \to \mathbb{R}^m$, such that

$$\sup\{\|u(x) - v(x)\| \mid x \in A\} \leq \delta,$$

there exists a 1-Lipschitz extension $\tilde{u}\colon B \to \mathbb{R}^m$ of u, that is $\tilde{u}(x) = u(x)$ for $x \in A$, such that

$$\sup\{\|v(x) - \tilde{u}(x)\| \mid x \in B\} \leq \sqrt{\delta^2 + 2\delta d_v(A, B)}. \tag{4.1}$$

Here by $d_v(A, B)$ we denote the number

$$\sup\{\|v(x) - v(y)\| \mid x \in A, y \in B\}. \tag{4.2}$$

Note that for 1-Lipschitz functions v we have $d_v(A, B) \leq \operatorname{diam}(B)$. We shall also give an example of functions u, v such that the bound is attained. This is to say, u, v are such that for any 1-Lipschitz extension $\tilde{u}$ of u we have equality in (4.1). Moreover, as we shall show, we cannot hope, in general, for any bound, if $d_v(A, B)$ is infinite.

Proposition 4.1 (C.). *Let $A \subset B \subset \mathbb{R}^n$ and let*

$$u\colon A \to \mathbb{R}^m,\ v\colon B \to \mathbb{R}^m$$

be 1-Lipschitz maps. Assume that $\|u(x) - v(x)\| \leq \delta$ for $x \in A$. Then there exists a 1-Lipschitz map $\tilde{u}\colon B \to \mathbb{R}^m$ such that $\tilde{u}(x) = u(x)$ for $x \in A$ and

$$\|v(x) - \tilde{u}(x)\| \leq \sqrt{\delta^2 + 2\delta d_v(A, B)}$$

for all $x \in B$.

Proof. Let $\epsilon^2 = \delta^2 + 2\delta d_v(A, B)$. Let us define a map

$$h\colon B \times \{0\} \cup A \times \{\epsilon\} \to \mathbb{R}^m$$

by the formulae $h(x, 0) = v(x)$ for $x \in B$ and $h(x, \epsilon) = u(x)$ for $x \in A$. Then h is a 1-Lipschitz map on a subset of $\mathbb{R}^{n+1}$. Indeed, if $x \in A$ and $y \in B$, then

$$\begin{aligned}
&\|h(y, 0) - h(x, \epsilon)\|^2 = \|v(y) - u(x)\|^2 = \\
&= \|v(y) - v(x)\|^2 + \|v(x) - u(x)\|^2 + 2\langle v(y) - v(x), v(x) - u(x)\rangle \leq \\
&\leq \|x - y\|^2 + \delta^2 + 2\delta d_v(A, B) = \|x - y\|^2 + \epsilon^2.
\end{aligned}$$

For other points of the domain of h the 1-Lipschitz condition follows from 1-Lipschitzness of u and v.

Using Theorem 2.1 we may extend h to a 1-Lipschitz map $\tilde{h}\colon \mathbb{R}^{n+1} \to \mathbb{R}^m$. Set for $x \in B$, $\tilde{u}(x) = \tilde{h}(x, \epsilon)$. Then $\tilde{u}$ is a 1-Lipschitz extension of u and moreover, for $x \in B$,

$$\|v(x) - \tilde{u}(x)\| = \|\tilde{h}(x, 0) - \tilde{h}(x, \epsilon)\| \leq \epsilon.$$

□

The following example [**7**] illustrates that the answer to the question posed at the begining of the current section is in general negative.

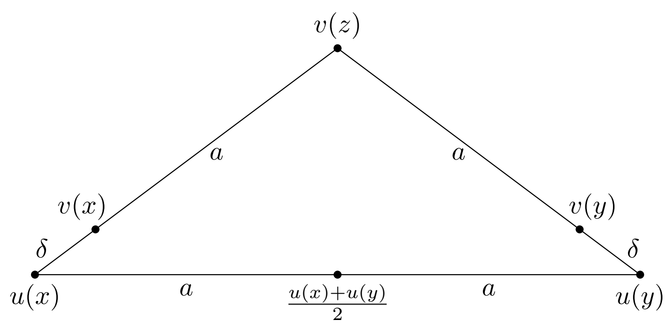
EXAMPLE 4.2 (C.). Let $m > 1$ and let $x, y \in \mathbb{R}^n$, $x \neq y$, $z = \frac{x+y}{2}$. Let $a = \|x - z\| = \|y - z\|$, let $\delta > 0$. Define $u\colon \{x, y\} \to \mathbb{R}^m$ by setting $u(x)$ and $u(y)$ so that $\|u(x) - u(y)\| = \|x - y\|$. Map u defined in this way is 1-Lipschitz. For the definition of v consider the triangle whose vertices are $u(x), u(y)$ and a point, called $v(z)$, such that

$$\|v(z) - u(x)\| = \|v(z) - u(y)\| = a + \delta.$$

Set $v(x), v(y)$ to be the points on the triangle's edges containing $u(x)$ and $u(y)$ respectively such that $\|v(x) - u(x)\| = \delta$ and $\|v(y) - u(y)\| = \delta$. If we define $v\colon \{x, y, z\} \to \mathbb{R}^m$ in this manner, then it is 1-Lipschitz. By Kirszbraun theorem we may extend it to $\mathbb{R}^n$ in such a way that the extension is still 1-Lipschitz. We shall call this extension $v\colon \mathbb{R}^n \to \mathbb{R}^m$. Moreover, $\sup\{\|u(t) - v(t)\| \mid t \in A\} = \delta$. Here $A = \{x, y\}$. Observe that any 1-Lipschitz extension $\tilde{u}$ of u to the point z must satisfy $\tilde{u}(z) = \frac{u(x)+u(y)}{2}$. Thus, if we set $B = \{x, y, z\}$, then any 1-Lipschitz extension $\tilde{u}$ of u to B satisfies

$$\|v(z) - \tilde{u}(z)\| = \sqrt{\delta^2 + 2\delta a}.$$

The situation is illustrated below.



Note now that

$$a = d_v(A, B) \text{ if } \delta \geq a \text{ and } a = \frac{1}{4} d_v(A, B) + \sqrt{\frac{1}{16} d_v(A, B)^2 + \frac{1}{2} d_v(A, B)\delta} \text{ if } \delta \leq a.$$

This exhibits that the bound (4.1) is indeed sharp, if $\delta \geq d_v(A, B)$. Note that $\delta \leq a$ if and only if $\delta \leq d_v(A, B)$. Hence we have shown that for all extensions $\tilde{u}$ of u we have

$$\sup\{\|v(z) - \tilde{u}(z)\| \mid z \in B\} = \sqrt{\delta^2 + 2\delta d_v(A, B)}$$

if $\delta \geq d_v(A, B)$ and

$$\sup\{\|v(z) - \tilde{u}(z)\| \mid z \in B\} = \sqrt{\delta^2 + \frac{1}{2}\delta d_v(A, B) + \delta\sqrt{\frac{1}{4} d_v(A, B)^2 + 2 d_v(A, B)\delta}}$$

if $\delta \leq d_v(A, B)$.

Let us consider the case when the target space is one-dimensional.

PROPOSITION 4.3 (C.). *Let X be a metric space. Let $v\colon X \to \mathbb{R}$ be a 1-Lipschitz function. Then for any set $A \subset X$ and for any 1-Lipschitz function $u\colon A \to \mathbb{R}$ such that for all $x \in A$*

$$|u(x) - v(x)| \leq \delta, \tag{4.3}$$

there exists 1-Lipschitz extension $\tilde{u}\colon X \to \mathbb{R}$ of v such that for all $x \in X$

$$|v(x) - \tilde{u}(x)| \leq \delta. \tag{4.4}$$

PROOF. Take any 1-Lipschitz extension $\tilde{u}_0\colon X \to \mathbb{R}$ of u. Existence of such function follows from Theorem 1.3. Define now [1]

$$\tilde{u}(x) = \tilde{u}_0(x) \wedge (v(x) + \delta) \vee (v(x) - \delta).$$

Employing Lemma 1.1 it is readily verifiable that $\tilde{u}$ satisfies the desired properties. ☐

The following theorem characterises the situation for Euclidean spaces. Note that the situation in the multi-dimensional case differs strikingly from the one-dimensional case (see [**7**]).

THEOREM 4.4 (C.). *Let X, Y be real Hilbert spaces such that Y is of dimension at least two. Let $v\colon X \to Y$ be a map. The following conditions are equivalent:*

i) for any $A \subset X$ and for any 1-Lipschitz map $u\colon A \to Y$ there exists a 1-Lipschitz extension $\tilde{u}\colon X \to Y$ of u such that for all $x \in X$

$$v(x) - \tilde{u}(x) \in \overline{\operatorname{Conv}}\big\{v(z) - u(z) \mid z \in A\big\}.$$

ii) for any $\delta > 0$, any $A \subset X$ and for any 1-Lipschitz map $u\colon A \to Y$ such that for all $x \in A$

$$\|v(x) - u(x)\| \leq \delta,$$

there exists 1-Lipschitz extension $\tilde{u}\colon X \to Y$ of u such that for all $x \in X$

$$\|v(x) - \tilde{u}(x)\| \leq \delta,$$

iii) v is affine and 1-Lipschitz.

[1] Here, symbols $a \wedge b$ and $a \vee b$ stand for minimum and maximum of two real numbers a, b respectively.

CHAPTER 2

Whitney's extension theorem

In this lecture we shall consider another problem concerning extensions of functions. The exposition is based on [**8**].

Suppose that $f\colon \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable. We shall denote by Df the derivative of f. Then for any compact set $K \subset \mathbb{R}^n$ there is

$$\lim_{\delta\to 0^+} \sup\Big\{\Big|\frac{f(y)-f(x)-Df(x)(y-x)}{\|y-x\|}\Big| \mid 0<\|x-y\|\leq \delta, x,y\in K\Big\} = 0.$$

It turns out that this condition for any compact set $K \subset C \subset \mathbb{R}^n$ implies that f, defined on a set $C \subset \mathbb{R}^n$, is a restriction of a continuously differentiable function defined on the entire $\mathbb{R}^n$. This is the statement of the Whitney's extension theorem.

Theorem 0.1 (Whitney). *Suppose that $C \subset \mathbb{R}^n$ is a closed set. Suppose that $f\colon C \to \mathbb{R}$ and $v\colon C \to (\mathbb{R}^n)^*$ are continuous and that for any compact set $K \subset C$ there is*

$$\lim_{\delta\to 0^+} \sup\Big\{\Big|\frac{f(y)-f(x)-v(x)(y-x)}{\|y-x\|}\Big| \mid 0<\|x-y\|\leq \delta, x,y\in K\Big\} = 0.$$

Then there exists a continuosly differentiable extension $\tilde{f}\colon \mathbb{R}^n \to \mathbb{R}$ of f such that $D\tilde{f} = v$ on C.

1. Covering theorems

Before we proceed to a proof of the Whintey's extension theorem we shall construct a suitable covering which shall allow us to construct an extension with the desired properties.

1.1. Vitali's covering theorem. We shall first provide a construction of the Vitali's covering theorem.

If B is a closed ball in $\mathbb{R}^n$ we shall deonte by $\hat{B}$ a concentric ball of radius five times the radius of B. If $\mathcal{F}$ is a collection of balls we shall denote by $\hat{\mathcal{F}}$ the family $\{\hat{B} \mid B \in \mathcal{F}\}$.

Definition 1.1. Let $A \subset \mathbb{R}^n$. Then a family $\mathcal{F}$ of balls in $\mathbb{R}^n$ is a covering of A if $A \subset \bigcup \mathcal{F}$. We shall say that $\mathcal{F}$ is a fine covering of A if additionally there is

$$\inf\{\mathrm{diam} B \mid x \in B, B \in \mathcal{F}\} = 0$$

for each $x \in A$.

