Optimal transport in Lorentzian synthetic spaces, synthetic timelike Ricci curvature lower bounds and applications

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Abstract

The goal of the present work is three-fold. The first goal is to set foundational results on optimal transport in Lorentzian synthetic spaces, including cyclical monotonicity, stability of optimal couplings and Kantorovich duality (several results are new even for smooth Lorentzian manifolds). The second one is to give a synthetic notion of "timelike Ricci curvature bounded below and dimension bounded above" for a Lorentzian space using optimal transport. The key idea being to analyse convexity properties of Entropy functionals along future directed timelike geodesics of probability measures. Such a notion is proved to be stable under a suitable weak convergence of Lorentzian synthetic spaces. The third goal is to draw applications, most notably extending volume comparisons and Hawking singularity Theorem (in sharp form) to the synthetic setting.
The framework of Lorentzian synthetic spaces includes as remarkable classes of examples: space-times endowed with a causally plain (or, more strongly, locally Lipschitz) continuous Lorentzian metric, closed cone structures, some approaches to quantum gravity (e.g. causal Fermion systems).

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Introduction

As the title suggests, the goal of the present work is three-fold. The first goal is to set foundational results on optimal transport in Lorentzian synthetic spaces. The second one is to use optimal transport to give a notion of "timelike Ricci curvature bounded below and dimension bounded above" for a Lorentzian synthetic space. The third goal is to draw applications, most notably extending volume comparisons and Hawking singularity Theorem (in sharp form) to the synthetic framework.

The Lorentzian synthetic framework adopted in the paper is the one of *Lorentzian pre-legth (and geodesic) spaces* introduced by Kunzinger and Sämann in [50] (see also an independent approach by Sormani and Vega [69]). The basic idea is that Lorentzian pre-length (resp. geodesic) spaces are the non-smooth analog of Lorentzian manifolds, in the same spirit as classical metric (resp. geodesic) spaces are the non-smooth analog of Riemannian manifolds (see Section 1.1 for the precise notions).

In the metric (measured) framework, the celebrated work of Sturm [72, 73] and Lott-Villani [54] laid the foundations for a theory of metric measure spaces satisfying Ricci curvature lower bounds and dimension upper bounds in a synthetic sense via optimal transport, the so-called $\mathsf{CD}(K,N)$ spaces. The theory of $\mathsf{CD}(K,N)$ spaces flourished in the last years with strong connections with analysis, geometry and probability. The ambition of the present paper is to lay the foundations for a parallel theory in the Lorentzian setting, which is the natural geometric framework for general relativity.

Motivations

Before discussing the main results, let us motivate the questions that we address. Apart from the intrinsic interest in a Lorentzian analog of Lott-Sturm-Villani $\mathsf{CD}(K,N)$ spaces, a main motivation for this work is the need to consider Lorentzian metrics/spaces of low regularity. Such a necessity is clear both from the PDE point of view in general relativity (i.e. the Cauchy initial value problem for the Einstein equations) and from physically relevant models.

From the PDE point of view, the standard local existence results for the vacuum Einstein equations assume the metric to be of Sobolev regularity H^s_{loc}, with $s > \frac{5}{2}$ (see for instance [66]). The Sobolev regularity of the metric has been lowered even further (e.g. [49]). Related to the initial value problem for the Einstein equations, one of the main open problems in the field is the so called (weak/strong) censorship conjecture (see e.g. [23, 25]). Such a conjecture (strong form) states roughly that the maximal globally hyperbolic development of generic initial data for the Einstein equations is inextendible as a suitably regular Lorentzian manifold. Formulating a precise statement of the conjecture is itself non-trivial since one needs to give a precise meaning to "generic initial data" and "suitably regular Lorentzian manifold". Understanding the latter is where Lorentzian metrics of low regularity and related inextendibility results become significant. The strongest form of the conjecture would prove inextendibility for a C^0 metric. As pointed out by Chrusciel-Grant [24], causality theory for C^0 metrics departs significantly from classical theory (e.g. the lightlike curves emanating from a point may span a set with non-empty interior, a phenomenon called "bubbling"). Nevertheless, Sbierski [68] gave a clever proof of C^0-inextendibility of Schwarzschild, [60] showed C^0-inextendibility for timelike geodesically complete spacetimes, and [39] pushed the inextendibility to Lorentzian length spaces.

From the point of view of physically relevant models, several types of matter in a spacetime may give a discontinuous energy-momentum tensor and thus, via the Einstein's equations, lead to a Lorentzian metric of regularity lower than C^2 (e.g. [52]). Examples of such a behaviour are spacetimes that model the inside and outside of a star, matched spacetimes [56], self-gravitating compressible fluids [12], or shock waves. Some physically relevant models require even lower regularity, for instance: spacetimes with conical singularities [76], cosmic strings [75] and (impulsive) gravitational waves (see for instance [65], [40, Chapter 20]).

Finally, a long term motivation for studying non-regular Lorentzian spaces is the desire of understanding the ultimate nature of spacetime. The rough picture is that at the quantum level (and thus in extreme physical conditions e.g. gravitational collapse, origin of the universe), the spacetime may be very singular and possibly not approximable by smooth structures (see Remark 1.13).

In case of a metric of low regularity, the approach to curvature used so far is distributional, taking advantage that the underline spacetime is a differentiable manifold. This permits [34] (see also [71]) to define distributional curvature tensors for $W^{1,2}_{loc}$-Lorentzian metrics satisfying a suitable non-degeneracy condition (satisfied for instance when the metric is C^1, see [37]). One of the goals of the present work is to address the question of (timelike Ricci) curvature when *not only the the metric tensor, but the space itself is singular.*

A lower bound on the timelike Ricci curvature of a spacetime (M^n, g), i.e.

$$\text{There exists } K \in \mathbb{R} \text{ such that } \mathrm{Ric}_g \geq -Kg(v,v) \text{ for all timelike vectors } v \in TM, \tag{0.1}$$

is quite a natural assumption in general relativity. Of course, for a C^2-metric g, (0.1) is satisfied on compact subsets of the space-time. Recalling that the Einstein's equations postulate proportionality of Ric_g and $T - \frac{1}{n-2}\mathrm{tr}_g(T)g$ (where T is the so-called energy-momentum tensor), for a general cosmological constant $\Lambda \in \mathbb{R}$, (0.1) is equivalent to require that

$$T(v,v) \geq -\frac{1}{n-2}\mathrm{tr}_g(T) + \frac{1}{8\pi}\left(K - \frac{2\Lambda}{n-2}\right), \text{ for all } v \in TM \text{ with } g(v,v) = -1.$$

In particular, if $\inf_M \mathrm{tr}_g(T) > -\infty$ (or, equivalently, $\inf_M \mathrm{R}_g > -\infty$ where R_g is the scalar curvature of g), then the *weak* energy condition $T(v,v) \geq 0$ for all timelike v (which is believed to hold for most physically reasonable T, according to [79, pag. 218]) implies (0.1).
The case $K = 0$ in (0.1) corresponds to the *strong energy condition* of Hawking and Penrose [64, 42, 44]. Even if the dominant energy condition or the (more general) weak energy condition have a wider physical range of validity (see e.g. [13, Chap. 4.6]), the strong energy condition plays an important role in gravitational theory: for instance, it typically appears as an assumption in singularity theorems and it is interpreted to be responsible for the attractive nature of gravity [13, 79].

Outline of the content of the paper

General synthetic setting

We now pass to discuss the content of the paper. The synthetic framework is the one of *measured Lorentzian pre-length spaces* $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ where X is a set endowed with a proper metric d (i.e. closed and bounded subsets are compact), a preorder $\leq$ (playing the role of *causal relation*) and a transitive relation $\ll$ contained in $\leq$ (playing the role of *chronological/timelike relation*), a lower semicontinuous function $\tau : X \times X \to [0, \infty]$ (called *time-separation function*) with $\{\tau > 0\} = \{(x,y) \in X^2 : x \ll y\}$ and satisfying reverse triangle inequality (1.1), and a non-negative Radon measure $\mathfrak{m}$ with $\mathrm{supp}\,\mathfrak{m} = X$.
The subset of causal pairs is denoted with $X^2_{\leq} := \{(x,y) \in X^2 \,:\, x \leq y\}$.

A curve $\gamma : [0,1] \to \mathbb{R}$ is *causal* if for every $t_0 \leq t_1$ it holds $\gamma_{t_0} \leq \gamma_{t_1}$. One can naturally associate a τ-length to γ, denoted by $\mathrm{L}_\tau(\gamma)$ (see Definition 1.4). A causal curve is a *geodesic* if it *maximises* the

τ-length, i.e. if $\mathrm{L}_\tau(\gamma) = \tau(\gamma_0, \gamma_1)$. The space X is said to be *geodesic* if for all $(x, y) \in X^2_{\leq}$ there is a geodesic γ from x to y.

Important classes of examples entering the framework of measured Lorentzian pre-length/geodesic spaces are spacetimes with a causally plain (or, more strongly, locally Lipschitz) C^0-metric (see Remark 1.12), closed cone structures as well as some approaches to quantum gravity (see Remark 1.13).

Optimal transport in Lorentzian pre-length spaces

Our approach to synthetic timelike Ricci curvature lower bounds is via optimal transport of causally related probability measures. To this aim, in Section 2 we throughly analyse optimal transport in Lorentz pre-length spaces. A key object is the space of Borel probability measures $\mathcal{P}(X)$ on X, and the subspace $\mathcal{P}_c(X)$ of Borel probability measures with compact support. In order to lift the causal structure of X to $\mathcal{P}(X)$, it is useful to consider the set of *causal couplings* between two probability measures $\mu, \nu \in \mathcal{P}(X)$:

$$\Pi_{\leq}(\mu, \nu) := \{\pi \in \mathcal{P}(X^2) : \pi(X^2_{\leq}) = 1, \, (P_1)_\sharp \pi = \mu, \, (P_2)_\sharp \pi = \nu\},$$

where $P_i : X \times X \to X$ is the projection on the i^{th} factor, and $(P_i)_\sharp : \mathcal{P}(X^2) \to \mathcal{P}(X)$ is the associated push-forward map defined as $((P_i)_\sharp \pi)(B) := \pi\left(P_i^{-1}(B)\right)$ for every Borel subset $B \subset X$. We say that (μ, ν) are *causally related* if $\Pi_{\leq}(\mu, \nu) \neq \emptyset$. The rough picture is that μ and ν represent some Random distribution of events in the spacetime X, and the two are causally related if it is possible to causally match events described by μ with events described by ν (possibly in a multi-valued way) via the casual coupling π. We endow $\mathcal{P}(X)$ with the *p-Lorentz-Wasserstein distance* defined by

$$\ell_p(\mu, \nu) := \sup_{\pi \in \Pi_{\leq}(\mu, \nu)} \left(\int_{X \times X} \tau(x, y)^p \, \pi(dxdy) \right)^{1/p}, \quad p \in (0, 1]. \tag{0.2}$$

When $\Pi_{\leq}(\mu, \nu) = \emptyset$ we set $\ell_p(\mu, \nu) := -\infty$. The name p-Lorentz-Wasserstein distance is motivated by the fact that ℓ_p satisfies a reversed triangle inequality (see Proposition 2.5).
Note that (0.2) extends to Lorentzian pre-length spaces the corresponding notion given in the smooth Lorentzian setting in [26] (see also [58, 61], and [74] for $p = 1$). A coupling $\pi \in \Pi_{\leq}(\mu, \nu)$ maximising in (0.2) is said *ℓ_p-optimal*. The set of *ℓ_p-optimal* couplings from μ to ν is denoted by $\Pi^{p\text{-opt}}_{\leq}(\mu, \nu)$.

Maximising over *causal* couplings $\Pi_{\leq}(\mu, \nu)$ instead of all the couplings can be modelled with an auxiliary cost (denoted with ℓ^p) taking value $-\infty$ outside of $X^2_{\leq}$ (see Remark 2.2); this complicates the associated optimal transport properties, and several fundamental results (see e.g. [2, 77, 78]) need to be re-established in the present setting. These include: cyclical monotonicity, stability of optimal couplings, Kantorovich duality. Indeed, this is the goal of Section 2. It is beyond the scopes of the introduction to give a detailed account of the results (several are new even in the smooth Lorentzian setting), we only mention few notions (in a slightly simplified form) that will be useful for analysing timelike Ricci curvature bounds.

We say that $(\mu, \nu) \in \mathcal{P}_c(X)^2$ is *timelike p-dualisable (by $\pi \in \Pi_{\leq}(\mu, \nu)$)* if $\ell_p(\mu, \nu) \in (0, \infty)$, $\pi \in \Pi^{p\text{-opt}}_{\leq}(\mu, \nu)$ and $\operatorname{supp} \pi \subset \{\tau > 0\}$. The pair $(\mu, \nu) \in \mathcal{P}_c(X)^2$ is *strongly timelike p-dualisable* if in addition there exists a subset $\Gamma \subset \{\tau > 0\} \subset X^2$ such that *every* p-optimal coupling $\pi' \in \Pi^{p\text{-opt}}_{\leq}(\mu, \nu)$ is concentrated on Γ, i.e. $\pi'(\Gamma) = 1$ (see Definitions 2.18 and 2.27 for the precise notions).

Let us also mention that if X is geodesic (plus a compactness condition that we call $\mathcal{K}$-global hyperbolicity, satisfied for globally hyperbolic Lorentzian C^0-metrics) then $(\mathcal{P}_c(X), \ell_p)$ is geodesic as well, for $p \in (0, 1)$. More precisely (see Proposition 2.32), if $(\mu_0, \mu_1) \in \mathcal{P}_c(X)^2$ is timelike p-dualisable, then there exists an ℓ_p-geodesic $(\mu_t)_{t \in [0,1]} \subset \mathcal{P}_c(X)$ joining them, i.e.

$$\ell_p(\mu_0, \mu_t) = t \ell_p(\mu_0, \mu_1), \quad \forall t \in [0, 1].$$

Synthetic timelike Ricci curvature lower bounds via optimal transport

The relation between optimal transport and timelike Ricci curvature bounds in the *smooth* Lorentzian setting has been the object of recent works by McCann [58] and Mondino-Suhr [61]. The key idea is that timelike Ricci curvature lower bounds can be equivalently characterised in terms of convexity properties of the Bolzmann-Shannon entropy functional $\mathrm{Ent}(\cdot|\mathfrak{m})$ along ℓ_p-geodesics of probability measures. Recall that, for a probability measure $\mu \in \mathcal{P}(X)$, the entropy $\mathrm{Ent}(\mu|\mathfrak{m})$ is defined by

$$\mathrm{Ent}(\mu|\mathfrak{m}) = \int_X \rho \log(\rho)\, \mathfrak{m},$$

if $\mu = \rho\, \mathfrak{m}$ is absolutely continuous with respect to $\mathfrak{m}$ and $(\rho \log(\rho))_+$ is $\mathfrak{m}$-integrable; otherwise we set $\mathrm{Ent}(\mu|\mathfrak{m}) = +\infty$. We denote $\mathrm{Dom}(\mathrm{Ent}(\cdot|\mathfrak{m})) := \{\mu \in \mathcal{P}(X) : \mathrm{Ent}(\mu|\mathfrak{m}) < \infty\}$.
The following definition is thus natural.

Definition ($\mathsf{TCD}^e_p(K,N)$ and $\mathsf{wTCD}^e_p(K,N)$ conditions)**.** Fix $p \in (0,1)$, $K \in \mathbb{R}$, $N \in (0,\infty)$. We say that a measured pre-length space $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ satisfies $\mathsf{TCD}^e_p(K,N)$ (resp. $\mathsf{wTCD}^e_p(K,N)$) if the following holds. For any couple $(\mu_0,\mu_1) \in (\mathrm{Dom}(\mathrm{Ent}(\cdot|\mathfrak{m})) \cap \mathcal{P}_c(X))^2$ which is (resp. strongly) timelike p-dualisable by some $\pi \in \Pi^{p\text{-opt}}_{\leq}(\mu_0,\mu_1)$, there exists an ℓ_p-geodesic $(\mu_t)_{t\in[0,1]}$ such that the function $[0,1] \ni t \mapsto e(t) := \mathrm{Ent}(\mu_t|\mathfrak{m})$ is semi-convex and it satisfies

$$e''(t) - \frac{1}{N} e'(t)^2 \geq K \int_{X\times X} \tau(x,y)^2\, \pi(dxdy), \quad \text{in the distributional sense on } [0,1].$$

Remark (Notation)**.** The notation $\mathsf{TCD}^e_p(K,N)$ comes by analogy with the corresponding Lott-Sturm-Villani theory of curvature dimension conditions in metric-measure spaces. Here the superscript e refers to the so-called "entropic" formulation of the CD condition by Erbar, Kuwada and Sturm [28]; such a formulation is slightly simpler, but equivalent under suitable technical assumptions. The possibility $p \in (1,\infty), p \neq 2$ was investigated by Kell [45] in the metric-measure setting. The leading T is a mnemonic for "timelike", following the notation of Woolgar and Wylie in their paper on N-Bakry-Émery spacetimes [80], and of McCann [58]. The symbol w in wTCD has to be read "weak TCD" and it is justified by the comparison with TCD requiring convexity estimates for the entropy along a smaller family of ℓ_p-geodesics.

The $\mathsf{TCD}^e_p(K,N)$ (resp. $\mathsf{wTCD}^e_p(K,N)$) satisfies the following natural compatibility conditions:

- $\mathsf{TCD}^e_p(K,N)$ (resp. $\mathsf{wTCD}^e_p(K,N)$) implies $\mathsf{TCD}^e_p(K',N')$ (resp. $\mathsf{wTCD}^e_p(K',N')$) for all $K' \leq K$, $N' \geq N$, see Lemma 3.10;
- A smooth globally hyperbolic Lorentzian manifold (M^n, g) has $\dim(M) = n \leq N$ and $\mathrm{Ric}_g(v,v) \geq -Kg(v,v)$ for every timelike $v \in TM$ if and only if it satisfies $\mathsf{TCD}^e_p(K,N)$, if and only if it satisfies $\mathsf{wTCD}^e_p(K,N)$, see Theorem 3.1 and Corollary A.2.

We show that $\mathsf{wTCD}^e_p(K,N)$ spaces satisfy the following geometric properties:

- a timelike Brunn-Minkowski inequality, see Proposition 3.4;
- a timelike Bishop-Gromov inequality, Proposition 3.5;
- a timelike Bonnet-Myers inequality, Proposition 3.6.

Moreover, we show a (weak) stability property for TCD stating roughly that if a sequence of $\mathsf{TCD}^e_p(K,N)$ spaces converges weakly to a limit Lorentzian pre-length space, then the limit space satisfies $\mathsf{wTCD}^e_p(K,N)$ (see Theorem 3.14 for the precise statement).

A weaker variant of the $\mathsf{TCD}^e_p(K,N)$ condition is obtained by considering (K,N)-convexity properties only for those ℓ_p-geodesics $(\mu_t)_{t\in[0,1]}$ where μ_1 is a Dirac delta. In the metric measure setting, such a variant goes under the name of Measure Contraction Property (MCP for short) and was developed independently by Sturm [73] and Ohta [62]. We call "Timelike Measure Contraction Property" ($\mathsf{TMCP}^e_p(K,N)$ for short) such a weaker variant of $\mathsf{TCD}^e_p(K,N)$, see Definition 3.7 for the precise notion. The following holds:

- under mild conditions on the space X (satisfied for instance for causally plain, globally hyperbolic spacetimes with a C^0 metric) $\mathsf{wTCD}^e_p(K,N)$ implies $\mathsf{TMCP}^e_p(K,N)$, see Proposition 3.11;
- $\mathsf{TMCP}^e_p(K,N)$ implies $\mathsf{TMCP}^e_p(K',N')$ for all $K' \leq K$, $N' \geq N$, see Lemma 3.10;
- a smooth globally hyperbolic Lorentzian manifold (M^n, g) with $\dim(M) = n \geq 2$ satisfies $\mathsf{TMCP}^e_p(K,n)$ if and only if $\mathrm{Ric}_g(v,v) \geq -Kg(v,v)$ for every timelike $v \in TM$, see Theorem A.1;
- if a smooth globally hyperbolic Lorentzian manifold (M^n, g) satisfies $\mathsf{TMCP}^e_p(K,N)$, then $\dim(M) = n \leq N$, see Corollary A.2;
- the aforementioned timelike Bishop-Gromov inequality (Proposition 3.5) and timelike Bonnet-Myers inequality (Proposition 3.6) remain valid for $\mathsf{TMCP}^e_p(K,N)$ spaces;
- The $\mathsf{TMCP}^e_p(K,N)$ condition is stable under weak convergence of Lorentzian synthetic spaces, see Theorem 3.12.

Let us mention that the stability of $\mathsf{TCD}^e_p(K,N)$ and $\mathsf{TMCP}^e_p(K,N)$ is more subtle than the one for the corresponding metric versions CD and MCP. Indeed, the fact that the convexity of the entropy is required to hold *only* along ℓ_p-geodesics connecting *timelike p-dualisable* measures does not permit to repeat verbatim the known stability arguments of [54, 72, 73, 35] for CD and MCP, where there is (almost) no restriction on the endpoints of the Wasserstein geodesic.
Another difference between the Lorentzian and the Riemannian/metric setting is that, while in the latter the CD/MCP conditions imply a control on the volume growth of metric balls and thus compactness in pointed-measured-Gromov-Hausdorff topology (which is thus the natural notion for weak convergence of spaces), in the former the τ-balls typically have infinite volume (for instance in Minkowski space, τ-spheres are hyperboloids) and thus we cannot expect to have compactness in pointed-measured-Gromov-Hausdorff topology (which is thus not anymore the clearly natural notion for weak convergence of spaces).