Theorem 1.2 (Vitali). *Let $\mathcal{F}$ be any family of non-degenerate closed balls in $\mathbb{R}^n$ with bounded radii. Then there exists a countable subfamily $\mathcal{G}$ of $\mathcal{F}$ of pairwise disjoint balls in $\mathcal{F}$ such that*

$$\bigcup \mathcal{F} \subset \bigcup \hat{\mathcal{G}}. \tag{1.1}$$

PROOF. Let D be the upper bound for the radii of balls in $\mathcal{F}$. Define for $j = 0, 1, 2, \ldots$

$$\mathcal{F}_j = \Big\{ B \in \mathcal{F} \mid \frac{1}{2^j} D < \mathrm{diam} B \leq \frac{1}{2^{j-1}} D \Big\}.$$

Define $\mathcal{G}_j$ for $j = 0, 1, 2, \ldots$ in the following manner. Let $\mathcal{G}_0$ be a maximal pairwise disjoint subfamily of balls in $\mathcal{F}_0$.

Suppose that we have already chosen $\mathcal{G}_0, \ldots, \mathcal{G}_{k-1}$. Let now $\mathcal{G}_k$ be a maximal pairwise disjoint subfamily of balls in $\mathcal{F}_k$ that are disjoint from $\bigcup_{j=0}^{k-1} \bigcup \mathcal{G}_j$.

Define $\mathcal{G} = \bigcup_{j=0}^{\infty} \mathcal{G}_j$. Then $\mathcal{G}$ is a subfamily of balls in $\mathcal{F}$ that are pairwise disjoint.

We need to prove (1.1). Take any ball $B \in \mathcal{F}$. Then $B \in \mathcal{F}_j$ for some $j = 0, 1, 2, \ldots$. Then, by the maximality of the considered families, there exists a ball $B' \in \bigcup_{l=0}^{j} \mathcal{G}_l$ such that $B' \cap B \neq \emptyset$. Then $\mathrm{diam} B' > \frac{1}{2^j} D$. Since $B \in \mathcal{F}_j$, $\mathrm{diam} B \leq \frac{1}{2^{j-1}} D$, so that $\mathrm{diam} B < 2 \mathrm{diam} B'$. Therefore $\hat{B}' \supset B$.

The family $\mathcal{G}$ is at most countable by separability of $\mathbb{R}^n$. The proof is complete. $\square$

1.2. Whitney's covering theorem. We refer the reader to [**20**] for the original formulation of the following theorem. The original formulation deals with cubes in Euclidean spaces. Below we shall provide a formulation that suffices for a proof of the Whitney's extension theorem and that can be extended to the setting of metric-measure spaces that satisfy the doubling condition.

THEOREM 1.3 (Whitney). *Let $U \subset \mathbb{R}^n$ be an open set. Let $c \in (0,1)$ and $\lambda > 0$. Then there exists a family $\mathcal{F}$ of pairwise disjoint balls in U such that $U = \hat{\mathcal{F}}$. For $y \in U$ let $r(y) = c(1 \wedge \mathrm{dist}(y, U^c))$ and*

$$S_y = \{B(x,r) \in \mathcal{F} \mid B(x, \lambda r) \cap B(y, \lambda r(y)) \neq \emptyset\}.$$

Then

$$\# S_y \leq \left(\lambda + (1+\lambda) \frac{1+\lambda c}{1-\lambda c} \right)^n \left(\frac{1+\lambda c}{1-\lambda c} \right)^n.$$

Moreover, for each ball $B(x,r) \in S_y$ there is

$$\frac{1-\lambda c}{1+\lambda c} \leq \frac{r}{r(y)} \leq \frac{1+\lambda c}{1-\lambda c}.$$

PROOF. Consider a family of balls

$$\mathcal{F}_0 = \Big\{ B(x, r(x)) \mid x \in U \Big\}.$$

By an application of Vitali's covering theorem, there exists a countable subfamily $\mathcal{F}$ of $\mathcal{F}_0$ such that

$$U = \bigcup \hat{\mathcal{F}}.$$

and the balls in $\mathcal{F}$ are pairwise disjoint. Suppose that

$$B(x, \lambda r(x)) \cap B(y, \lambda r(y)) \neq \emptyset \text{ for some } x, y \in U.$$

Then

$$|r(x) - r(y)| \leq c\|x - y\| \leq c\lambda(r(x) + r(y))$$

Therefore

$$\frac{1-\lambda c}{1+\lambda c} r(y) \leq r(x) \leq \frac{1+\lambda c}{1-\lambda c} r(y).$$

Moreover

$$B(x, r(x)) \subset B\Big(y, \Big(\lambda + (1+\lambda)\frac{1+\lambda c}{1-\lambda c}\Big) r(y)\Big),$$

as

$$\|x - y\| + r(x) \leq \lambda(r(x) + r(y)) + r(x) \leq \Big(\lambda + (1+\lambda)\frac{1+\lambda c}{1-\lambda c}\Big) r(y)$$

The fact that the balls in $\mathcal{F}$ are pairwise disjoint and the fact that radii of balls in $\mathcal{F}$ that intersect $B(y, \lambda r(y))$ are bounded from below by $\frac{1-\lambda c}{1+\lambda c} r(y)$ imply that

$$\# S_y \leq \Big(\lambda + (1+\lambda)\frac{1+\lambda c}{1-\lambda c}\Big)^n \Big(\frac{1+\lambda c}{1-\lambda c}\Big)^n.$$

This follows by a calculation of the Lebesgue measure. □

1.3. Besicovitch's covering theorem. A much more complicated and intricate is Besicovitch's covering theorem. Here we state it without a proof. For a proof we refer the reader to [**8**].

Theorem 1.4 (Besicovitch). *For each $n \in \mathbb{N}$ there exists a number N_n such that if $\mathcal{F}$ is a family of non-degenerate closed balls in $\mathbb{R}^n$ with bounded radii and if A is the set of centres of balls in $\mathcal{F}$, then there exist subfamilies $\mathcal{G}_1, \ldots, \mathcal{G}_{N_n} \subset \mathcal{F}$ of pairwise disjoint balls in $\mathcal{F}$ such that*

$$A \subset \bigcup_{i=1}^{N_n} \bigcup \mathcal{G}_i.$$

2. Partitions of unity

Using the Whitney's covering theorem we shall construct a smooth partition of unity, subordinate to the covering.

Lemma 2.1. *Let $\rho > 1$. There exists a smooth function $\mu \colon \mathbb{R} \to \mathbb{R}$ such that $0 \leq \mu \leq 1$, $\mu(t) = 1$ for $t \leq 1$, $\mu(t) = 0$ for $t \geq \rho$.*

Lemma 2.2. *Suppose that U is an open set in $\mathbb{R}^n$. Then there exist smooth, non-negative functions $(v_j)_{j=1}^{\infty}$ on $\mathbb{R}^n$ such that*

$$\sum_{j=1}^{\infty} v_j = 1, \ \sum_{j=1}^{\infty} Dv_j = 0 \text{ on } U \text{ and } \|Dv_j(x)\| \leq \frac{C}{c(1 \wedge \operatorname{dist}(x, U^c))} \text{ for } x \in U.$$

Moreover, for any $x \in U$, the number of indices $j = 1, 2, 3, \ldots$ such that the support of v_j intersects $B(x, \rho r(x))$ is at most C.

Proof. Let $c \in (0,1)$ and $\rho > 1$, $\lambda = 5\rho$. Pick a covering $\mathcal{F}$ of U constructed as in Theorem 1.3. Let μ be a function of Lemma 2.1 for ρ. For a ball $B \in \mathcal{F}$ define

$$u_B(x) = \mu\Big(\frac{\|x - x_B\|}{5 r_B}\Big), x \in \mathbb{R}^n,$$

where x_B is the centre of B and r_B is its radius. Then each u_B is a smooth function with values in $[0,1]$, $u_B = 1$ on $B(x_B, 5r_B)$ and $u_B = 0$ on $B(x_B, 5\rho r_B)$. Moreover

$$\|Du_B(x)\| \leq \frac{C}{r_B} \leq \frac{C'}{r(x)} \tag{2.1}$$

whenever $B(x, \lambda r(x)) \cap B(x_B, \lambda r_B) \neq \emptyset$, and $u_B = 0$ on $B(x, \lambda r(x))$ if $B(x, \lambda r(x)) \cap B(x_B, \lambda r_B) = \emptyset$. Set

$$\sigma = \sum_{B \in \mathcal{F}} u_B.$$

Then σ is a smooth function on U, as the sum is locally finite, $\sigma \geq 1$ on U and for $x \in U$

$$\|D\sigma(x)\| \leq \frac{C}{r(x)}.$$

Set $v_B = \frac{u_B}{\sigma}$. Observe that

$$Dv_B = \frac{Du_B}{\sigma} - \frac{u_B D\sigma}{\sigma^2},$$

so the bound on the derivative of v_B is satisfied. □

3. Whitney's extension theorem

In this section we shall provide a proof of the Whitney's extension theorem.

Proof of Theorem 0.1. Pick a covering $\mathcal{F}$ of the complement U of the set C, constructed as in Theorem 1.3 and an associated partition of unity $(v_B)_{B \in \mathcal{F}}$ of Lemma 2.2. For each centre x_B of a ball $B \in \mathcal{F}$ pick $s_B \in C$ such that

$$\|x_B - s_B\| = \operatorname{dist}(x_B, C).$$

Define $\tilde{f} \colon \mathbb{R}^n \to \mathbb{R}$ by the formula

$$\tilde{f}(x) = f(x) \text{ for } x \in C$$

and

$$\tilde{f}(x) = \sum_{B \in \mathcal{F}} \Big(f(s_B) + v(x_B)(x - s_B) \Big) v_B(x) \text{ for } x \in U.$$

Clearly, $\tilde{f}$ is a smooth function on U with

$$D\tilde{f}(x) = \sum_{B \in \mathcal{F}} \Bigg(\big(f(s_B) + v(x_B)(x - s_B) \big) Dv_B(x) + v_B(x) v(x_B) \Bigg) \text{ for } x \in U.$$

We claim that $D\tilde{f} = v$ on C. Fix a point $a \in C$ and let $K = C \cap B(a, 1)$. Then K is a compact set. Define for $\delta > 0$

$$\phi(\delta) = \sup \Big\{ \Big| \frac{f(y) - f(x) - v(x)(y - x)}{\|y - x\|} \Big| + \|v(x) - v(y)\| \mid x, y \in K, 0 < \|x - y\| \leq \delta \Big\}.$$

Then, by the assumption, $\lim_{\delta \to 0^+} \phi(\delta) = 0$. Let $x \in C$, $\|x - a\| \leq 1$. Then

$$|\tilde{f}(x) - \tilde{f}(a) - v(a)(x - a)| = |f(x) - f(a) - v(a)(x - a)| \leq \phi(\|x - a\|) \|x - a\|,$$

and

$$\|v(x) - v(a)\| \leq \phi(\|x - a\|).$$

Suppose now that $x \in U$ and that $\|x - a\| < 1$. Then

$$|\tilde{f}(x) - \tilde{f}(a) - v(a)(x - a)| = |\tilde{f}(x) - f(a) - v(a)(x - a)| \leq$$

$$\leq \sum_{B \in \mathcal{F}} \Big| v_B(x) \big(f(s_B) - f(a) + v(s_B)(x - s_B) - v(a)(x - a) \big) \Big| \leq$$

$$\sum_{B \in \mathcal{F}} v_B(x) |f(s_B) - f(a) - v(s_B)(a - s_B)| + \sum_{B \in \mathcal{F}} v_B(x) |(v(s_B) - v(a))(x - a)|.$$

If $B(x_B, \lambda r_B) \cap B(x, \lambda r(x)) \neq \emptyset$, then

$$\|a - s_B\| \leq \|a - x_B\| + \|x_B - s_B\| \leq 2\|a - x_B\| \leq 2(\|a - x\| + \|x - x_B\|) \leq$$
$$\leq 2(\|a - x\| + \lambda(r(x) + r_B)) \leq C\|x - a\|,$$

for some constant C, depending on c and ρ. Hence

$$|\tilde{f}(x) - \tilde{f}(a) - v(a)(x - a)| \leq C\phi(C\|x - a\|)\|x - a\|.$$

Therefore $\tilde{f}$ is differentiable at a and $D\tilde{f}(a) = v(a)$.