Timelike non-branching $\mathsf{TMCP}^e_p(K,N)$ and applications

An important subclass of Lorentzian geodesic spaces is the the one of *timelike non-branching* structures: roughly the ones for which timelike geodesic do not branch (both forward and backward in time), see Definition 1.10 for the precise notion. In the classical Lorentzian setting, this is satisfied for $C^{1,1}$ metrics and it is expected to fail for lower regularity. The same phenomenon happens in the Riemannian/metric setting, where the non-branching assumption (or slightly weaker variants) is rather standard in the recent literature of CD/MCP spaces.

For timelike non-branching $\mathsf{TMCP}^e_p(K,N)$ spaces we obtain:

- solution to the ℓ_p-Monge problem: if (μ_0, μ_1) are timelike p-dualisable with $\mu_0 \in \mathrm{Dom}(\mathrm{Ent}(\cdot|\mathfrak{m}))$, then there exists a unique ℓ_p-optimal coupling $\pi \in \Pi^{p\text{-opt}}_{\leq}(\mu_0,\mu_1)$ such that $\pi(\{\tau > 0\}) = 1$ and it is induced by a map; see Theorem 3.19.
  Under the same assumptions, there exists a unique ℓ_p-geodesic from μ_0 to μ_1; see Theorem 3.20;
- a synthetic notion of mean curvature bounds for achronal Borel sets having locally finite "area", see Section 5.1;
- a sharp version (holding for every $K \in \mathbb{R}, N \in [1,\infty)$) of Hawking singularity Theorem, see Theorem 5.5. Let us mention that the statement of Theorem 5.5 is sharp, as for $N \in \mathbb{N}$ the bounds are attained in the smooth model spaces identified in [38] (see also [36]);
- sharp versions of timelike Bishop-Gromov, Poincaré, and Bonnet-Myers inequalities (see Propositions 5.6, 5.7, 5.8, 5.9; for the sharpness see Remark 5.10).

In order to obtain the applications in the last three bullet points, in Section 4, we study the ℓ_1-optimal transport problem associated to the τ-distance function τ_V from an achronal set V (see (1.8) for the definition of τ_V). The rough idea is that τ_V induces a partition of $I^+(V)$, namely "the chronological future of V", into timelike geodesics (also called "rays"). In the smooth setting (outside the cut locus) such rays correspond to the gradient flow curves of τ_V. Such a partition of $I^+(V)$ induces a disintegration of $\mathfrak{m}$ into one-dimensional conditional measures, which satisfy $\mathsf{MCP}(K,N)$ (see Theorems 4.16 and 4.17 for the precise statements). In Section 5.1, the disintegration is used to construct an "area measure" as well as "normal variations" of V, and thus define synthetic notions of mean curvature bounds. At this point, the above applications will follow.

The fact that the one-dimensional conditional measures satisfy $\mathsf{MCP}(K,N)$ is not trivial: recall indeed that the $\mathsf{TMCP}^e_p(K,N)$ and $\mathsf{TCD}^e_p(K,N)$ conditions are expressed in terms of ℓ_p (not ℓ_1) optimal transport, while here we are dealing with an ℓ_1-optimal transport problem. The key idea to overcome this issue is to include ℓ^p-cyclically monotone sets inside ℓ-cyclically monotone sets; this technique was introduced in [14] and pushed further in [16, 17] for the metric setting. In the present setting, since the cost ℓ^p may take the value $-\infty$, ℓ^p-cyclical monotonicity does not directly imply optimality. Nontheless using the work of Bianchini-Caravenna [9] and its consequences included in Proposition 2.8, we will use cyclically monotone sets to construct *locally optimal* couplings and to deduce local estimates on the disintegration that will be then globalized. Another useful idea is that there is a natural way to construct ℓ_p-geodesics with $0 < p < 1$: translate along transport rays by a constant "distance". Notice that $0 < p < 1$ plays a crucial role here, as an analogous statement in the Riemannian setting does not hold true for W_2.

Let us conclude the introduction by pointing out that the reader interested in space-times with continuous metrics can find the main applications specialised to such a framework in Section 5.4.
In analogy with the huge impact that the synthetic theory of Ricci curvature lower bounds had in the geometric analysis of metric measure spaces, it is natural to expect several other geometric and analytic applications of the tools developed here; for instance, in a forthcoming paper [20], we will obtain timelike isoperimetric inequalities, timelike Sobolev and log-Sobolev inequalities, among others.

1 Preliminaries

1.1 Basics on Lorentzian synthetic spaces

In this section we briefly recall some basic notions and results from the theory of Lorentzian length (resp. geodesic) spaces. We follow the approach of Kunzinger-Sämann [50] and we refer to their paper for further details and proofs.

Definition 1.1 (Causal space $(X, \ll, \leq)$)**.** A *causal space* $(X, \ll, \leq)$ is a set X endowed with a preorder $\leq$ and a transitive relation $\ll$ contained in $\leq$.

We write $x < y$ when $x \leq y, x \neq y$. We say that x and y are *timelike* (resp. *causally*) related if $x \ll y$ (resp. $x \leq y$). Let $A \subset X$ be an arbitrary subset of X. We define the *chronological* (resp. *causal*) future of A the set

$$I^+(A) := \{y \in X \,:\, \exists x \in A,\, x \ll y\}$$
$$J^+(A) := \{y \in X \,:\, \exists x \in A,\, x \leq y\}$$

respectively. Analogously, we define $I^-(A)$ (resp. $J^-(A)$) the *chronological* (resp. *causal*) past of A. In case $A = \{x\}$ is a singleton, with a slight abuse of notation, we will write $I^\pm(x)$ (resp. $J^\pm(x)$) instead of $I^\pm(\{x\})$ (resp. $J^\pm(\{x\})$).

Definition 1.2 (Lorentzian pre-length space $(X, \mathsf{d}, \ll, \leq, \tau)$)**.** A *Lorentzian pre-length space* $(X, \mathsf{d}, \ll, \leq, \tau)$ is a casual space $(X, \ll, \leq)$ additionally equipped with a proper metric d (i.e. closed and bounded subsets are compact) and a lower semicontinuous function $\tau : X \times X \to [0, \infty]$, called *time-separation function*, satisfying

$$\begin{aligned} \tau(x,y) + \tau(y,z) \leq \tau(x,z) \quad \forall x \leq y \leq z \quad \text{reverse triangle inequality} \\ \tau(x,y) = 0, \text{ if } x \not\leq y, \quad \tau(x,y) > 0 \Leftrightarrow x \ll y. \end{aligned} \tag{1.1}$$

Note that the lower semicontinuity of τ implies that $I^{\pm}(x)$ is open, for any $x \in X$.
We endow X with the metric topology induced by d. All the topological concepts on X will be formulated in terms of such metric topology. We say that X is (resp. *locally*) *causally closed* if $\{x \leq y\} \subset X \times X$ is a closed subset (resp. if every point $x \in X$ has neighbourhood U such that $\{x \leq y\} \cap \bar{U} \times \bar{U}$ is closed in $\bar{U} \times \bar{U}$).

If $(X, \mathsf{d}, \ll, \leq, \tau)$ is a Lorentzian pre-length space, notice that setting $x \tilde{\leq} y$ (resp. $x \tilde{\ll} y$) if and only if $y \leq x$ (resp. $y \ll x$) and $\tilde{\tau}(x,y) := \tau(y,x)$, we obtain a new Lorentzian pre-length space $(X, \mathsf{d}, \tilde{\ll}, \tilde{\leq}, \tilde{\tau})$. The latter is said to be the *causally reversed* of the former.

Throughout the paper, $I \subset \mathbb{R}$ will denote an arbitrary interval.

Definition 1.3 (Causal/timelike curves)**.** A non-constant curve $\gamma : I \to X$ is called (future-directed) *timelike* (resp. *causal*) if γ is locally Lipschitz continuous (with respect to d) and if for all $t_1, t_2 \in I$, with $t_1 < t_2$, it holds $\gamma_{t_1} \ll \gamma_{t_2}$ (resp. $\gamma_{t_1} \leq \gamma_{t_2}$). We say that γ is a *null* curve if, in addition to being causal, no two points on $\gamma(I)$ are related with respect to $\ll$.

It was proved in [50, Proposition 5.9] that for strongly causal continuous Lorentzian metrics, this notion of causality coincides with the classical one.

The length of a causal curve is defined via the time separation function, in analogy to the theory of length metric spaces.

Definition 1.4 (Length of a causal curve)**.** For $\gamma : [a,b] \to X$ future-directed causal we set

$$\mathrm{L}_\tau(\gamma) := \inf \left\{ \sum_{i=0}^{N-1} \tau(\gamma_{t_i}, \gamma_{t_{i+1}}) \,:\, a = t_0 < t_1 < \ldots < t_N = b,\ N \in \mathbb{N} \right\}.$$

In case the interval is half-open, say $I = [a,b)$, then the infimum is taken over all partitions with $a = t_0 < t_1 < \ldots < t_N < b$ (and analogously for the other cases).

It was proved in [50, Proposition 2.32] that for smooth strongly causal spacetimes (M,g), this notion of length coincides with the classical one: $\mathrm{L}_\tau(\gamma) = \mathrm{L}_g(\gamma)$.
A future-directed causal curve $\gamma : [a,b] \to X$ is *maximal* (also called *geodesic*) if it realises the time separation, i.e. if $\mathrm{L}_\tau(\gamma) = \tau(\gamma_a, \gamma_b)$.
In case the time separation function is continuous with $\tau(x,x) = 0$ for every $x \in X$ (as it will be throughout the paper, since we will assume that X is a globally hyperbolic geodesic Lorentzian space), then any timelike geodesic γ with finite τ-length has a (continuous, monotonically strictly increasing) reparametrisation λ by τ-arc-length, i.e. $\tau(\gamma_{\lambda(s_1)}, \gamma_{\lambda(s_2)}) = s_2 - s_1$ for all $s_2 \leq s_1$ in the corresponding interval (see [50, Corollary 3.35]). We denote the set of causal (resp. timelike) geodesics as:

$$\mathrm{Geo}(X) := \{\gamma : [0,1] \to X \,:\, \tau(\gamma_s, \gamma_t) = (t-s)\tau(\gamma_0, \gamma_1) \,\forall s < t\}, \tag{1.2}$$

$$\mathrm{TGeo}(X) := \{\gamma \in \mathrm{Geo}(X) :\, \tau(\gamma_0, \gamma_1) > 0\}. \tag{1.3}$$

Given $x \leq y \in X$ we also set

$$\mathrm{Geo}(x,y) := \{\gamma \in \mathrm{Geo}(X) \,:\, \gamma_0 = x,\, \gamma_1 = y\} \tag{1.4}$$

$$\mathfrak{I}(x,y,t) := \{\gamma_t \,:\, \gamma \in \mathrm{Geo}(x,y)\} \tag{1.5}$$

respectively the space of geodesics, and the set of t-intermediate points from x to y.
If $x \ll y \in X$ we also set

$$\mathrm{TGeo}(x,y) := \{\gamma \in \mathrm{TGeo}(X) \,:\, \gamma_0 = x,\, \gamma_1 = y\}.$$

Given two subsets $A, B \subset X$ we call

$$\mathfrak{I}(A,B,t) := \bigcup_{x\in A, y\in B} \mathfrak{I}(x,y,t) \tag{1.6}$$

the subset of t-intermediate points of geodesics from points in A to points in B.