We need now show that $\tilde{f}$ has continuous derivative. Fix $a \in C$ and $x \in \mathbb{R}^n$, $\|x - a\| < 1$. If $x \in C$, then

$$\|D\tilde{f}(x) - D\tilde{f}(a)\| = \|v(x) - v(a)\| \leq \phi(\|x - a\|).$$

Suppose that $x \in U$. Take $b \in C$ such that $\|x - b\| = \operatorname{dist}(x, C)$. Then

$$\|D\tilde{f}(x) - D\tilde{f}(a)\| = \|D\tilde{f}(x) - v(a)\| \leq \|D\tilde{f}(x) - v(b)\| + \|v(b) - v(a)\|.$$

As $\|b - a\| \leq \|b - x\| + \|x - a\| \leq 2\|x - a\|$, we see that $\|v(b) - v(a)\| \leq \phi(2\|x - a\|)$. We need to estimate the first summand. We calculate

$$\|D\tilde{f}(x) - v(b)\| = \Big| \sum_{B \in \mathcal{F}} \big(f(s_B) + v(s_B)(x - s_B)\big) Dv_B(x) + v_B(x)(v(s_B) - v(b))\Big| \leq$$
$$\leq \Big| \sum_{B \in \mathcal{F}} \big(- f(b) + f(s_B) + v(s_B)(b - s_B)\big) Dv_B(x)\Big| +$$
$$+ \Big| \sum_{B \in \mathcal{F}} \big((v(s_B) - v(b))(x - b)\big) Dv_B(x)\Big| + \Big| \sum_{B \in \mathcal{F}} v_B(x)(v(s_B) - v(b))\Big| \leq$$
$$\leq \frac{C}{r(x)} \sum_{B \in S_x} \phi(\|b - s_B\|)\big(\|b - s_B\| + \|x - b\|\big) + \sum_{B \in S_x} \phi(\|b - s_B\|).$$

Now, $\|x - b\| \leq \|x - a\| < 1$, so that $r(x) = c(1 \wedge \|x - b\|) < c$. Moreover, if $B(x_B, \lambda r_B) \cap B(x, \lambda r(x)) \neq \emptyset$, then $r_B \leq Cr(x) \leq C'$. Therefore,

$$\|b - s_B\| \leq \|b - x\| + \|x - x_B\| + \|x_B - s_B\| \leq$$
$$\leq \frac{1}{c} r(x) + \lambda(r(x) + r_B) + \frac{1}{c} r_B \leq Cr(x) = C\|x - b\| \leq C\|x - a\|.$$

We infer that

$$\|D\tilde{f}(x) - d(b)\| \leq C\phi(c\|x - a\|)$$

and in turn

$$\|D\tilde{f}(x) - D\tilde{f}(a)\| \leq C\phi(c\|x - a\|).$$

This proves the asserted continuity and completes the proof of the theorem. □

Remark 3.1. The constants in the proof of the above theorem, in Theorem 1.3 and in Lemma 2.2 depend on $\rho > 1$ and $c \in (0, 1)$, which might be picked arbitrarily.

CHAPTER 3

Minimal Lipschtiz extensions to differentiable functions

In this chapter we shall provide another proof of the Whitney's extension theorem. The proof will be in the spirit of the proof of Kirszbraun's theorem. Our exposition is based on [**12**].

1. Affine jets

The formulation that we shall deal with considers fields of affine jets. Such a field is an association to any point of its domain of an affine, real-valued function. Let $A \subset B \subset \mathbb{R}^n$. Suppose that we are given a field of affine jets T on A. That is, T is a map $A \ni a \mapsto T_a$, where T_a is an affine function on $\mathbb{R}^n$. We say that a field U, defined on B, extends T whenever for any $a \in A$, $T_a = U_a$.

A differentiable function u on $\mathbb{R}^n$ extends a field T provided that the field of Taylor expansions of u extends T.

2. Extension problem

Suppose that T is a field of affine jets defined on some set $A \subset \mathbb{R}^n$. The Lipschitz extension problem is to find necessary and sufficient condition on the field T that will ensure that T admits an extension to a differentiable function on $\mathbb{R}^n$ which has Lipschitz derivative.

For a field T defined on $A \subset \mathbb{R}^n$ define

$$\Gamma^1(T) = 2 \sup \left\{ \frac{T_a(y) - T_b(y)}{\|a-y\|^2 + \|b-y\|^2} \mid a \neq b, a, b \in A, y \in \mathbb{R}^n \right\}.$$

We shall show that if T is the field of Taylor expansions of a differentiable function u, then $\Gamma^1(T)$ is equal to the Lipschitz constant of the derivative Du of u.

Let us now turn to the minimal Lipschitz extension problem. A differentiable function u on $\mathbb{R}^n$ is said to be a minimal Lipschitz extension of a field T if the field of its Taylor expansions extends T and for any other extension v of T, the Lipschtiz constant of Dv is at least as large as that of Du.

We will show that $\Gamma^1(T)$ is equal to the infimum of all Lipschitz constants of derivatives of functions extending T.

The aim of this lecture is to prove the following theorem.

Theorem 2.1. *Γ^1 is a unique functional such that:*

i) if U extends T, then $\Gamma^1(U) \geq \Gamma^1(T)$,

ii) if U is defined on $\mathbb{R}^n$ and $\Gamma^1(U) < \infty$, then the function defined by the formula

$$u(x) = U_x(x) \text{ for } x \in \mathbb{R}^n$$

is differentiable and its derivative is Lipschitz, *iii) if u is differentiable function on $\mathbb{R}^n$ with Lipschitz derivative, then $\Gamma^1(U)$ is equal to the Lipschitz constant of Du, where U is the field of jets of Taylor expansions of u,*

iv) for any field T such that $\Gamma^1(T) < \infty$, there exists an extension of T to a field of affine jets defined on $\mathbb{R}^n$ with $\Gamma^1(U) = \Gamma^1(T)$.

COROLLARY 2.2. *If T is a filed of affine jets satisfying $\Gamma^1(T) < \infty$, then there exists a differentiable function u defined on $\mathbb{R}^n$, which is a minimal extension of T and such that its derivatve has Lipschitz constant equal to $\Gamma^1(T)$*

The above theorem holds true also in the case of Hilbert spaces, separable or not. The previous proof of the Whitney's extension theorem does not convey to this setting.

The proof of i) is immediate, as well as the proof of uniqueness. The proof of the main part of the theorem, iv), relies on an approach similar to the one employed in the proof of Kirszbraun's theorem.

3. Proofs

In what follows we shall denote by H a Hilbert space equipped with scalar product $\langle \cdot, \cdot \rangle$. For any field T of affine jets we write

$$T_a(x) = u_a + \langle D_a u, x - a \rangle \text{ for all } a \text{ in the domain of } T \text{ and } x \in H,$$

for some $u_a \in \mathbb{R}$ and some $D_a u \in H$.

DEFINITION 3.1. For a field of affine jets T defined on $A \subset H$ we define its Lipschitz constant $\Gamma^1(T)$ by the formula

$$\Gamma^1(T) = 2 \sup \left\{ \frac{T_a(y) - T_b(y)}{\|a - y\|^2 + \|b - y\|^2} \mid y \in H, a \neq b, a, b, \in A \right\},$$

whenever A has at least two elements and $\Gamma^1(T) = 0$ if has not.

PROPOSITION 3.2. *If A has at least two elements, then*

$$\Gamma^1(T) = \sup \left\{ \sqrt{A_{a,b}^2 + B_{a,b}^2} + |A_{a,b}| \mid a \neq b, a, b \in A \right\},$$

where

$$A_{a,b} = \frac{2(u_a - u_b) + \langle D_a u + D_b u, b - a \rangle}{\|a - b\|^2}, B_{a,b} = \frac{D_a u - D_b u}{\|a - b\|}.$$

PROOF. For $a \neq b$ and $y \in H$ denote

$$g_{a,b}(y) = \frac{T_a(y) - T_b(y)}{\|a - y\|^2 + \|b - y\|^2}.$$

We need to prove that for any $a \neq b$ there is

$$\sup\{g_{a,b}(y) \mid y \in H\} = \sqrt{A_{a,b}^2 + B_{a,b}^2} + |A_{a,b}|.$$

For any $y \in H$ set

$$t = \frac{2}{\|a - b\|} \left(y - \frac{a + b}{2} \right).$$

Then

$$T_a(y) - T_b(y) = \frac{1}{2} \|a - b\|^2 A_{a,b} + \frac{1}{2} \|a - b\| \langle D_a u - D_b u, t \rangle$$

and

$$\|a-y\|^2+\|b-y\|^2=\frac{1}{2}\|a-b\|^2\big(1+\|t\|^2\big).$$

Thus

$$\sup\big\{|g_{a,b}(y)| \mid y\in H\big\}=\sup\left\{\frac{|A_{a,b}+\frac{\langle D_a u-D_b u,t\rangle}{\|a-b\|}|}{1+\|t\|^2}\mid t\in H\right\}.$$

Suppose that $D_a u = D_b u$. Then the maximum is attained at $t=0$ and is equal to $|A_{a,b}|$. Let now $D_a u\neq D_b u$. Let

$$e=\frac{D_a u-D_b u}{\|D_a u-D_b u\|}.$$

Then we may write $t=\alpha e+f$ for some $\alpha\in\mathbb{R}$ and f perpendicular to e. Then

$$\sup\big\{|g_{a,b}(y)| \mid y\in H\big\}=\sup\left\{\frac{|A_{a,b}+\alpha B_{a,b}|}{1+\alpha^2}\mid \alpha\in\mathbb{R}\right\}.$$

An elementary calculation completes the proof. □

REMARK 3.3. In the above proof for computation of the supremum it suffices to take $|\alpha|\leq 1$, so that $\|t\|\leq 1$ also suffices. Thus

$$\Gamma^1(T)=2\sup\left\{\frac{T_a(y)-T_b(y)}{\|a-y\|^2+\|b-y\|^2}\mid y\in B\Big(\frac{a+b}{2},\frac{\|a-b\|}{2}\Big)\right\}.$$

PROPOSITION 3.4. *Let u be differentiable function on H with Lipschitz derivative Du. Then $\Gamma^1(U)$ is equal to the Lipschitz constant of Du. Here U is the field of Taylor expansions of u.*

PROOF. Let L denote the Lipschitz constant of Du. For $x,y\in H$ we have

$$U_x(y)=u_x+\langle D_x u, y-x\rangle.$$

We may write

$$u_x-u_y=\int_0^1\langle D_{y+t(x-y)}u, x-y\rangle dt.$$

For any $x,y,z\in H$ there is

$$|U_x(z)-u_z|=|u_x-u_z+\langle D_x u, z-x\rangle|=\Big|\int_0^1\langle D_{z+t(x-z)}u-D_x u, x-z\rangle dt\Big|\leq\frac{1}{2}L\|z-x\|^2.$$

Therefore

$$|U_x(z)-U_y(z)|\leq|U_x(z)-u_z|+|u_z-U_y(z)|\leq\frac{1}{2}L(\|x-z\|^2+\|x-y\|^2).$$

Hence $\Gamma^1(U)\leq L$. Conversely, by Proposition 3.2, there is

$$\Gamma^1(U)\geq B_{x,y}=\frac{\|D_x u-D_y u\|}{\|x-y\|}$$

for all $x,y\in H$, $x\neq y$. Therefore $\Gamma^1(U)\geq L$. □

PROOF OF IV) OF THEOREM 2.1. Let T be a field of affine jets such that $\Gamma^1(T)<\infty$. Let $K=\frac{1}{2}\Gamma^1(T)$ and let A be the domain of T, i.e., the set where T is defined. As in the proof of the Kirszbraun theorem, it is enough to extend T to a point $x \notin A$. Thus, we need to show that there exist $u_x \in \mathbb{R}, D_x u \in H$ such that for all $a \in A$ and all $y \in H$ there is

$$-K \leq \frac{T_x(y) - T_a(y)}{\|x-y\|^2 + \|a-y\|^2} \leq K, \tag{3.1}$$

where $T_x(y) = u_x + \langle D_x u, y - x\rangle$. If $K = 0$, we see that T is constant on A, so that the extension to x is trivial. In what follows we assume that $K > 0$. Then (3.1) is, by Proposition 3.2, equivalent to

$$|A_{a,x}| + \sqrt{A_{a,x}^2 + B_{a,x}^2} \leq 2K \text{ for any } a \in A.$$

In other words

$$|A_{a,x}| \leq K - \frac{B_{a,x}^2}{4K}. \tag{3.2}$$

Note that u_x appears only in $A_{a,x}$, so that it will be possible to eliminate it in the following way. The condition (3.2) is equivalent to that for any $a \in A$ there is

$$u_x \leq u_a + \frac{1}{2}\langle D_a u + D_x u, x - a\rangle + \frac{K}{2}\|a - x\|^2 - \frac{1}{8K}\|D_a u - D_x u\|^2$$

and that for any $b \in A$ there is

$$u_x \geq u_b + \frac{1}{2}\langle D_b u + D_x u, x - b\rangle - \frac{K}{2}\|b - x\|^2 + \frac{1}{8K}\|D_b u - D_x u\|^2.$$

There exists such u_x if and only if for all $a, b \in A$ there is

$$\begin{aligned} u_b + \frac{1}{2}\langle D_b u + D_x u, x - b\rangle - \frac{K}{2}\|b - x\|^2 + \frac{1}{8K}\|D_b u - D_x u\|^2 \leq \\ \leq u_a + \frac{1}{2}\langle D_a u + D_x u, x - a\rangle + \frac{K}{2}\|a - x\|^2 - \frac{1}{8K}\|D_a u - D_x u\|^2. \end{aligned}$$

We need to prove existence of D_x that satisfies the above inequality.

The above inequality, for any $a, b \in A$, may be restated that some quadratic form in $D_x u$ is non-positive. More precisely, the above is equivalent to

$$\|D_x u - V_{a,b}\|^2 \leq \alpha_{a,b} + \beta_{a,b} \text{ for all } a, b \in A, \tag{3.3}$$

where

$$\begin{aligned} V_{a,b} &= \frac{1}{2}(D_a u + D_b u) + K(b - a), \\ \alpha_{a,b} &= 4K(u_a - u_b) + 2K\langle D_a u + D_b u, b - a\rangle - \frac{1}{2}\|D_a u - D_b u\|^2 + 2K^2\|a - b\|^2, \\ \beta_{a,b} &= \left\|\frac{1}{2}(D_a u - D_b u) + K(2x - a - b)\right\|^2. \end{aligned}$$

Observe that

$$\alpha_{a,b} = \frac{1}{2}\big(4KA_{a,b} - B_{a,b}^2 + 4K^2\big)\|a - b\|^2 \geq 0,$$

by the definition of K, see (3.2). Set $r_{a,b} = \sqrt{\alpha_{a,b} + \beta_{a,b}}$. Then (3.3) becomes

$$\|D_x u - V_{a,b}\| \leq r_{a,b} \text{ for all } a, b \in A,$$

or, in other words, $D_x u \in \bigcap_{a,b\in A} B(V_{a,b}, r_{a,b})$. By compactness it is enough to veriify this condition for any finite subset of A. Thus, without of generality, we assume that A is finite.

In what follows we shall need two lemmata below. □

For $a, b, c, d \in A$ and $y \in H$ set

$$\Phi((a,b),(c,d)) = r_{a,b}^2 + r_{c,d}^2 - \|V_{a,b}\|^2 - \|V_{c,d}\|^2$$

and

$$X_{a,y} = \frac{1}{2} D_a u + K(y-a), Y_{b,y} = \frac{1}{2} D_b u + K(b-y).$$

LEMMA 3.5. *For any $a, b, c, d \in A$ and $y \in H$ there is*

$$\Phi((a,b),(c,d)) \geq -4\langle X_{a,y}, Y_{d,y}\rangle - 4\langle X_{c,y}, Y_{b,y}\rangle.$$

PROOF. We write $\alpha_{a,b} + \alpha_{c,d} = \delta_{11} + \delta_{12} + \delta_2 + \delta_3 + \delta_4$, where

$$\delta_{11} = 4K(u_a - u_d) + 2K\langle D_a u + D_d u, d-a\rangle - \frac{1}{2}\|D_a u - D_d u\|^2,$$

$$\delta_{12} = 4K(u_c - u_b) + 2K\langle D_c u + D_d u, b-c\rangle - \frac{1}{2}\|D_c u - D_b u\|^2,$$

$$\delta_2 = 2K\big(\langle D_a u, b-d\rangle + \langle D_b u, c-a\rangle + \langle D_c u, d-b\rangle + \langle D_d u, a-c\rangle\big),$$

$$\delta_3 = \frac{1}{2}\big(- \|D_a u - D_b u\|^2 - \|D_c u - D_d u\|^2 + \|D_a u - D_d u\|^2 + \|D_c u - D_b u\|^2\big),$$

$$\delta_4 = 2K^2(\|a-b\|^2 + \|c-d\|^2).$$

Now, using (3.2) for pairs (a,d) and (b,c), we infer that

$$\delta_{11} + \delta_{12} \geq -2K^2\big(\|d-a\|^2 + \|b-c\|^2\big).$$

Therefore

$$\alpha_{a,b} + \alpha_{c,d} \geq \delta_2 + \delta_3 + \rho_1,$$

where

$$\rho_1 = 2K^2\big(\|a-b\|^2 + \|c-d\|^2 - \|a-d\|^2 - \|b-c\|^2\big).$$

Observe that

$$\rho_1 + \delta_2 + \delta_3 = 4\big(- \langle X_{a,y}, Y_{d,y}\rangle - \langle X_{c,y}, Y_{b,y}\rangle + \langle X_{a,y}, Y_{b,y}\rangle + \langle X_{c,y}, Y_{d,y}\rangle\big).$$

Since

$$\beta_{a,b} + \beta_{c,d} = \|X_{a,y} - Y_{b,y}\|^2 + \|X_{c,y} - Y_{d,y}\|^2,$$

and

$$-\|V_{a,b}\|^2 - \|V_{c,d}\|^2 = -\|X_{a,y} + Y_{b,y}\|^2 - \|X_{c,y} + Y_{d,x}\|^2,$$

we have

$$\beta_{a,b} + \beta_{c,d} - \|V_{a,b}\|^2 - \|V_{c,d}\|^2 = -4\big(\langle X_{a,y}, Y_{b,y}\rangle + \langle X_{c,y}, Y_{d,y}\rangle\big).$$

This completes the proof. □

LEMMA 3.6. *Suppose that there is $a, b \in A$ such that $r_{a,b} = 0$. Then*

$$\bigcup_{c,d \in A} B(V_{c,d}, r_{c,d}) \neq \emptyset.$$

PROOF. Let $c, d \in A$. Then

$$r_{a,b}^2 + r_{c,d}^2 - \|V_{a,b} - V_{c,d}\|^2 = \Phi((a,b),(c,d)) + 2\langle V_{a,b}, V_{c,d}\rangle. \tag{3.4}$$

The condition $r_{a,b} = 0$ implies that

$$X_{a,y} = Y_{b,y} \text{ and } V_{a,b} = X_{a,y} + Y_{b,y}.$$

Lemma 3.5 implies that

$$\Phi((a,b),(c,d)) \geq -4\big(\langle X_{a,y}, Y_{d,y}\rangle + \langle X_{c,y}, Y_{b,y}\rangle\big) = -2\langle V_{c,d}, V_{a,b}\rangle.$$

By (3.4) and the above we see that

$$\|V_{a,b} - V_{c,d}\| \le r_{c,d} \text{ for all } c, d \in A.$$

Hence $V_{a,b}$ belongs to the intersection in the statement of the lemma. □

By the lemma above, it is now enought to handle situation when $r_{a,b} > 0$ for all $a, b \in A$.

PROOF OF IV) OF THEOREM 2.1. From the proof of Kirszbraun's theorem it follows that if $\lambda \ge 0$ is a minimal number such that

$$\bigcap_{a,b\in A} B(V_{a,b}, \lambda r_{a,b}) \neq \emptyset,$$

then the intersection contains a single element V_m which belongs to the convex hull of $V_{a,b}$, with (a, b) in the set

$$E = \{(a, b) \in A \times A \mid \|V_m - V_{a,b}\| = \lambda r_{a,b}\}.$$

Therefore there exist non-negative $\xi_{a,b}$ that sum up to one such that

$$V_m = \sum_{a,b\in E} \xi_{a,b} V_{a,b}. \tag{3.5}$$

We want to prove that $\lambda \le 1$. By (3.5) we see that

$$0 = \sum_{(a,b),(c,d)\in E} \xi_{a,b}\xi_{c,d}\langle V_m - V_{a,b}, V_m - V_{c,d}\rangle. \tag{3.6}$$

Note that

$$\|V_{c,d} - V_{a,b}\|^2 = \|V_m - V_{a,b}\|^2 + \|V_m - V_{c,d}\|^2 - 2\langle V_m - V_{a,b}, V_m - V_{c,d}\rangle.$$

Therefore for $(a, b), (c, d) \in E$ we have

$$\|V_{c,d} - V_{a,b}\|^2 = \lambda^2 r_{a,b}^2 + \lambda^2 r_{c,d}^2 - 2\langle V_m - V_{a,b}, V_m - V_{c,d}\rangle.$$

Multipying these equations by respective coefficients $\xi_{a,b}\xi_{c,d}$ and employing (3.6) we get

$$0 = \sum_{(a,b),(c,d)\in E} \xi_{a,b}\xi_{c,d}\big(- \|V_{c,d} - V_{a,b}\|^2 + \lambda^2 r_{a,b}^2 + \lambda^2 r_{c,d}^2\big).$$

Let us denote

$$\Delta = \sum_{(a,b),(c,d)\in E} \xi_{a,b}\xi_{c,d}\big(- \|V_{c,d} - V_{a,b}\|^2 + r_{a,b}^2 + r_{c,d}^2\big).$$

Then

$$\Delta = (1 - \lambda^2) \sum_{(a,b),(c,d)\in E} \xi_{a,b}\xi_{c,d}(r_{a,b}^2 + r_{c,d}^2).$$

The assumption on the radii implies that

$$\sum_{(a,b),(c,d)\in E} \xi_{a,b}\xi_{c,d}(r_{a,b}^2 + r_{c,d}^2) > 0.$$

To complete the proof we thus need to show that $\Delta \ge 0$.