A Lorentzian pre-length space $(X, \mathsf{d}, \ll, \leq, \tau)$ is called

- *non-totally imprisoning* if for every compact set $K \Subset X$ there is constant $C > 0$ such that the d-arc-length of all causal curves contained in K is bounded by C;
- *globally hyperbolic* if it is non-totally imprisoning and for every $x, y \in X$ the set $J^+(x) \cap J^-(y)$ is compact in X;
- *$\mathcal{K}$-globally hyperbolic* if it is non-totally imprisoning and for every $K_1, K_2 \Subset X$ compact subsets, the set $J^+(K_1) \cap J^-(K_2)$ is compact in X;
- *geodesic* if for all $x, y \in X$ with $x \leq y$ there is a future-directed causal curve γ from x to y with $\tau(x,y) = L_\tau(\gamma)$, i.e. a (maximizing) geodesic from x to y.

It was proved in [50, Theorem 3.28] that for a globally hyperbolic Lorentzian geodesic (actually length would suffice) space $(X, \mathsf{d}, \ll, \leq, \tau)$, the time-separation function τ is finite and continuous. Moreover, any globally hyperbolic Lorentzian length space (for the definition of Lorenzian length space see [50, Definition 3.22], we skip it for brevity since we will not use it) is geodesic [50, Theorem 3.30].

Let us mention that, in the setting of C^0-Lorentzian metrics, global hyperbolicity implies causal closedness and $\mathcal{K}$-global hyperbolicity [67, Proposition 3.3 and Corollary 3.4].

It is readily seen that if X is $\mathcal{K}$-hyperbolic and $K_1, K_2 \Subset X$ are compact subsets then

$$\mathfrak{I}(K_1,K_2,t) \Subset \bigcup_{t\in[0,1]} \mathfrak{I}(K_1,K_2,t) \Subset X, \quad \forall t \in [0,1]. \tag{1.7}$$

In the next lemma we show that, under mild assumptions, global hyperbolicity implies $\mathcal{K}$-global hyperbolicity also in the synthetic setting.

Lemma 1.5. *Let $(X, \mathsf{d}, \ll, \leq, \tau)$ be a globally hyperbolic locally causally closed Lorentzian geodesic space, and let $K_1, K_2 \Subset X$ be compact subsets satisfying the following property: $I^-(x) \neq \emptyset$ for every $x \in K_1$ and $I^+(y) \neq \emptyset$ for every $y \in K_2$. Then $J^+(K_1) \cap J^-(K_2)$ is compact in X.*
In particular, if $(X, \mathsf{d}, \ll, \leq, \tau)$ is a globally hyperbolic, locally causally closed Lorentzian geodesic space satisfying $I^\pm(x) \neq \emptyset$ for all $x \in X$, then it is also $\mathcal{K}$-globally hyperbolic.

Proof. **Step 1.** Let $(z_n) \subset J^+(K_1) \cap J^-(K_2)$ be an arbitrary sequence. By definition there exist $(x_n) \subset K_1, (y_n) \subset K_2$ such that $x_n \leq z_n \leq y_n$ for every $n \in \mathbb{N}$. Since K_1, K_2 are compact, there exists $x \in K_1, y \in K_2$ such that $x_n \to x, y_n \to y$, up to a subsequence.
By assumption there exist $x', y' \in X$ such that $\tau(x', x) > 0, \tau(y, y') > 0$. Since $I^+(x'), I^-(y') \subset X$ are open subsets, it holds that $x_n \in I^+(x'), y_n \in I^-(y')$ for n large enough. In particular we have $x' \leq x_n \leq z_n \leq y_n \leq y'$ and thus $(z_n) \subset J^+(x') \cap J^-(y')$. Since $J^+(x') \cap J^-(y')$ is compact by global hyperbolicity, the sequence (z_n) has a subsequence converging to some $z \in X$. We claim that $x \leq z \leq y$.

Step 2. Let $\gamma^n : [0,1] \to X$ be causal curves (which, by global hyperbolicity, can be chosen to be broken geodesics) starting at x', ending at y' and passing through x_n, z_n and y_n. Clearly $\gamma^n([0,1]) \subset J^+(x') \cap J^-(y')$ which is compact by global hyperbolicity. Thus $\sup_n \mathrm{L}_\mathsf{d}(\gamma^n) < \infty$ by the non-total imprisoning property. Thus, without loss of generality we can assume that each γ^n is parametrized with constant d-metric speed over $[0,1]$. By the metric Arzelá-Ascoli Theorem, there exists a Lipschitz curve $\gamma : [0,1] \to X$ such that $\gamma^n \to \gamma$ uniformly on $[0,1]$, up to a subsequence. Since by assumption X is locally causally closed, we can cover $\gamma([0,1])$ with finitely many causally closed neighbourhoods. Using that $\gamma^n \to \gamma$ uniformly and that each γ^n is a causal curve, it easily follows that γ is causal as well.

Since γ passes through x, z, y (in this order), it follows that $x \leq z \leq y$. Hence $z \in J^+(K_1) \cap J^-(K_2)$ giving the claim. □

Lemma 1.6. *Let $(X, \mathsf{d}, \ll, \leq, \tau)$ be a $\mathcal{K}$-globally hyperbolic locally causally closed Lorentzian pre-length space such that for every $x, y \in X$ with $x \leq y$ there is a causal curve from x to y (in particular, this holds if X is geodesic). Then X is causally closed.*

Proof. Let $x_n \to x, y_n \to y$, with $x_n \leq y_n$. By $\mathcal{K}$-globally hyperbolicity, it holds that $J^+(\{x, x_n\}_{n\in\mathbb{N}}) \cap J^-(\{y, y_n\}_{n\in\mathbb{N}}) \subset X$ is compact. For each $n \in \mathbb{N}$, let γ^n be a causal curve from x_n to y_n. Arguing as in Step 2 in the proof of Lemma 1.5, we obtain that γ^n converges uniformly (up to a subsequence) to a limit causal curve γ from x to y. Thus $x \leq y$, as desired. □

In the proof of the singularity theorem, we will use a slight variation of the time separation function associated to a subset $V \subset X$. Recall that a subset $V \subset X$ is called *achronal* if $x \not\ll y$ for every $x, y \in V$. In particular, if V is achronal, then $I^+(V) \cap I^-(V) = \emptyset$, so we can define the *signed time-separation* to V, $\tau_V : X \to [-\infty, +\infty]$, by

$$\tau_V(x) := \begin{cases} \sup_{y\in V} \tau(y,x), & \text{for } x \in I^+(V) \\ \sup_{y\in V} -\tau(x,y), & \text{for } x \in I^-(V) \;. \\ 0 & \text{otherwise} \end{cases} \tag{1.8}$$

Note that τ_V is lower semi-continuous, as supremum of (lower semi-)continuous functions.
In order for these suprema to be attained, global hyperbolicity and geodesic property of X alone are not sufficient. One should rather demand additional compactness properties of the set V. The following notion, introduced by Galloway [33] in the smooth setting, is well suited to this aim.

Definition 1.7 (Future timelike complete (FTC) subsets)**.** A subset $V \subset X$ is *future timelike complete* (FTC), if for each point $x \in I^+(V)$, the intersection $J^-(x) \cap V \subset V$ has compact closure (w.r.t. d) in V. Analogously, one defines *past timelike completeness* (PTC). A subset that is both FTC and PTC is called *timelike complete*.

We denote with $\overline{C}$ the topological closure (with respect to d) of a subset $C \subset X$.

Lemma 1.8. *Let $(X, \mathsf{d}, \ll, \leq, \tau)$ be a globally hyperbolic Lorentzian geodesic space and let $V \subset X$ be an achronal FTC (resp. PTC) subset. Then for each $x \in I^+(V)$ (resp. $x \in I^-(V)$) there exists a point $y_x \in V$ with $\tau_V(y_x) = \tau(y_x, x) > 0$ (resp. $\tau_V(y_x) = -\tau(x, y_x) < 0$).*

Proof. Fix a point $x \in I^+(V)$ (for $x \in I^-(V)$ the proof is analogous). By the very defnition of τ_V and (1.1), it holds $\tau_V(x) > 0$ and $\tau(\cdot, x) \equiv 0$ outside of $J^-(x)$. Since by global hyperbolicity [50, Theorem 3.28] the function $\tau(\cdot, x) : X \to \mathbb{R}$ is finite and continuous, then it admits maximum on the compact set $K := \overline{J^-(x) \cap V} \subset V$ at some point y_x . Thus

$$\tau(y_x, x) = \max_{y\in K} \tau(y,x) = \sup_{y\in V} \tau(y,x) = \tau_V(x) > 0.$$

□

Remark 1.9. Lemma 1.8 and reverse triangle inequality (1.1)implies that

$$\tau_V(x) - \tau_V(z) \geq \tau(y_z, x) - \tau(y_z, z) \geq \tau(z, x), \quad \forall x, z \in I^+(V),\, z \leq x.$$

In analogy to the metric setting, it is natural to introduce the next notion of timelike non-branching.

Definition 1.10 (Timelike non-branching)**.** A Lorentzian pre-length space $(X, \mathsf{d}, \ll, \leq, \tau)$ is said to be *forward timelike non-branching* if and only if for any $\gamma^1, \gamma^2 \in \mathrm{TGeo}(X)$, it holds:

$$\exists\, \bar{t} \in (0,1) \text{ such that } \ \forall t \in [0, \bar{t}] \quad \gamma^1_t = \gamma^2_t \quad \Longrightarrow \quad \gamma^1_s = \gamma^2_s, \quad \forall s \in [0,1].$$

It is said to be *backward timelike non-branching* if the reversed causal structure is forward timelike non-branching. In case it is both forward and backward timelike non-branching it is said *timelike non-branching.*

By Cauchy Theorem, it is clear that if (M, g) is a space-time whose Christoffel symbols are locally-Lipschitz (e.g. in case $g \in C^{1,1}$) then the associated synthetic structure is timelike non-branching. It is expected that for spacetimes with a metric of lower regularity (e.g. $g \in C^1$ or $g \in C^0$) timelike branching can occur. It is also expected that timelike branching can occur in closed cone structures (see Remark 1.13) when the Lorentz-Finsler norm is not strictly convex (see [59, Remark 2.8]).