Observe that

$$\Delta = 2\|V_m\|^2 + \sum_{(a,b),(c,d)\in E} \xi_{a,b}\xi_{c,d}\Phi((a, b), (c, d)).$$

Set

$$X = \sum_{(a,b)\in E} \xi_{a,b}\big(\frac{1}{2}D_a u + K(y-a)\big) = \sum_{(a,b)\in E} \xi_{a,b} X_{a,x}$$

and

$$Y = \sum_{(a,b)\in E} \xi_{a,b}\big(\frac{1}{2}D_b u + K(b-y)\big) = \sum_{(a,b)\in E} \xi_{a,b} Y_{b,y}.$$

Then

$$X + Y = \sum_{(a,b)\in E} \xi_{a,b} V_{a,b},$$

so that

$$\|X+Y\|^2 = \|V_m\|^2.$$

By Lemma 3.5 it follows that

$$\sum_{(a,b),(c,d)\in E} \xi_{a,b}\xi_{c,d}\Phi((a,b),(c,d)) \geq -8\langle X, Y\rangle.$$

Therefore

$$\Delta \geq 2\|X+Y\|^2 - 8\langle X, Y\rangle = 2\|X-Y\|^2 \geq 0.$$

This completes the proof. $\square$

CHAPTER 4

Ball's extension theorem

In this lecture we shall provide an extension theorem in the situations where it is not possible to preserve the Lipshitz constant.

EXAMPLE 0.1. An example of such situation is the case of the space ℓ^1. Suppose we consider a map $f\colon \{x_1, x_2, x_3\} \to \ell^1$, defined on the set $\{x_1, x_2, x_3\}$ of vertices of an equilateral triangle of sidelength 2 by the formulae

$$f(x_i) = e_i \text{ for } i = 1, 2, 3,$$

where $(e_i)_{i=1}^3$ are basis elements of ℓ^1. Then f is an isometry. However, it does not extend to a 1-Lipschitz map defined on $\{x_1, x_2, x_3, c\}$ where c is the barycentre of $\{x_1, x_2, x_3\}$. For if such $f(c)$ existed, then $\|f(c) - e_i\|_1 \leq \frac{\sqrt{3}}{3}$. Then $|f(c)_i - 1| \leq \frac{\sqrt{3}}{3}$, so $f(c)_i \geq 1 - \frac{\sqrt{3}}{3}$ for $i = 1, 2, 3$. We get a contradiction $\frac{\sqrt{3}}{3} \geq \|f(c) - e_1\|_1 \geq 2 - \frac{2\sqrt{3}}{3}$.

Therefore in such situations we cannot hope to preserve the Lipschitz constant, and we need to allow for more flexibility. Note that the techniques of extension of a function point by point do not apply readily. We need to come up with another approach. Our exposition will be based on [**1**] and [**15**].

The approach in this lecture will be analgous to the one used in the proof of the classical theorem of Maurey concerning extension of linear operators between Banach spaces enjoying finite type or cotype. The type and cotype describe the behaviour of sums of independent random variables in Banach spaces. The analogues of type and cotype introduced in [**1**] describe the behaviour of Markov chains in metric spaces.

Let us recall that the theorem of Maurey says that whenever X, Y are Banach spaces, X having type 2 with constant $T_2(X)$ and Y having cotype 2 with constant $C_2(Y)$, then any bounded operator $T\colon Z \to Y$ on a subspace $Z \subset X$ admits an extension to a bounded operator on X with norm at most $T_2(X)C_2(Y)\|T\|$.

Let us note a connection of these topics with the Ribe programme, proposed by Bourgain [**5**]; see also [**2**] and [**14**].

1. Markov type and cotype

Let us recall the definitions of type and cotype.

DEFINITION 1.1. Let X be a Banach space. Let $\epsilon_1, \epsilon_2, \ldots$ be independent Bernoulli random variables. Let $p \in (1, 2]$. We say that X has type p whenever there is $K < \infty$ such that for all $n \in \mathbb{N}$ and all $x_1, x_2, \ldots \in X$ there is

$$\mathbb{E}\Big\| \sum_{i=1}^{n} \epsilon_i x_i \Big\|^p \leq K^p \sum_{i=1}^{n} \|x_i\|^p.$$

Similarly, X has cotype $q \in [2, \infty)$, whenever for some constant L, all $n \in \mathbb{N}$ and all sequences $x_1, x_2, \ldots \in X$ there is

$$\mathbb{E}\Big\| \sum_{i=1}^n \epsilon_i x_i \Big\|^q \geq L^q \sum_{i=1}^n \|x_i\|^q.$$

In what follows we shall denote

$$\Delta = \Big\{ (x_1, \ldots, x_n) \in [0,1]^n \mid \sum_{i=1}^n x_i = 1 \Big\}.$$

We think of Δ as the space of probabilites on the set $\{1, \ldots, n\}$.

We say that an $n \times n$ matrix A is stochastic if it has non-negative entries and all of its rows add up to one.

Given $\pi \in \Delta$ a stochastic matrix A is reversible relative to π if for all $i, j = 1, \ldots, n$ there is $\pi_i a_{ij} = \pi_j a_{jii}$.

These notions stem from the Markov chains theory. A sequence $(X_i)_{i=0}^\infty$ of random variables with values in $\{1, \ldots, n\}$ is a Markov chain provided that

$$\mathbb{P}(X_{k+1} = t_{k+1} \mid X_k = t_k, \ldots, X_0 = t_0) = \mathbb{P}(X_{k+1} = t_{k+1} \mid X_k = t_k)$$

for all $t_0, \ldots, t_{k+1} \in \{1, \ldots, n\}$. A Markov chain is called stationary if for all $k, l \in \mathbb{N}$ and all $t_0, \ldots, t_{k+1} \in \{1, \ldots, n\}$ there is

$$\mathbb{P}(X_{k+1} = t_{k+1}, X_k = t_k, \ldots, X_0 = t_0) = \mathbb{P}(X_{k+l+1} = t_{k+1}, X_{k+l} = t_k, \ldots, X_l = t_0)$$

Then a stochastic matrix A is its transition matrix if for all $i, j \in \{1, \ldots, n\}$ there is

$$a_{ij} = \mathbb{P}(X_{k+1} = i \mid X_k = j) \text{ for all } k \in \mathbb{N}.$$

A measure π on the state space $\{1, \ldots, n\}$ is a stationary measure of the chain if any X_j is distributed according to π, for $j \in \mathbb{N}$. A stationary Markov chain with stationary measure π and transition matrix A is reversible if the detailed balance condition holds true. That is if

$$\pi_i a_{ij} = \pi_j a_{ji} \text{ for } i, j = 1, \ldots, n.$$

Definition 1.2. Let (X, d) be a metric space. We say that X has a Markov type $p \in (0, \infty)$ with constant M if for every stationary and reversible Markov chain $(X_i)_{i=0}^\infty$ on $\{1, \ldots, n\}$ and any function $f \colon \{1, \ldots, n\} \to X$, any $k \in \mathbb{N}$, there is

$$\mathbb{E} d(f(X_k), f(X_0))^p \leq M^p k \mathbb{E} d(f(X_1), f(X_0))^p.$$

An optimal constant M is the Markov type-p constant of X. We denote it by $M_p(X)$.

Clearly, any metric space has Markov type 1 with constant 1, by the triangle inequality.

Remark 1.3. The above definition may be reformulated as follows. Metric space X has Markov type $p \in (0, \infty)$ with constant M if for any $\pi \in \Delta$, any $n \times n$ stochastic matrix A reversible relative to π and any $x_1, \ldots, x_n$ there is

$$\sum_{i,j=1}^n \pi_i A_{ij}^k d(x_i, x_j)^p \leq M^p k \sum_{i,j=1}^n \pi a_{ij} d(x_i, x_j)^p$$

for all $k \in \mathbb{N}$.

Lemma 1.4. *Let H be a Hilbert space. Then H has Markov type 2 with constant $M_2(H) = 1$.*

Proof. Let $\pi \in \Delta$ and let A be an $n \times n$ stochastic matrix reversible relative to π. We need to prove that

$$\sum_{i,j=1}^{n} \pi A_{ij}^k \|x_i - x_i\|^2 \le t \sum_{i,j=1}^{n} \pi a_{ij} \|x_i - x_i\|^2. \tag{1.1}$$

Clearly, we may assume that H is one-dimensional, as the general case follows from this one.

Consider the space $L^2(\pi)$; i.e. the space $\mathbb{R}^n$ with scalar product defined by the formula

$$\langle v, w\rangle = \sum_{i=1}^{n} v_i w_i \pi_i \text{ for } v, w \in \mathbb{R}^n.$$

Then the right-hand side of (1.1) may be written as, by reversibility and stochasticity,

$$k \sum_{i,j=1}^{n} \pi_i a_{ij}\big(x_i^2 + x_j^2 - 2x_i x_j\big) = 2k \sum_{i,j=1}^{n} \pi_i a_{ij} x_i^2 - 2\langle Ax, x\rangle = 2k\langle (\mathrm{Id} - A)x, x\rangle.$$

Similarly we express the left-hand side, with A replaced by A^k. Then the inequality we are to prove is

$$\langle k(\mathrm{Id} - A) - (\mathrm{Id} - A^k)x, x\rangle \ge 0.$$

The reversibility of A is equivalent to A being self-adjoint on $L^2(\pi)$. Therefore it is diagonalisable and has real eigenvalues. It is therefore enough to prove that for any eigenvalue λ of A there is

$$k(1 - \lambda) \ge 1 - \lambda^k.$$

It is enough to prove that $\lambda \le 1$, as then $k \ge 1 + \lambda + \ldots + \lambda^{k-1} = \frac{1-\lambda^k}{1-\lambda}$. For this we employ stochasticity of A:

$$\|Ay\|_{L^1(\pi)} = \sum_{i=1}^{n} \pi_i \Big| \sum_{j=1}^{n} a_{ij} y_j \Big| \le \sum_{i,j=1}^{n} \pi_i a_{ij} |y_j| = \sum_{i,j=1}^{n} \pi_j a_{ji} |y_j| = \|y\|_{L^1(\pi)}.$$

Thus $|\lambda| \le 1$ and the lemma is proven. □

For the extension theorem we shall also need a dual notion of Markov cotype, analogously to the situation of extension of operators acting on Banach spaces.

Definition 1.5. A metric space (X, d) has metric Markov cotype $q \in (0, \infty)$ with constant N if for every $n, k \in \mathbb{N}$, every $\pi \in \Delta$, any stochastic $n \times n$ matrix A reversible relative to π, any $x_1, \ldots, x_n \in X$ there exist $y_1, \ldots, y_n \in X$ such that

$$\sum_{i=1}^{n} \pi_i d(x_i, y_i)^q + k \sum_{i,j=1}^{n} \pi_i a_{ij} d(y_i, y_j)^q \le N^q \sum_{i,j=1}^{n} \pi_i \left(\frac{1}{k} \sum_{l=1}^{k} A_{ij}^l\right) d(x_i, x_j)^q.$$

An optimal constant is the metric Markov cotype-q constant of X and is denoted by $N_p(X)$.

REMARK 1.6. In the language of Markov chains, the above condition may be rewritten in the following form: for any $f\colon \{1,\dots,n\} \to X$, any stationary and reversible Markov chain $(X_i)_{i=0}^{\infty}$ there exists a function $g\colon \{1,\dots,n\} \to X$ such that for all $k \in \mathbb{N}$ there is

$$\mathbb{E}d(f(X_0), g(X_0))^q + k\mathbb{E}d(g(X_1), g(X_0))^q \leq N^q \frac{1}{k}\sum_{l=1}^{k} \mathbb{E}d(f(X_l), f(X_0))^q.$$

REMARK 1.7. Note the apperance of function g, or points $y_1,\dots,y_n \in X$ in the above definition. If $g = f$, or $y_i = x_i$ for $i = 1,\dots,n$, then we would simply get a reverse inequality to the one defining the Markov type, but with Cesàro average, for technical reasons. However, this introduction is necessary, as otherwise the inequality would be false, by consideration of constant chain distributed on two points of X.