Definition 1.11 (Measured Lorentzian pre-length space $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$)**.** A *measured Lorentzian pre-length space* $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ is a Lorentzian pre-length space endowed with a Radon non-negative measure $\mathfrak{m}$ with $\operatorname{supp} \mathfrak{m} = X$. We say that $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ is globally hyperbolic (resp. geodesic) if $(X, \mathsf{d}, \ll, \leq, \tau)$ is so.

Recall that a Radon measure $\mathfrak{m}$ on a proper metric space X is a Borel-regular measure which is finite on compact subsets. In this framework, it is well known (see for instance [47, Section 1.6]) that Suslin sets are $\mathfrak{m}$-measurable. For the sake of this paper it will be enough to recall that Suslin sets (also called analytic sets) are precisely images via continuous mappings of Borel subsets in complete and separable metric spaces (for more details see [47, 70]).

Remark 1.12 (Case of a spacetime with a continuous Lorentzian metric)**.** Let M be a smooth manifold, g be a continuous Lorentzian metric over M and assume that (M, g) is time-oriented (i.e. there is a continuous timelike vector field). Note that, for C^0-metrics, the natural class of differentiability of the manifolds is C^1; now, C^1 manifolds always possess a C^∞ subatlas, and one can choose some such sub-atlas whenever convenient.

A causal (respectively timelike) curve in M is by definition a locally Lipschitz curve whose tangent vector is causal (resp. timelike) almost everywhere. It would also be possible to start from absolutely continuous (AC for short) curves, but since causal AC curves always admit a re-parametrisation that is Lipschitz [59, Sec. 2.1, Rem. 2.3], we do not loose in generality with the above convention.

Denote with $\mathrm{L}_g(\gamma)$ the g-length of a causal curve $\gamma : I \to M$, i.e. $\mathrm{L}_g(\gamma) := \int_I \sqrt{-g(\dot\gamma, \dot\gamma)}\, dt$. The time separation function $\tau : M \times M \to [0, \infty]$ is then defined in the usual way, i.e.

$$\tau(x, y) := \sup\{\mathrm{L}_g(\gamma) : \gamma \text{ is future directed causal from } x \text{ to } y\}, \quad \text{if } x \leq y,$$

and $\tau(x, y) = 0$ otherwise. Note that the reverse triangle inequality (1.1) follows directly from the definition. It is easy to check that an L_g-maximal curve γ is also L_τ-maximal, and $\mathrm{L}_g(\gamma) = \mathrm{L}_\tau(\gamma)$ (see for instance [50, Remark 5.1]). Also, we fix a complete Riemannian metric h on M and denote by d^h the associated distance function.

For a spacetime with a Lorentzian C^0-metric:

- Global hyperbolicity implies causal closedness and $\mathcal{K}$-global hyperbolicity [67, Proposition 3.3 and Corollary 3.4].
- Recall that a Cauchy hypersurface is a subset which is met exactly once by every inextendible causal curve. It is a well known fact that, even for C^0-metrics, a Cauchy hypersurface is a closed acasual topological hypersurface [67, Proposition 5.2]. Global hyperbolicity is equivalent to the existence of a Cauchy hypersurface [67, Theorem 5.7, Theorem 5.9] which in turn implies strong causality [67, Proposition 5.6].
- By [50, Proposition 5.8], if g is a causally plain (or, more strongly, locally Lipschitz) Lorentzian C^0-metric on M then the associated synthetic structure is a pre-length Lorentzian space. More

strongly, from [50, Theorem 3.30 and Theorem 5.12] and combining the above items, if g is a globally hyperbolic and causally plain Lorentzian C^0-metric on M then the associated synthetic structure is a causally closed, $\mathcal{K}$-globally hyperbolic Lorentzian geodesic space.

- Any Cauchy hypersurface is causally complete. More strongly, if $V \subset M$ is Cauchy hypersurface then for every $x \in J^+(V)$ it holds that $J^-(x) \cap J^+(V)$ is compact (and analogous statement for $x \in J^-(V)$). This fact is classical and well known in the smooth setting (see for instance [63, Lemma 14.40] or [79, Theorem 8.3.12]) and extendable to C^0-metrics along the lines of the proof of [67, Theorem 5.7].

Remark 1.13 (Other classes of examples).

- **Closed cone structures**. Several results from smooth causality theory can be extended to cone structures on smooth manifolds. One of the motivations for such generalisations comes from the problem of constructing smooth time functions in stably causal or globally hyperbolic spacetimes. Fathi and Siconolfi [29] analysed continuous cone structures with tools from weak KAM theory, Bernard and Suhr [6] studied Lyapunov functions for closed cone structures and showed (among other results) the equivalence between global hyperbolicity and the existence of steep temporal functions in this framework, Minguzzi [59] gave a deep and comprehensive analysis of causality theory for closed cone structures, including embedding and singularity theorems in this framework. Closed cone structures provide a rich source of examples of Lorentzian pre-length and length spaces, which can be seen as the synthetic-Lorentzian analogue of Finsler manifolds (see [50, Section 5.2] for more details).

- **Outlook on examples, towards quantum gravity**. The framework of Lorentzian synthetic spaces allows to handle situations where one may not have the structure of a manifold or a Lorentz(-Finsler) metric. The optimal transport tools developed in the paper can provide a new perspective on curvature in those cases where there is no classical notion of curvature (Riemann tensor, Ricci and sectional curvature, etc.). A remarkable example of such a situation is given by certain approaches to quantum gravity, see for instance [57] where it is shown that from only a countable dense set of events and the causality relation, it is possible to reconstruct a globally hyperbolic spacetime in a purely order theoretic manner. In particular, two approaches to quantum gravity are linked to Lorentzian synthetic spaces: the one of *causal Fermion systems* [30, 31] and the *theory of causal sets* [11]. The basic idea in both cases is that the structure of spacetime needs to be adjusted on a microscopic scale to include quantum effects. This leads to non-smoothness of the underlying geometry, and the classical structure of Lorentzian manifold emerges only in the macroscopic regime. For the connection to the theory of Lorentzian (pre-)length spaces we refer to [50, Section 5.3], [30, Section 5.1]. Let us mention that the link with causal Fermion systems looks particularly promising: indeed the two cornerstones, used to define synthetic timelike-Ricci curvature lower bounds, are *Loretzian-distance* and *measure*, and a causal Fermion system is naturally endowed with both (the reference measure in this setting is called *universal measure*).

1.2 Measures and weak/narrow convergence

In this subsection we briefly recall some basic notions of convergence of measures that will be used in the paper. Standard references for the topic are [2, 78].

Given a complete and separable (in particular, everything hold for proper) metric space (X, d), we denote by $\mathscr{B}(X)$ the collection of its Borel sets and by $\mathcal{P}(X)$ (resp. $\mathcal{P}_c(X)$) the collection of all Borel probability measures (resp. with compact support).

We say that $(\mu_n) \subset \mathcal{P}(X)$ *narrowly converges* to $\mu_\infty \in \mathcal{P}(X)$ provided

$$\lim_{n\to\infty} \int f\, \mu_n = \int f\, \mu_\infty \qquad \text{for every } f \in \mathrm{C}_b(X), \tag{1.9}$$

where $\mathrm{C}_b(X)$ denotes the space of bounded and continuous functions.

Relative narrow compactness in $\mathcal{P}(X)$ can be characterized by Prokhorov's Theorem. Let us first recall that a set $\mathcal{K} \subset \mathcal{P}(X)$ is said to be tight provided for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset X$ such that

$$\mu(X \setminus K_\varepsilon) \leq \varepsilon \quad \text{for every } \mu \in \mathcal{K}.$$

The we have the following classical result:

Theorem 1.14 (Prokhorov)**.** *Let (X, d) be complete and separable. A subset $\mathcal{K} \subset \mathcal{P}(X)$ is tight if and only if it is precompact in the narrow topology.*

We next recall a useful tightness criterion for measures in $\mathcal{P}(X \times X)$ (for the proof see for instance [2, Lemma 5.2.2]). To this aim, denote with $P_1, P_2 : X \times X \to X$ the projections onto the first and second factor. The push-forward is defined as $(P_i)_\sharp \pi(A) := \pi(P_i^{-1}(A))$ for any $A \in \mathscr{B}(X)$.

Lemma 1.15 (Tightness criterion in $\mathcal{P}(X \times X)$)**.** *A subset $\mathcal{K} \subset \mathcal{P}(X \times X)$ is tight if and only if $(P_i)_\sharp \mathcal{K} \subset \mathcal{P}(X)$ is tight for $i = 1, 2$.*

We next recall a useful property concerning passage to the limit in (1.9) when f is possibly unbounded, but a "uniform integrability" condition holds.

Definition 1.16 (Uniform integrability)**.** We say that a Borel function $g : X \to [0, +\infty]$ is *uniformly integrable* w.r.t. a given set $\mathcal{K} \subset \mathcal{P}(X)$ if

$$\limsup_{k \to \infty} \sup_{\mu \in \mathcal{K}} \int_{\{g \geq k\}} g \, \mu = 0. \tag{1.10}$$

Lemma 1.17 (Lemma 5.1.7 [2])**.** *Let $(\mu_n) \subset \mathcal{P}(X)$ be narrowly convergent to $\mu_\infty \in \mathcal{P}(X)$. If $f : X \to [0, \infty)$ is continuous and uniformly integrable with respect to the set $\{\mu_n\}_{n \in \mathbb{N}}$, then*

$$\lim_{n \to \infty} \int f \, \mu_n = \int f \, \mu_\infty.$$

Conversely, if $f : X \to [0, \infty)$ is continuous, $f \in L^1(\mu_n)$ for every $n \in \mathbb{N}$ and

$$\limsup_{n \to \infty} \int_X f \, \mu_n \leq \int_X f \, \mu_\infty < +\infty, \tag{1.11}$$

then f is uniformly integrable with respect to the set $\{\mu_n\}_{n \in \mathbb{N}}$.

1.3 Relative entropy and basic properties

We denote $\mathcal{P}_{ac}(X)$ the space of probability measures absolutely continuous with respect to $\mathfrak{m}$.

Definition 1.18. Given a probability measure $\mu \in \mathcal{P}(X)$ we define its relative entropy by

$$\mathrm{Ent}(\mu|\mathfrak{m}) = \int_X \rho \log(\rho) \, \mathfrak{m}, \tag{1.12}$$

if $\mu = \rho \, \mathfrak{m}$ is absolutely continuous with respect to $\mathfrak{m}$ and $(\rho \log(\rho))_+$ is $\mathfrak{m}$-integrable. Otherwise we set $\mathrm{Ent}(\mu|\mathfrak{m}) = +\infty$.

A simple application of Jensen inequality using the convexity of $(0, \infty) \ni t \mapsto t \log t$ gives

$$\mathrm{Ent}(\mu|\mathfrak{m}) \geq -\log \mathfrak{m}(\operatorname{supp} \mu) > -\infty, \quad \forall \mu \in \mathcal{P}_c(X). \tag{1.13}$$

We set $\mathrm{Dom}(\mathrm{Ent}(\cdot|\mathfrak{m})) := \{\mu \in \mathcal{P}(X) : \mathrm{Ent}(\mu|\mathfrak{m}) \in \mathbb{R}\}$ to be the finiteness domain of the entropy. An important property of the relative entropy is the (joint) lower-semicontinuity under narrow convergence in case the reference measures are probabilities (for a proof, see for instance [2, Lemma 9.4.3]):

$$\mathfrak{m}_n, \mathfrak{m}_\infty \in \mathcal{P}(X), \ \mathfrak{m}_n \to \mathfrak{m}_\infty, \ \mu_n \to \mu_\infty \text{ narrowly} \quad \Longrightarrow \quad \liminf_{n \to \infty} \mathrm{Ent}(\mu_n|\mathfrak{m}_n) \geq \mathrm{Ent}(\mu_\infty|\mathfrak{m}_\infty). \tag{1.14}$$

In particular, for a general fixed reference measure $\mathfrak{m}$ it holds:

$$\mu_n \to \mu_\infty \text{ narrowly and } \mathfrak{m}\Big(\bigcup_{n \in \mathbb{N}} \operatorname{supp} \mu_n\Big) < \infty \quad \Longrightarrow \quad \liminf_{n \to \infty} \mathrm{Ent}(\mu_n|\mathfrak{m}) \geq \mathrm{Ent}(\mu_\infty|\mathfrak{m}). \tag{1.15}$$

2 Optimal transport in Lorentzian synthetic spaces

2.1 The ℓ_p-optimal transport problem

Given $\mu, \nu \in \mathcal{P}(X)$, denote

$$\begin{aligned}\Pi(\mu,\nu) &:= \{\pi \in \mathcal{P}(X\times X) \,:\, (P_1)_\sharp \pi = \mu,\, (P_2)_\sharp \pi = \nu\},\\ \Pi_{\leq}(\mu,\nu) &:= \{\pi \in \Pi(\mu,\nu) \,:\, \pi(X^2_{\leq}) = 1\},\\ \Pi_{\ll}(\mu,\nu) &:= \{\pi \in \Pi(\mu,\nu) \,:\, \pi(X^2_{\ll}) = 1\}\end{aligned}$$

where $X^2_{\leq} := \{(x,y) \in X^2 \,:\, x \leq y\}$ and $X^2_{\ll} := \{(x,y) \in X^2 \,:\, x \ll y\}$.

Note that for $(X, \mathsf{d}, \ll, \leq, \tau)$ causally closed (i.e. $X^2_{\leq} \subset X^2$ is a closed subset), $\pi \in \Pi_{\leq}(\mu,\nu)$ if and only if $\operatorname{supp}\pi \subset X^2_{\leq}$.

Definition 2.1. Let $(X, \mathsf{d}, \ll, \leq, \tau)$ be a Lorentzian pre-length space and let $p \in (0,1]$. Given $\mu, \nu \in \mathcal{P}(X)$, the *p-Lorentz-Wasserstein distance* is defined by

$$\ell_p(\mu,\nu) := \sup_{\pi \in \Pi_{\leq}(\mu,\nu)} \left(\int_{X\times X} \tau(x,y)^p \, \pi(dxdy) \right)^{1/p}. \tag{2.1}$$

When $\Pi_{\leq}(\mu,\nu) = \emptyset$ we set $\ell_p(\mu,\nu) := -\infty$.

Note that Definition 2.1 extends to Lorentzian pre-length spaces the corresponding notion given in the smooth Lorentzian setting in [26] (see also [58, 61], and [74] for $p = 1$); when $\Pi_{\leq}(\mu,\nu) = \emptyset$ we adopt the convention of McCann [58] (note that [26] set $\ell_p(\mu,\nu) = 0$ in this case). A coupling $\pi \in \Pi_{\leq}(\mu,\nu)$ maximising in (2.1) is said ℓ_p*-optimal*. The set of ℓ_p*-optimal* couplings from μ to ν is denoted by $\Pi^{p\text{-opt}}_{\leq}(\mu,\nu)$.

Remark 2.2 (An equivalent formulation of (2.1))**.** Set

$$\ell^p(x,y) := \begin{cases} \tau(x,y)^p & \text{if } x \leq y \\ -\infty & \text{otherwise} \end{cases}. \tag{2.2}$$

Notice that for every $\pi \in \Pi_{\leq}(\mu,\nu)$ it holds $\int_{X\times X} \tau(x,y)^p \, \pi(dxdy) = \int_{X\times X} \ell(x,y)^p \, \pi(dxdy) \in \mathbb{R}_{\geq 0}$. Moreover, if $\pi \in \Pi(\mu,\nu)$ satisfies $\int_{X\times X} \ell(x,y)^p \, \pi(dxdy) > -\infty$ then $\pi \in \Pi_{\leq}(\mu,\nu)$. Thus the maximization problem (2.1) is equivalent (i.e. the sup and the set of maximisers coincide) to the maximisation problem

$$\sup_{\pi \in \Pi(\mu,\nu)} \left(\int_{X\times X} \ell^p(x,y) \, \pi(dxdy) \right)^{1/p}. \tag{2.3}$$

The advantage of the formulation (2.3) is that, when X is causally closed and globally hyperbolic geodesic (so that τ is continuous) then ℓ^p is upper semi-continuous on $X \times X$. Similarly, when X is *locally* causally closed globally hyperbolic geodesic, if μ and ν have compact support then ℓ is upper semi-continuous on $\operatorname{supp}\mu \times \operatorname{supp}\nu$.
In both cases, one can apply to the Monge-Kantorovich problem (2.3) standard optimal transport techniques (e.g. [78]).

We will adopt the following standard notation: given $\mu, \nu \in \mathcal{P}(X)$, we denote with $\mu \otimes \nu \in \mathcal{P}(X^2)$ the product measure; given $u, v : X \to \mathbb{R} \cup \{+\infty\}$ we denote with $u \oplus v : X^2 \to \mathbb{R} \cup \{+\infty\}$ the function $u \oplus v(x,y) := u(x) + v(y)$.

Proposition 2.3. *Let $(X, \mathsf{d}, \ll, \leq, \tau)$ be a causally closed (resp. locally causally closed) globally hyperbolic Lorentzian geodesic space and let $\mu, \nu \in \mathcal{P}(X)$ (resp. $\mathcal{P}_c(X)$). If $\Pi_{\leq}(\mu,\nu) \neq \emptyset$ and if there exist measurable functions $a, b : X \to \mathbb{R}$, with $a \oplus b \in L^1(\mu \otimes \nu)$ such that $\ell^p \leq a \oplus b$ on $\operatorname{supp}\mu \times \operatorname{supp}\nu$ (e.g. when $\mu, \nu \in \mathcal{P}_c(X)$) then the $\sup$ in (2.1) is attained and finite.*

Proof. The claim follows from Remark 2.2 combined with [78, Theorem 4.1] (see also [77, Theorem 1.3]). □

We next show that ℓ_p satisfies the reverse triangle inequality. This was proved in the smooth Lorentzian setting by Eckstein-Miller [26, Theorem 13], and it is the natural Lorentzian analogue of the fact that the Kantorovich-Rubinstein-Wasserstein distances W_p, $p \geq 1$, in the metric space setting satisfy the usual triangle inequality (see for instance [78, Section 6]).

We first isolate the causal version of the Gluing Lemma, a classical tool in Optimal Transport theory (see for instance [78]).

Lemma 2.4 (Gluing Lemma)**.** *Let $(X, \mathsf{d}, \ll, \leq, \tau)$ be a Lorentzian pre-length space and let $\mu_i \in \mathcal{P}(X)$ for $i = 1, 2, 3$. If $\pi_{12} \in \Pi_{\leq}(\mu_1, \mu_2)$ and $\pi_{23} \in \Pi_{\leq}(\mu_2, \mu_3)$ are given, then there exists $\pi_{123} \in \mathcal{P}(X \times X)$ such that*

$$(P_{12})_\sharp \pi_{123} = \pi_{12}, \quad (P_{23})_\sharp \pi_{123} = \pi_{23}, \quad (P_{13})_\sharp \pi_{123} \in \Pi_{\leq}(\mu_1, \mu_3).$$

Proof. The proof goes along the same lines of the classical Gluing Lemma (see for instance [77, Lemma 7.6]). Disintegrate the coupling π_{12} with respect to P_2 and the coupling π_{23} with respect to P_1 and obtain the following formula

$$\pi_{12} = \int_X (\pi_{12})_x \, \mu_2(dx), \quad \pi_{23} = \int_X (\pi_{23})_x \, \mu_2(dx), \qquad (\pi_{12})_x, (\pi_{23})_x \in \mathcal{P}(X \times X),$$

with $(\pi_{12})_x(X \times \{x\}) = (\pi_{23})_x(\{x\} \times X) = 1$, μ_2-a.e. . Since π_{12} and π_{23} are causal couplings, we have

$$(\pi_{12})_x(X^2_{\leq}) = (\pi_{23})_x(X^2_{\leq}) = 1, \quad \text{for } \mu_2\text{-a.e. } x \in X.$$

In particular, for $(\pi_{12})_x$-a.e. (z, x) and for $(\pi_{23})_x$-a.e. (x, y), the transitive property of $\leq$ gives that $z \leq y$. Hence defining

$$\pi_{123} = \int_X (P_{14})_\sharp((\pi_{12})_x \otimes (\pi_{23})_x) \, \mu_2(dx),$$

the first two claims are obtained by the classical Gluing Lemma [77, Lemma 7.6] (or [78, Chapter 1]), while the last one follows from the previous argument. □

Proposition 2.5 (ℓ_p satisfies the reverse triangle inequality)**.** *Let $(X, \mathsf{d}, \ll, \leq, \tau)$ be a Lorentzian pre-length space and let $p \in (0, 1]$. Then ℓ_p satisfies the reverse triangle inequality:*

$$\ell_p(\mu_0, \mu_1) + \ell_p(\mu_1, \mu_2) \leq \ell_p(\mu_0, \mu_2), \quad \forall \mu_0, \mu_1, \mu_2 \in \mathcal{P}(X), \tag{2.4}$$

where we adopt the convention that $\infty - \infty = -\infty$ to interpret the left hand side of (2.4).