DEFINITION 1.8. Let (X,d) be a metric space and let $\mathcal{P}(X)$ denote the space of finitely supported probability measures on X. We say that (X,d) is W^p-barycentric with constant Γ if there exists a map $B\colon \mathcal{P}(X) \to X$, called the barycentre map, such that for all $x \in X$ there is $B(\delta_x) = x$ and

$$d\Big(B\Big(\sum_{i=1}^{n} \lambda_i \delta_{x_i}\Big), B\Big(\sum_{i=1}^{n} \lambda_i \delta_{y_i}\Big)\Big)^p \leq \Gamma^p \sum_{i=1}^{n} \lambda_i d(x_i,y_i)^p$$

for all $x_i, y_i \in X$, $i = 1,\dots,n$ and all non-negative λ_i, $i = 1,\dots,n$ that sum up to one.

Note that Banach spaces are 1-barycentric with constant 1. The barycentre map is simply the expectation of the measure.

2. Statement of the theorem

In this section we shall state and prove a generalisation by Naor and Mendel,see e.g. [**15**], of the Ball's extension theorem of [**1**].

THEOREM 2.1. *Let (X,d_x) and (Y,d_Y) be two metric spaces and let $p \in (0,\infty)$. Suppose that X has metric Markov type p with constant $M_p(X)$ and Y has metric Markov cotype p with constant $N_p(Y)$ and that Y is W^p-barycentric with constant Γ.*

Suppose that $Z \subset X$ and that $f\colon Z \to Y$ is a Lipschitz map with Lipschitz constant L. Then for any finite set $S \subset X$ there is a map $f_S\colon S \to Y$ such that $f_S|_{S\cap Z} = f_{S\cap Z}$ and the Lipschitz constant of f_S is at most $C\Gamma M_p(X)N_p(Y)L$, where C is a universal constant.

REMARK 2.2. If Y is a reflexive Banach space, then the finite extension property ensures that there also exist an extension to entire space X.

LEMMA 2.3. *Suppose that (X,d_X) and (Y,d_Y) are two metric spaces such that Y is W^p-barycentric with constant Γ, $Z \subset X$, $f\colon Z \to Y$ and $\epsilon > 0$.*

Suppose also that there exists a constant $K > 0$ such that for any $n \in \mathbb{N}$, any $x_1,\dots,x_n \in X$ and every symmetric $n \times n$ matrix H with non-negative entries there exists $\Phi^H\colon \{x_1,\dots,x_n\} \to Y$ such that:

i)

$$\Phi^H|_{\{x_1,\dots,x_n\}\cap Z} = f_{\{x_1,\dots,x_n\}\cap Z},$$

ii)
$$\sum_{i,j=1}^{n} h_{ij} d_Y(\Phi^H(x_i), \Phi^H(x_j)) \leq K^p L^p \sum_{i,j=1}^{n} h_{ij} d_X(x_i, x_j)^p,$$
where L is the Lipschitz constant of f.

Then there exists $F\colon \{x_1, \ldots, x_n\} \to Y$, which agrees with f on $\{x_1, \ldots, x_n\} \cap Z$ and whose Lipschitz constant is at most $(1+\epsilon)\Gamma K L$.

Proof. Let us fix points $x_1, \ldots, x_n \in X$. Consider sets C, D of $n \times n$ symmetric matrices given by the formulae

$$C = \Big\{ \big(d_Y(\Phi(x_i), \Phi(x_j))\big)_{i,j=1}^n \mid \Phi\colon \{x_1, \ldots, x_n\} \to Y \text{ agrees with } f \text{ on } \{x_1, \ldots, x_n\} \cap Z \Big\},$$
$$D = \big\{ M \mid M \text{ has non-negative entries} \big\}.$$

Let E be the closed convex hull of the sum $C + D$. Consider matrix T with entries

$$t_{ij} = K^p L^p d_X(x_i, x_j)^p \text{ for } i, j = 1, \ldots, n.$$

We need to prove that $T \in E$. Indeed, if $T \in E$, then there are non-negative $\lambda_1, \ldots, \lambda_m$ that add up to one and maps $\Phi_1, \ldots, \Phi_m \colon \{x_1, \ldots, x_m\} \to Y$, which all agree with f on $\{x_1, \ldots, x_n\} \cap Z$ and such that for each $i, j = 1, \ldots, n$ there is

$$\sum_{k=1}^{m} \lambda_k d_Y(\Phi_k(x_i), \Phi_k(x_j)) \leq (1+\epsilon)^p K^p L^p d_X(x_i, x_j)^p.$$

Define now $\mu_i = \sum_{k=1}^m \lambda_k \delta_{\Phi_k(x_i)}$ for each $i = 1, \ldots, n$ and $F\colon \{x_1, \ldots, x_n\} \to Y$ by the formula $F(x_i) = B(\mu_i)$, where B is the barycentre map of Y. Then F agrees with f on $\{x_1, \ldots, x_n\} \cap Z$, as all $\Phi_1, \ldots, \Phi_m$ did. Moreover,

$$d_Y(F(x_i), F(y_i))^p \leq \Gamma^p \sum_{k=1}^{m} \lambda_k d_Y(\Phi_k(x_i), \Phi_k(x_j))^p \leq (1+\epsilon)^p \Gamma^p K^p L^p d_X(x_i, x_j)^p.$$

This is the desired inequality.

We shall now prove that $T \in E$. If not, then by the Hahn–Banach theorem there would exist a symmetric matrix $H = (h_{ij})_{i,j=1}^n)$ such that

$$\sum_{i,j=1}^{n} h_{ij} t_{ij} < \inf \Big\{ \sum_{i,j=1}^{n} h_{ij} m_{ij} \mid M \in E \Big\}.$$

Testing the above inequality by a matrix $\rho(e_i e_j^* + e_j e_i^*)$ for large ρ, we see that necessarily there is $h_{ij} \geq 0$ for all $i, j = 1, \ldots, n$. Moreover, as $C \subset E$, for any $\Phi\colon \{x_1, \ldots, x_n\} \to Y$ that agrees with f on $\{x_1, \ldots, x_n\} \cap Z$ there is

$$K^p L^p \sum_{i,j=1}^{n} h_{ij} d_X(x_i, x_j)^p < \sum_{i,j=1}^{n} h_{ij} d_Y(\Phi(x_i), \Phi(x_j))^p$$

which stands in contradiction with the assumption. □

Before we pass to the proof of the Ball's extension theorem, let us prove the following lemma.

Lemma 2.4. *Let $m, n \in \mathbb{N}$, $p \geq 1$. Let C be an $n \times n$ stochastic matrix which is reversible relative to some $\pi \in \Delta$. Let also B be an $m \times n$ stochastic matrix. Then* *for any metric space (X, d_X), any $z_1, \dots, z_m \in X$ there are $w_1, \dots, w_n \in X$ such that*

$$3^p \sum_{i,j=1}^{m} (B^* D_\pi CB)_{ij} d_X(z_i, z_j)^p \geq \max\Big\{ \sum_{i=1}^{n} \sum_{r=1}^{m} \pi_i b_{ir} d_X(w_i, z_r)^p, \sum_{i,j=1}^{n} \pi_i c_{ij} d_X(w_i, w_j)^p \Big\}.$$

Here D_π denotes the diagonal matrix with elements $\pi_1, \dots, \pi_n$ on the diagonal.

Proof. Let $f\colon \{z_1, \dots, z_m\} \to \ell^\infty$ be an isometric embedding, given e.g. by the formula $f(z) = (d(z, z_i))_{i=1}^m$. Define

$$y_i = \sum_{r=1}^{m} b_{ir} f(z_r) \text{ for } i = 1, \dots, n.$$

Let $w_i \in \{z_1, \dots, z_m\}$ be such that

$$\|y_i - w_i\|_{\ell^\infty} = \min\big\{ \|y_i - f(z_j)\|_{\ell^\infty} \mid j = 1, \dots, m \big\}.$$

By the triangle inequality and by Jensen's inequality we have

$$d_X(w_i, w_j)^p = \|f(w_i) - f(w_j)\|_{\ell^\infty}^p \leq 3^{p-1}\big(\|f(w_i) - y_i)\|_{\ell^\infty}^p + \|f(w_j) - y_j\|_{\ell^\infty}^p + \|y_i - y_j\|_{\ell^\infty}^p \big).$$

By the reversibility and stochasticity of C we get

$$\begin{aligned} &\sum_{i,j=1}^{n} \pi_i c_{iji} d_X(w_i, w_j)^p \leq \\ &\leq 3^{p-1} \sum_{i,j=1}^{n} \pi_i c_{ij} \|y_i - y_j\|_{\ell^\infty}^p + 2 \cdot 3^{p-1} \sum_{i=1}^{n} \pi_i \|y_i - f(w_i)\|_{\ell^\infty}^p. \end{aligned} \tag{2.1}$$

Therefore

$$\begin{aligned} \sum_{i,j=1}^{n} \pi_i c_{ij} \|y_i - y_j\|_{\ell^\infty}^p &= \sum_{i,j=1}^{n} c_{ij} \Big\| \sum_{r,s=1}^{m} b_{ir} b_{js} \big(f(z_r) - f(z_s)\big) \Big\|_{\ell^\infty}^p \leq \\ &\leq \sum_{i,j=1}^{n} \sum_{r,s=1}^{m} \pi_i c_{ij} b_{ir} b_{js} \|f(z_r) - f(z_s)\|_{\ell^\infty}^p = \sum_{r,s=1}^{m} (B^* D_\pi CB)_{rs} d_X(z_r, z_s)^p. \end{aligned}$$

To bound the second sum in (2.1) we take into account that $\|y_i - f(w_i)\|_{\ell^\infty} \leq \|y_i - f(z_r)\|_{\ell^\infty}$ for any $i = 1, \dots, n$ and $r = 1, \dots, m$. Moreover, as matrices C, B are stochastic,

$$\sum_{r=1}^{m} (CB)_{ir} = 1 \text{ for each } i = 1, \dots, n.$$

Thus, by Jensen's inequality,

$$\begin{aligned} &\sum_{i=1}^{n} \pi_i \|y_i - f(w_i)\|_{\ell^\infty}^p = \sum_{i=1}^{n} \sum_{r=1}^{m} \pi_i (CB)_{ir} \|y_i - f(w_i)\|_{\ell^\infty}^p \leq \\ &\leq \sum_{i=1}^{n} \sum_{r=1}^{m} \pi_i (CB)_{ir} \|y_i - f(z_r)\|_{\ell^\infty}^p = \sum_{i=1}^{n} \sum_{r=1}^{m} \pi_i (CB)_{ir} \Big\| \sum_{s=1}^{m} b_{is} \big(f(z_s) - f(z_r)\big) \Big\|_{\ell^\infty}^p \leq \\ &\leq \sum_{i=1}^{n} \sum_{r,s=1}^{m} \pi_i (CB)_{ir} b_{is} \|f(z_s) - f(z_r)\|_{\ell^\infty}^p = \sum_{r,s=1}^{m} (B^* D_\pi CB)_{rs} d_X(z_r, z_s)^p. \end{aligned}$$

This proves the bound for the second term in the inequality in the statement of the lemma. We now pass to a proof of the bound for the first term.