Proof. We assume $\ell_p(\mu_0, \mu_1), \ell_p(\mu_1, \mu_2) > -\infty$, otherwise the claim is trivial.
We first consider the case when $\ell_p(\mu_0, \mu_1), \ell_p(\mu_1, \mu_2) < \infty$. By the very definition (2.1) of ℓ_p, for any $\varepsilon > 0$ we can find $\pi_{01} \in \Pi_{\leq}(\mu_0, \mu_1)$ $\pi_{12} \in \Pi_{\leq}(\mu_1, \mu_2)$ such that

$$\ell_p(\mu_0, \mu_1) \leq \left(\int_{X \times X} \tau(x, y)^p \, \pi_{01}(dxdy) \right)^{1/p} + \varepsilon, \quad \ell_p(\mu_1, \mu_2) \leq \left(\int_{X \times X} \tau(x, y)^p \, \pi_{12}(dxdy) \right)^{1/p} + \varepsilon.$$

We denote with $\pi_{012} \in \mathcal{P}(X^3)$ the measure given by the Gluing Lemma 2.4. Recalling that for π_{012}-a.e.

$(x,z,y)\in X^3$ it holds $x\le z\le y$, we can use (1.1) to compute

$$\begin{aligned}
\ell_p(\mu_0,\mu_2) &\geq \left(\int_{X\times X} \tau(x,y)^p\,(P_{13})_\sharp\pi_{012}(dxdy)\right)^{1/p}\\
&= \left(\int_{X\times X\times X} \tau(x,y)^p\,\pi_{012}(dxdzdy)\right)^{1/p}\\
&\geq \left(\int_{X\times X\times X} [\tau(x,z)+\tau(z,y)]^p\,\pi_{012}(dxdzdy)\right)^{1/p}\\
&\geq \left(\int_{X\times X} \tau(x,z)^p\,\pi_{012}(dxdzdy)\right)^{1/p}+\left(\int_{X\times X} \tau(z,y)^p\,\pi_{012}(dxdzdy)\right)^{1/p}\\
&= \left(\int_{X\times X} \tau(x,z)^p\,\pi_{01}(dxdz)\right)^{1/p}+\left(\int_{X\times X} \tau(z,y)^p\,\pi_{12}(dzdy)\right)^{1/p}\\
&\geq \ell_p(\mu_0,\mu_1)+\ell_p(\mu_1,\mu_2)-2\varepsilon,
\end{aligned}$$

proving the inequality, by the arbitrariness of $\varepsilon>0$. If one of $\ell_p(\mu_0,\mu_1)$, $\ell_p(\mu_1,\mu_2)$ is not bounded from above, then simply take a sequence of couplings with diverging cost; repeating the above calculations we obtain that also $\ell_p(\mu_0,\mu_2)=\infty$, proving the claim. □

2.2 Cyclical monotonicity

The notion of cyclical monotonicity is very useful to relate an optimal coupling with its support.

Definition 2.6 (τ^p-cyclical monotonicity and ℓ^p-cyclical monotonicity)**.** Fix $p\in(0,1]$ and let $(X,\mathsf{d},\ll,\le,\tau)$ be a Lorentzian pre-length space. A subset $\Gamma\subset X^2_{\le}$ is said to be τ^p-cyclically monotone (resp. ℓ^p-cyclically monotone) if, for any $N\in\mathbb{N}$ and any family $(x_1,y_1),\dots,(x_N,y_N)$ of points in Γ, the next inequality holds:

$$\sum_{i=1}^N \tau(x_i,y_i)^p \geq \sum_{i=1}^N \tau(x_{i+1},y_i)^p, \tag{2.5}$$

(resp. $\sum_{i=1}^N \ell(x_i,y_i)^p \geq \sum_{i=1}^N \ell(x_{i+1},y_i)^p)$ with the convention $x_{N+1}=x_1$. A coupling is said to be τ^p-cyclically monotone (resp. ℓ^p-cyclically monotone) if it is concentrated on an τ^p-cyclically monotone set (resp. ℓ^p-cyclically monotone set).

Remark 2.7. Notice that $\Gamma\subset X^2_{\le}$ is ℓ^p-cyclically monotone if and only if (2.5) holds for those families with $x_{i+1}\le y_i$ for all $i\in\{1,\dots,N\}$. It is then clear that

$$\tau^p\text{-cyclical monotonicity} \;\Rightarrow\; \ell^p\text{-cyclical monotonicity.} \tag{2.6}$$

Note if $P_1(\Gamma)\times P_2(\Gamma)\subset X^2_{\le}$ then ℓ^p-cyclical monotonicity is equivalent to τ^p-cyclical monotonicity.

Proposition 2.8 (Optimality $\Leftrightarrow$ cyclical monotonicity)**.** *Fix $p\in(0,1]$. Let $(X,\mathsf{d},\ll,\le,\tau)$ be a Lorentzian pre-length space and let $\mu,\nu\in\mathcal{P}(X)$. Assume that $\Pi_{\le}(\mu,\nu)\neq\emptyset$ and that there exist measurable functions $a,b:X\to\mathbb{R}$, with $a\oplus b\in L^1(\mu\otimes\nu)$ such that $\ell^p\le a\oplus b$, $\mu\otimes\nu$-a.e.. Then the following holds.*

1. *If π is ℓ_p-optimal then π is ℓ^p-cyclically monotone.*
2. *If $\pi(X^2_{\ll})=1$ and π is ℓ^p-cyclically monotone then π is ℓ_p-optimal.*

Proof. The result follows from [9], dealing with optimal transport (minimisation) problems associated to general Borel cost functions $c(\cdot,\cdot):X^2\to[0,+\infty]$. Of course, the (maximising) optimal couplings in $\Pi(\mu,\nu)$ for the cost ℓ^p are the same as for the cost $\ell^p-(a\oplus b)$, which is non-positive $\mu\otimes\nu$-a.e.;

hence we enter in the framework of [9].
The first claim thus follows from [9, Lemma 5.2] (see also [9, Proposition B.16]).
For the second claim, notice that [9, Theorem 5.6] provides a general condition to ensure that an ℓ^p-cyclically monotone coupling is ℓ^p-optimal. Thanks to [9, Corollary 5.7, Proposition 5.8] it will be enough to verify the existence of countably many Borel sets $A_i, B_i \subset X$ such that

$$\pi\Big(\bigcup_{i\in\mathbb{N}} A_i \times B_i\Big) = 1, \qquad \bigcup_{i\in\mathbb{N}} A_i \times B_i \subset X^2_{\leq}.$$

The existence of such sets (that can actually chosen to be open) follows directly from the fact that $X^2_{\ll} = \{\tau > 0\} \subset X^2$ is open by the lower semicontinuity of τ. □

Remark 2.9. Thanks to [50, Proposition 5.8], Proposition 2.8 is valid for a causally plain (so, in particular, for a locally-Lipschitz) Lorentzian C^0-metric g on a space-time M.
In case $(X, \mathsf{d}, \ll, \leq, \tau)$ is a causally closed globally hyperbolic Lorentzian geodesic space (as it will be for most of the paper), the first claim in Proposition 2.8 follows from more standard literature (see e.g. [3, Theorem 3.2]), thanks to Remark 2.2.

We will later see that for τ^p-cyclically monotone causal couplings, ℓ_p-optimality holds true (Theorem 2.26). To conclude we report a standard fact about optimal couplings.

Lemma 2.10 (Restriction)**.** *Fix $p \in (0, 1]$. Let $(X, \mathsf{d}, \ll, \leq, \tau)$ be a Lorentzian pre-length space and let $\mu, \nu \in \mathcal{P}(X)$. Then for every $\pi \in \Pi^{p\text{-}opt}_{\leq}(\mu, \nu)$ and every measurable function $f : X \times X \to [0, \infty)$ with $\int f\,\pi = 1$ and $f \in L^\infty(\pi)$, also the coupling $f\pi$ is optimal, i.e. denoting with*

$$\mu_f := (P_1)_\sharp f\pi, \qquad \nu_f := (P_2)_\sharp f\pi,$$

it holds true $f\pi \in \Pi^{p\text{-}opt}_{\leq}(\mu_f, \nu_f)$.

Proof. Trivially $f\pi \in \Pi_{\leq}(\mu_f, \nu_f)$ hence we will only be concerned about optimality. Assume by contradiction the existence of $\hat\pi \in \Pi_{\leq}(\mu_f, \nu_f)$ with

$$\int_{X\times X} \tau(x,y)^p f(x,y)\pi(dxdy) < \int_{X\times X} \tau(x,y)^p \hat\pi(dxdy).$$

Consider then the new coupling

$$\bar\pi := \pi - \frac{f}{\|f\|_\infty}\pi + \frac{1}{\|f\|_\infty}\hat\pi.$$

By linearity, $\bar\pi$ has the same marginals of π and it is causal, i.e. $\bar\pi \in \Pi_{\leq}(\mu, \nu)$. Finally

$$\begin{aligned}\int_{X\times X} \tau(x,y)^p \bar\pi(dxdy) &= \int_{X\times X} \tau(x,y)^p \pi(dxdy) + \frac{1}{\|f\|_\infty}\int_{X\times X} \tau(x,y)^p (\hat\pi - f\pi)(dxdy) \\ &> \int_{X\times X} \tau(x,y)^p \pi(dxdy),\end{aligned}$$

giving a contradiction. □

2.3 Stability of optimal couplings

While in the Riemannian framework stability of optimal couplings follows by stability of cyclical monotonicity, in the Lorentzian setting, due to the upper semicontinuity of the cost function ℓ^p (opposed to continuity of the Riemannian cost d^p), a more refined analysis is needed.

Building on the previous Proposition 2.8, we can establish a first basic stability property with respect to narrow convergence valid for a special class of optimal couplings.

Lemma 2.11 (Stability of ℓ_p-optimal couplings I)**.** *Let $(X, \mathsf{d}, \ll, \leq, \tau)$ be a causally closed, globally hyperbolic Lorentzian geodesic space and fix $p \in (0,1]$. Let $(\mu_n^1), (\mu_n^2) \subset \mathcal{P}(X)$ be narrowly convergent to some $\mu_\infty^1, \mu_\infty^2 \in \mathcal{P}(X)$ and assume that, for every $n \in \mathbb{N}$, there exists an ℓ_p-optimal coupling $\pi_n \in \Pi_\leq^{p\text{-}opt}(\mu_n^1, \mu_n^2)$ which is also τ^p-cyclically monotone.*

Then (π_n) is narrowly relatively compact in $\mathcal{P}(X^2)$, moreover any narrow limit point π_∞ belongs to $\Pi_\leq(\mu_\infty^1, \mu_\infty^2)$ and is ℓ_p-optimal, provided $\pi_\infty(X_\ll^2) = 1$.