By the triangle inequality and Jensen's inequality we have for each $i, j = 1, \dots, n$

$$d_X(w_i, z_r)^p \le 3^{p-1}\big(\|f(w_i) - y_i\|_{\ell\infty}^p + \|y_j - f(z_r)\|_{\ell\infty}^p + \|y_i - y_j\|_{\ell\infty}^p\big).$$

Since C is stochastic, we get

$$\sum_{i=1}^n \sum_{r=1}^m \pi_i b_{ir} d_X(w_i, z_r)^p = \sum_{i,j=1}^n \sum_{r=1}^m \pi_i b_{ir} c_{ij} d_X(w_i, z_r)^p.$$

Observe that the quantities

$$\sum_{i,j=1}^n \sum_{r=1}^m \pi_i b_{ir} c_{ij} \|f(w_i) - y_i\|_{\ell\infty}^p = \sum_{i=1}^n \pi_i \|f(w_i) - y_i\|_{\ell\infty}^p$$

and

$$\sum_{i,j=1}^n \sum_{r=1}^m \pi_i b_{ir} c_{ij} \|y_i - y_j\|_{\ell\infty}^p = \sum_{i=1}^n \pi_i \|y_i - y_j\|_{\ell\infty}^p$$

have already been appropriately bounded. It is enough thus to bound

$$\sum_{i,j=1}^n \sum_{r=1}^m \pi_i b_{ir} c_{ij} \|f(z_r) - y_j\|_{\ell\infty}^p = \sum_{i,j=1}^n \sum_{r=1}^m \pi_i b_{ir} c_{ij} \Big\| \sum_{s=1}^m b_{js}\big(f(z_s) - f(z_r)\big)\Big\|_{\ell\infty}^p \le$$
$$\le \sum_{i,j=1}^n \sum_{r,s=1}^m \pi_i b_{ir} c_{ij} b_{js} \|f(z_s) - f(z_r)\|_{\ell\infty}^p = \sum_{r,s=1}^m (B^* D_\pi C B)_{rs} d_X(z_r, z_s)^p.$$

The proof is complete. □

We shall now provide a proof of the generalised Ball's extension theorem.

Proof of Theorem 2.1. Let $M > M_p(X), N > N_p(Y)$. Fix some $m, n \in \mathbb{N}$, some $x_1, \dots, x_n \in X \setminus Z$ and $z_1, \dots, z_m \in Z$. We shall prove that the assumptions of Lemma 2.3 are satisfied.

Consider a $(n+m) \times (n+m)$ symmetric matrix H with non-negative entries. Let $V(H), U(H)$ be the $n \times n$ and $m \times m$ symmetric block-diagonal submatrices of H and let $W(H)$ be the off-diagonal $n \times m$ matrix. Set

$$R_H = \sum_{r,s=1}^m u_{rs} d_X(z_r, z_s)^p + 2\sum_{i=1}^n \sum_{r=1}^m w_{ir} d_X(x_i, z_r)^p + \sum_{i,j=1}^n v_{ij} d_X(x_i, x_j)^p$$

and for $y_1, \dots, y_n \in Y$ let

$$L_H(y_1, \dots, y_n) = \sum_{r,s=1}^m u_{rs} d_Y(f(z_r), f(z_s))^p + 2\sum_{i=1}^n \sum_{r=1}^m w_{ir} d_Y(y_i, f(z_r))^p + \sum_{i,j=1}^n v_{ij} d_Y(y_i, y_j)^p.$$

By Lemma 2.3 it suffices to prove that for some $y_1, \dots, y_n \in Y$ there is

$$L_H(y_1, \dots, y_n) \le \Lambda R_H, \text{ where } \Lambda = \frac{18^p}{3}(N^p + 1) M^p L^p. \tag{2.2}$$

Let $\delta > 0$. We first observe that

$$\sum_{r,s=1}^m u_{rs} d_Y(f(z_r), f(z_s))^p \le L^p \sum_{r,s=1}^m u_{rs} d_X(z_r, z_s)^p,$$

so it suffices to prove existence of $y_1,\dots,y_n\in Y$ that satisfy

(2.3)
$$2\sum_{i=1}^{n}\sum_{r=1}^{m} w_{ir}d_Y(y_i,f(z_r))^p+\sum_{i,j=1}^{n} v_{ij}d_Y(y_i,y_j)^p\le\Lambda\Bigg(2\sum_{i=1}^{n}\sum_{r=1}^{m} w_{ir}d_X(x_i,z_r)^p+\sum_{i,j=1}^{n} v_{ij}d_X(x_i,x_j)^p+\delta\Bigg).$$

Since metrics vanish on the diagonal, we may assume that $V(H),U(H)$ vanish on the diagonal as well.

We claim that for $t>0$ large enough there exists $\theta>0$ and $\pi\in\Delta$ such that

$$w_{ir}=\theta\pi_i b_{ir} \text{ for every } i=1,\dots,n \text{ and } r=1,\dots,m$$

and

$$v_{ij}=\theta t\pi_i a_{ij} \text{ for every } i,j=1,\dots,n, i\neq j.$$

Here A,B are two stochastic matrices of dimensions $n\times n$ and $m\times n$. Moreover, A is reversible relative to π. Indeed, we take

$$\theta=\sum_{i=1}^{n}\sum_{j=1}^{r} w_{ir},\pi_i=\frac{1}{\theta}\sum_{s=1}^{m} w_{is}, b_{ir}=\frac{w_{ir}}{\sum_{s=1}^{m} w_{is}} \text{ for } i=1,\dots,n,r=1,\dots,m$$

and

$$a_{ii}=1-\frac{1}{t}\frac{\sum_{j=1}^{n} v_{ij}}{\sum_{r=1}^{m} w_{ir}} \text{ for all } i=1,\dots,n \text{ and } a_{ij}=\frac{1}{t}\frac{v_{ij}}{\sum_{r=1}^{m} w_{ir}} \text{ for } i\neq j.$$

Under this change of parameters the left-hand side of the inequality (2.3) may be rewritten as

$$\theta\Bigg(2\sum_{i=1}^{n}\sum_{r=1}^{m}\pi_i b_{ir}d_Y(y_i,f(z_r))^p+t\sum_{i,j=1}^{n}\pi_i a_{ij}d_Y(y_i,y_j)^p\Bigg). \tag{2.4}$$

Denote by $\tau=\lceil\frac{t}{2^p}\rceil$ and by $C_\tau(A)=\frac{1}{\tau}\sum_{s=1}^{\tau}A^s$ the Cesàro averages. Now $C_\tau(A)$ is reversible relative to π, so Lemma 2.4 guarantees existence of $w_1,\dots,w_n\in Y$ such that

$$\begin{aligned}3^p\sum_{r,s=1}^{m}(B^*D_\pi C_\tau(A)B)_{rs}d_Y(f(z_r),f(z_s))^p\ge\\ \ge\max\Bigg\{\sum_{i=1}^{n}\sum_{r=1}^{m}\pi_i b_{ir}d_Y(w_i,f(z_r))^p,\sum_{i,j=1}^{n}\pi_i C_\tau(A)_{ij}d_Y(w_i,w_j)^p\Bigg\}.\end{aligned}$$

From the definition of the Markov cotype there exist $y_1,\dots,y_n\in Y$ such that

$$\sum_{i=1}^{n}\pi_i d_Y(w_i,y_i)^p+\tau\sum_{i,j=1}^{n}\pi_i a_{ij}d_Y(y_i,y_j)^p\le N^p\sum_{i,j=1}^{n}\pi_i C_\tau(A)_{ij}d_Y(w_i,w_j)^q. \tag{2.5}$$

We will show that if t is large enough, then the appropriate $y_1,\dots,y_n$ will suffice. Observe that

$$d_Y(y_i,f(z_r))^p\le 2^{p-1}(d_Y(y_i,w_i)^p+d_Y(w_i,f(z_r))^p),$$

so we may bound the first sum in (2.4) as follows

$$2\sum_{i=1}^{n}\sum_{r=1}^{m}\pi_i b_{ir} d_Y(y_i, f(z_r))^p \le 2^p \sum_{i=1}^{n}\sum_{r=1}^{m}\pi_i b_{ir}\Big(d_Y(y_i,w_i)^p + d_Y(w_i, f(z_r))^p\Big) =$$
$$= 2^p \sum_{i=1}^{n}\pi_i d_Y(y_i,w_i)^p + 2^p \sum_{i=1}^{n}\sum_{r=1}^{m}\pi_i b_{ir} d_Y(w_i, f(z_r))^p.$$

Now, using corollary of Lemma 2.4 and the Markov cotype property, we get that the left-hand side of (2.4) may be bounded by

$$\theta\Bigg(2^p \sum_{i=1}^{n}\pi_i d_Y(y_i,w_i)^p + \tau \sum_{i,j=1}^{n}\pi_i a_{ij} d_Y(y_i,y_j)^p + 2^p \sum_{i=1}^{n}\sum_{r=1}^{m}\pi_i b_{ir} d_Y(w_i, f(z_r))^p\Bigg) \le$$
$$\le \theta\Bigg((2N)^p \sum_{i,j=1}^{n}\pi_i C_\tau(A)_{ij} d_Y(w_i,w_j)^p + 2^p \sum_{i=1}^{n}\sum_{r=1}^{m}\pi_i b_{ir} d_Y(w_i, f(z_r))^p\Bigg) \le$$
$$\le \theta\Bigg(6^p(N^p+1)\sum_{r,s=1}^{m}(B^* D_\pi C_\tau(A) B)_{rs} d_Y(f(z_r), f(z_s))\Bigg) \le$$
$$\le \theta\Bigg(6^p(N^p+1)L^p\sum_{r,s=1}^{m}(B^* D_\pi C_\tau(A) B)_{rs} d_X(z_r, z_s)\Bigg).$$

Now, we use the triangle inequality

$$d_X(z_r,z_s)^p \le 3^{p-1}\big(d_X(z_r,x_i)^p + d_X(x_i,x_j)^p + d_X(x_j,z_s)^p\big)$$

and the identitiy

$$(B^* D_\pi C_\tau(A) B)_{rs} = \sum_{i,j=1}^{n}\pi_i b_{ir} b_{js} C_\tau(A)_{ij}.$$

They give that the left-hand side of (2.4) may be continued to be bounded by

$$\frac{18^p}{3}(N^p+1)L^p\theta\sum_{i,j=1}^{n}\sum_{r,s=1}^{m}\pi_i b_{ir} b_{js} C_\tau(A)_{ij}\big(d_X(z_r,x_i)^p + d_X(x_i,x_j)^p + d_X(x_j,z_s)^p\big).$$

We have therefore three sums to deal with. We deal with the first and the third one analogously:

$$\sum_{i,j=1}^{n}\sum_{r,s=1}^{m}\pi_i b_{ir} b_{js} C_\tau(A)_{ij} d_X(z_r,x_i)^p = \sum_{i,j=1}^{n}\sum_{r=1}^{m}\pi_i b_{ir} C_\tau(A)_{ij} d_X(z_r,x_i)^p =$$
$$= \sum_{i=1}^{n}\sum_{r=1}^{m}\pi_i b_{ir} d_X(z_r,x_i)^p = \frac{1}{\theta}\sum_{i=1}^{n}\sum_{r=1}^{m} w_{ir} d_X(z_r,x_i)^p.$$

The third sum gives rise to the same quantity, by reversibility of $C_\tau(A)$ with respect to π. The second sum we bound using the Markov type property of X as follows

$$\begin{aligned}\sum_{i,j=1}^{n}\sum_{r,s=1}^{m}\pi_i b_{ir}b_{js}C_\tau(A)_{ij}d_X(x_i,x_j)^p &= \sum_{i,j=1}^{n}\pi_i C_\tau(A)_{ij}d_X(x_i,x_j)^p =\\ =\frac{1}{\tau}\sum_{\sigma=1}^{\tau}\sum_{i,j=1}^{n}\pi_i A^\sigma_{ij}d_X(x_i,x_j)^p &\le \frac{1}{\tau}\sum_{\sigma=1}^{\tau}M^p\sigma\sum_{i,j=1}^{n}\pi_i a_{ij}d_X(x_i,x_j)^p =\\ =\frac{\tau+1}{2}M^p\sum_{i,j=1}^{n}\frac{v_{ij}}{\theta t}d_X(x_i,x_j)^p &= \frac{1}{\theta}\frac{\tau+1}{2t}M^p\sum_{i,j=1}^{n}v_{ij}d_X(x_i,x_j)^p.\end{aligned}$$

Thus (2.4) is bounded from above by

$$\frac{18^p}{3}(N^p+1)L^p\left(2\sum_{i=1}^{n}\sum_{r=1}^{m}w_{ir}d_X(z_r,x_i)^p+M^p\sum_{i,j=1}^{n}v_{ij}d_X(x_i,x_j)^p\right).$$

Thus the assumptions of Lemma 2.3 are satisfied and the theorem is proven. □

3. Examples

In this section we shall provide some examples where the assumptions of Theorem 2.1 are satisfied, so that it is possible to extend Lipschitz maps without losing too much on their Lipschitz constant.

Examples of spaces satisfying the Markov type p property include p-smooth Banach spaces.

Recall that a Banach space X is called p-smooth provided that there is a constant $C>0$ such that

$$\frac{1}{2}\big(\|x+\tau y\|+\|x-\tau y\|\big)-1\le C\tau^p$$

for all $\tau>0$ and all unit vectors $x,y\in X$ Similarly, a Banach space is called q-convex whenever there is $C>0$ such that

$$C\epsilon^q\le 1-\left\|\frac{x+y}{2}\right\|$$

for all unit vectors x,y such that $\|x-y\|=\epsilon$.

Examples of spaces satisfying the Markov cotype q property are q-convex Banach spaces.

CHAPTER 5

Assesment and references

1. Reading list

The reading list consists of all the papers cited above, lecture notes [**15**], and parts of books [**19, 3**].

2. Assesment

Students are encouraged to give a short presentation on a topic related to the content of the course. The timing of this will be arranged by the lecturer with the student group. Suggested topics include:

i) Brehm's theorem [**6**],
ii) continuity of Kirszbraun's extension theorem [**38**],
iii) Kirszbraun's theorem for Alexandrov spaces [**39, 21**],
iv) two-dimensional Kneser-Poulsen conjecture [**4**],
v) origami [**30**],
vi) absolutely minimising Lipschitz extensions and infinity Laplacian [**35, 47, 48, 22, 23, 25, 24**],
vii) Fenchel duality and Fitzpatrick functions [**46, 27**],
viii) sharp form of Whitney's extension theorem [**31**],
ix) Whitney's extension theorem for C^m [**32**],
x) Markov type and cotype calculation [**15, 1, 44**],
xi) extending Lipschitz functions via random metric partitions [**41, 15**].

Additional references

Additional references for lectures:

i) Kirszbraun's theorem [**10, 18**],
ii) Kneser-Poulsen conjecture [**11, 16, 9**],
iii) Whitney's extension theorem [**20**].

References regarding applications:

i) clustering of data [**14, 40**],
ii) dimension reduction [**33**].

Bibliography

1. K. Ball, *Markov chains, Riesz transforms and Lipschitz maps*, Geometric & Functional Analysis GAFA **2** (1992), no. 2, 137–172.
2. ———, *The Ribe programme*, Séminaire Bourbaki volume 2011/2012 exposés 1043-1058, Astérisque, no. 352, Société mathématique de France, 2013, talk:1047 (en). MR 3087345
3. Y. Benyamini and J. Lindenstrauss, *Geometric nonlinear functional analysis*, Colloquium publications (American Mathematical Society) ; v. 48, American Mathematical Society, Providence, R.I., 2000 (eng).
4. K. Bezdek and R. Connelly, *Pushing disks apart – the Kneser-Poulsen conjecture in the plane*, Journal für die reine und angewandte Mathematik (2002), no. 553, 221 – 236.
5. J. Bourgain, *The metrical interpretation of superreflexivity in Banach spaces*, Israel Journal of Mathematics **56** (1986), 222–230.
6. U. Brehm, *Extensions of distance reducing mappings to piecewise congruent mappings on* $\mathbf{R}^m$, J. Geom. **16** (1981), no. 2, 187–193. MR 642266
7. K.J. Ciosmak, *Continuity of extensions of Lipschitz maps*, arXiv e-prints (2019), arXiv:1904.02993.
8. L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions; Rev. ed.*, Textbooks in mathematics, ch. 6, CRC Press, Oakville, 2015.
9. M. Gromov, *Monotonicity of the volume of intersection of balls*, Geometrical Aspects of Functional Analysis (Berlin, Heidelberg) (J. Lindenstrauss and V. D. Milman, eds.), Springer Berlin Heidelberg, 1987, pp. 1–4.
10. M. Kirszbraun, *Über die zusammenziehende und Lipschitzsche Transformationen*, Fundamenta Mathematicae **22** (1934), no. 1, 77–108 (ger).
11. M. Kneser, *Einige Bemerkungen über das Minkowskische Flächenmaß*, Archiv der Mathematik **6** (1955), no. 5, 382–390.
12. E. Le Gruyer, *Minimal Lipschitz extensions to differentiable functions defined on a Hilbert space*, Geometric and Functional Analysis **19** (2009), no. 4, 1101–1118. MR 2570317
13. E. J. McShane, *Extension of range of functions*, Bull. Amer. Math. Soc. **40** (1934), no. 12, 837–842. MR 1562984
14. A. Naor, *An introduction to the Ribe program*, Japanese Journal of Mathematics **7** (2012), no. 2, 167–233.
15. ———, *Metric embeddings and Lipschitz extensions*, Princeton University, Lecture Notes, 2015.
16. E. T. Poulsen, *Problem 10*, Mathematica Scandinavica **2** (1954), 346.
17. I. J. Schoenberg, *On a Theorem of Kirzbraun and Valentine*, The American Mathematical Monthly **60** (1953), no. 9, 620–622. MR 0058232
18. F. A. Valentine, *A Lipschitz condition preserving extension for a vector function*, Amer. J. Math. **67** (1945), 83–93. MR 0011702
19. J. H. Wells and L. R. Williams, *Embeddings and extensions in analysis*, Ergebnisse der Mathematik und ihrer Grenzgebiete ; Bd. 84, Springer-Verlag, Berlin, 1975 (eng).
20. H. Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Transactions of the American Mathematical Society **36** (1934), no. 1, 63–89. MR 1501735

Additionnal Topics and References

21. S. Alexander, V. Kapovitch, and A. Petrunin, *Alexandrov meets Kirszbraun*, 2017.
22. G. Aronsson, *Minimization problems for the functional* $\sup_x F(x, f(x), f'(x))$, Ark. Mat. **6** (1965), no. 1, 33–53.
23. ———, *Minimization problems for the functional* $\sup_x F(x, f(x), f'(x))$*(ii)*, Ark. Mat. **6** (1966), no. 4-5, 409–431.
24. ———, *Extension of functions satisfying lipschitz conditions*, Ark. Mat. **6** (1967), no. 6, 551–561.
25. ———, *Minimization problems for the functional* $\sup_x F(x, f(x), f'(x))$*(iii)*, Ark. Mat. **7** (1969), no. 6, 509–512.
26. K. Ball, *Markov chains, Riesz transforms and Lipschitz maps*, Geometric & Functional Analysis GAFA **2** (1992), no. 2, 137–172.
27. H. Bauschke, *Fenchel duality, Fitzpatrick functions and the extension of firmly nonexpansive mappings*, Proceedings of the American Mathematical Society **135** (2007), no. 1, 135–139. MR 2280182
28. K. Bezdek and R. Connelly, *Pushing disks apart – the Kneser-Poulsen conjecture in the plane*, Journal für die reine und angewandte Mathematik (2002), no. 553, 221 – 236.
29. U. Brehm, *Extensions of distance reducing mappings to piecewise congruent mappings on* $\mathbf{R}^m$, J. Geom. **16** (1981), no. 2, 187–193. MR 642266
30. B. Dacorogna, P. Marcellini, and E. Paolini, *Lipschitz-continuous local isometric immersions: rigid maps and origami*, Journal de Mathématiques Pures et Appliqués **90** (2008), no. 1, 66 – 81.
31. C. L. Fefferman, *A sharp form of Whitney's extension theorem*, Annals of Mathematics **161** (2005), no. 1, 509–577. MR 2150391
32. ———, *Whitney's extension problem for* C^m, Annals of Mathematics **164** (2006), no. 1, 313–359. MR 2233850
33. L.-A. Gottlieb and R. Krauthgamer, *A nonlinear approach to dimension reduction*, Weizmann Institute of Science.
34. M. Gromov, *Monotonicity of the volume of intersection of balls*, Geometrical Aspects of Functional Analysis (Berlin, Heidelberg) (J. Lindenstrauss and V. D. Milman, eds.), Springer Berlin Heidelberg, 1987, pp. 1–4.
35. R. Jensen, *Uniqueness of Lipschitz extensions: Minimizing the sup norm of the gradient*, Archive for Rational Mechanics and Analysis **123** (1993), no. 1, 51–74.
36. M. Kirszbraun, *Über die zusammenziehende und Lipschitzsche Transformationen*, Fundamenta Mathematicae **22** (1934), no. 1, 77–108 (ger).
37. M. Kneser, *Einige Bemerkungen über das Minkowskische Flächenmaß*, Archiv der Mathematik **6** (1955), no. 5, 382–390.
38. E. Kopecká, *Bootstrapping Kirszbraun's extension theorem*, Fund. Math. **217** (2012), no. 1, 13–19. MR 2914919
39. U. Lang and V. Schroeder, *Kirszbraun's theorem and metric spaces of bounded curvature*, Geometric & Functional Analysis GAFA **7** (1997), no. 3, 535–560. MR 1466337
40. J. Lee, *Jl lemma and Kirszbraun's extension theorem*, 2020, Sublinear Algorithms for Big Data Lectues Notes, Brown University.
41. J. R. Lee and A. Naor, *Extending Lipschitz functions via random metric partitions*, Inventiones mathematicae **160** (2005), no. 1, 59–95.
42. A. Naor, *Probabilistic clustering of high dimensional norms*, pp. 690–709.
43. ———, *Metric embeddings and Lipschitz extensions*, Princeton University, Lecture Notes, 2015.

44. A. Naor, Y. Peres, O. Schramm, and S. Sheffield, *Markov chains in smooth Banach spaces and Gromov-hyperbolic metric spaces*, Duke Math. J. **134** (2006), no. 1, 165–197.
45. E. T. Poulsen, *Problem 10*, Mathematica Scandinavica **2** (1954), 346.
46. S. Reich and S. Simons, *Fenchel duality, Fitzpatrick functions and the Kirszbraun–Valentine extension theorem*, Proceedings of the American Mathematical Society **133** (2005), no. 9, 2657–2660. MR 2146211
47. S. Sheffield and C. K. Smart, *Vector-valued optimal Lipschitz extensions*, Communications on Pure and Applied Mathematics **65** (2012), no. 1, 128–154. MR 2846639
48. P. V. Than, *Extensions lipschitziennes minimales*, Ph.D. thesis, INSA de Rennes, 2015.
49. F. A. Valentine, *A Lipschitz condition preserving extension for a vector function*, Amer. J. Math. **67** (1945), 83–93. MR 0011702
50. H. Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Transactions of the American Mathematical Society **36** (1934), no. 1, 63–89. MR 1501735