Proof. By Prokhorov Theorem 1.14, the subsets $\{\mu_n^1\}_{n\in\mathbb{N}}, \{\mu_n^2\}_{n\in\mathbb{N}} \subset \mathcal{P}(X)$ are tight. Lemma 1.15 implies that $\{\pi_n\}_{n\in\mathbb{N}} \subset \mathcal{P}(X \times X)$ is tight as well, and then (again by Theorem 1.14) it converges narrowly, up to a subsequence, to some $\pi_\infty \in \mathcal{P}(X \times X)$. Using the continuity of the projection maps, it is readily seen that $\pi_\infty \in \Pi(\mu_\infty^1, \mu_\infty^2)$. The casual closeness assumption further implies that $\pi_\infty \in \Pi_\leq(\mu_\infty^1, \mu_\infty^2)$. To conclude optimality it is enough to observe that τ^p is continuous and therefore τ^p-cyclical monotonicity is preserved under narrow convergence (note that the same claim would be false for ℓ^p-cyclically monotone sets) and apply the second point of Proposition 2.8 together with (2.6) (see also Theorem 2.26 below, for a more self-contained proof of the implication π is τ^p-cyclically monotone $\Rightarrow$ π is ℓ_p-optimal). □

To obtain stronger stability properties, we will use Γ-convergence techniques. For the reaming of this section, $(X, \mathsf{d}, \ll, \leq, \tau)$ will be a causally closed, globally hyperbolic Lorentzian geodesic space and we also fix $p \in (0,1]$. Let $(\mu_n^1), (\mu_n^2) \subset \mathcal{P}(X)$ be narrowly convergent to some $\mu_\infty^1, \mu_\infty^2 \in \mathcal{P}(X)$. Associated with them we define

$$F_n, F_\infty : \mathcal{P}(X^2) \to \mathbb{R} \cup \{-\infty\}, \quad F_i(\pi) = \begin{cases} \int_{X\times X} \tau(x,y)^p \, \pi(dxdy), & \pi \in \Pi_\leq(\mu_i^1, \mu_i^2) \\ -\infty, & \text{otherwise,} \end{cases}$$

for $i = n, \infty$.

Lemma 2.12. *($\limsup$-inequality) Let $\{\pi_i\}_{i\in\mathbb{N}\cup\{\infty\}} \subset \mathcal{P}(X^2)$ be such that $\pi_n \to \pi_\infty$ narrowly and τ^p is uniformly integrable with respect to $\{\pi_i\}_{i\in\mathbb{N}\cup\{\infty\}}$ (in particular, the second condition is satisfied if there exists a compact subset containing* $\operatorname{supp} \pi_n$ *for all $n \in \mathbb{N}$). Then*

$$F_\infty(\pi_\infty) \geq \limsup_{n\to\infty} F_n(\pi_n). \tag{2.7}$$

If moreover, $\pi_n(X_\leq^2) = 1$ for all $n \in \mathbb{N}$, then also $\pi_\infty(X_\leq^2) = 1$ and

$$F_\infty(\pi_\infty) = \lim_{n\to\infty} F_n(\pi_n).$$

Proof. Without loss of generality we can assume that $\pi_n(X_\leq^2) = 1$ definitively, otherwise the claim (2.7) is trivial. Since by assumption $X_\leq^2 \subset X^2$ is closed, it follows that

$$\pi_\infty(X_\leq^2) \geq \limsup_{n\to\infty} \pi_n(X_\leq^2) = 1.$$

Using that (from global hyperbolicity) τ^p is continuous on $X_\leq^2$ together with Lemma 1.17, we conclude that $F_n(\pi_n) \to F_\infty(\pi_\infty)$. □

For the liminf inequality we have to select a particular family of $(\mu_n^1), (\mu_n^2)$.

Lemma 2.13 (Existence of a recovery sequence)**.** *Assume that there exists a compact subset $\mathcal{K} \subset X$ such that* $\operatorname{supp} \mu_n^1$, $\operatorname{supp} \mu_n^2 \subset \mathcal{K}$ *for all $n \in \mathbb{N}$. Assume that, for each $n \in \mathbb{N}$, the sets $\Pi_\leq(\mu_n^1, \mu_\infty^1)$ and $\Pi_\leq(\mu_\infty^2, \mu_n^2)$ are both not empty. Then, for any $\pi \in \Pi(\mu_\infty^1, \mu_\infty^2)$, there exists a sequence $\pi_n \in \Pi(\mu_n^1, \mu_n^2)$ such that $F_n(\pi_n) \to F_\infty(\pi)$.*

Proof. Fix any $\pi \in \mathcal{P}(X^2)$. If $\pi \notin \Pi_\leq(\mu_\infty^1, \mu_\infty^2)$ then the claim is trivial (just take as recovering sequence π itself). Assume then $\pi \in \Pi_\leq(\mu_\infty^1, \mu_\infty^2)$. By assumption there exists $\pi_n^1 \in \Pi_\leq(\mu_n^1, \mu_\infty^1)$ and $\pi_n^2 \in$

$\Pi_{\leq}(\mu^2_\infty, \mu^2_n)$. Then, by Gluing Lemma 2.4 and transitivity of $\leq$, we obtain a $\hat{\pi}_n \in \mathcal{P}(X \times X \times X \times X)$ such that

$$(P_{12})_\sharp \hat{\pi}_n = \pi^1_n, \quad (P_{23})_\sharp \hat{\pi}_n = \pi, \quad (P_{34})_\sharp \hat{\pi}_n = \pi^2_n, \quad (P_{14})_\sharp \hat{\pi}_n \in \Pi_{\leq}(\mu^1_n, \mu^2_n).$$

Recalling that τ is non-negative and satisfies reverse triangle inequality, we get:

$$\begin{aligned}
F((P_{14})_\sharp \hat{\pi}_n) &= \int_{X \times X} \tau(x,y)^p \, (P_{14})_\sharp \hat{\pi}_n(dxdy) \\
&= \int_{X \times X \times X \times X} \tau(P_{14}(x,z,w,y))^p \, \hat{\pi}_n(dxdydzdw) \\
&\geq \int_{X \times X \times X \times X} (\tau(x,z) + \tau(z,w) + \tau(w,y))^p \, \hat{\pi}_n(dxdydzdw) \\
&\geq \int_{X \times X} \tau(z,w)^p \pi(dzdw) \\
&\geq F(\pi).
\end{aligned}$$

Using (2.7), it follows that the sequence $\pi_n := (P_{14})_\sharp \hat{\pi}_n$ satisfies the claim. □

Theorem 2.14 (Stability of ℓ_p-optimal couplings II)**.** *Assume that there exists a compact subset $\mathcal{K} \subset X$ such that* $\operatorname{supp} \mu^1_n$, $\operatorname{supp} \mu^2_n \subset \mathcal{K}$ *for all $n \in \mathbb{N}$, and that for each $n \in \mathbb{N}$ the sets $\Pi_{\leq}(\mu^1_n, \mu^1_\infty)$ and $\Pi_{\leq}(\mu^2_\infty, \mu^2_n)$ are both not empty.*

Then $\ell_p(\mu^1_n, \mu^2_n)$ converges to $\ell_p(\mu^1_\infty, \mu^2_\infty)$ and any narrow-limit point of $\Pi^{p\text{-}opt}_{\leq}(\mu^1_n, \mu^2_n)$ belongs to $\Pi^{p\text{-}opt}_{\leq}(\mu^1_\infty, \mu^2_\infty)$.

Proof. For the first claim, notice that from Lemma 2.12 and the equintegrability of τ^p granted by the assumptions, it readily follows that $\limsup_{n\to\infty} \ell_p(\mu^1_n, \mu^2_n) \leq \ell_p(\mu^1_\infty, \mu^2_\infty)$. Also, Lemma 2.13 gives $\ell_p(\mu^1_\infty, \mu^2_\infty) \leq \liminf_{n\to\infty} \ell_p(\mu^1_n, \mu^2_n)$. Hence, $\ell_p(\mu^1_n, \mu^2_n) \to \ell_p(\mu^1_\infty, \mu^2_\infty)$.

For the second claim, if $\pi_n \in \Pi^{p\text{-opt}}_{\leq}(\mu^1_n, \mu^2_n)$ converges narrowly to π then, by the continuity of the projections and the causal closedness of X, we have that $\pi \in \Pi_{\leq}(\mu^1_\infty, \mu^2_\infty)$ and

$$\int \tau(x,y)^p \, \pi(dxdy) = \lim_{n\to\infty} \int \tau(x,y)^p \, \pi_n(dxdy) = \lim_{n\to\infty} \ell_p(\mu^1_n, \mu^2_n)^p = \ell_p(\mu^1_\infty, \mu^2_\infty)^p,$$

where the first identity follows from Lemma 2.12 and last by the previous part of the proof. We conclude that $\pi \in \Pi^{p\text{-opt}}_{\leq}(\mu^1_\infty, \mu^2_\infty)$. □

Another simple criterion, based on ideas from Γ-convergence, to ensure stability of ℓ_p-optimal couplings is the following one.

Lemma 2.15. *Let $(X, \mathsf{d}, \ll, \leq, \tau)$ be a causally closed Lorentzian globally hyperbolic geodesic space and fix $p \in (0,1]$. Let $(\mu^1_n), (\mu^2_n) \subset \mathcal{P}(X)$ be narrowly convergent to some $\mu^1_\infty, \mu^2_\infty \in \mathcal{P}(X)$. Assume moreover the existence of an optimal $\bar{\pi}_\infty \in \Pi^{p\text{-}opt}_{\leq}(\mu^1_\infty, \mu^2_\infty)$ and of a sequence $\bar{\pi}_n \in \Pi_{\leq}(\mu^1_n, \mu^2_n)$ such that $F_n(\bar{\pi}_n) \to F_\infty(\bar{\pi}_\infty)$.*

Then for any τ^p-uniformly integrable sequence $\pi_n \in \Pi^{p\text{-}opt}_{\leq}(\mu^1_n, \mu^2_n)$, any limit measure π_∞ in the narrow topology is ℓ_p-optimal, i.e. $\pi_\infty \in \Pi^{p\text{-}opt}_{\leq}(\mu^1_\infty, \mu^2_\infty)$.

Proof. Consider any limit point π_∞ of a τ^p-uniformly integrable sequence $\pi_n \in \Pi^{p\text{-opt}}_{\leq}(\mu^1_n, \mu^2_n)$ and $\bar{\pi}_\infty \in \Pi^{p\text{-opt}}_{\leq}(\mu^1_\infty, \mu^2_\infty)$ limit point of $\bar{\pi}_n \in \Pi_{\leq}(\mu^1_n, \mu^2_n)$. From Lemma 2.12 we have that

$$F_\infty(\pi_\infty) \geq \limsup_{n\to\infty} F_n(\pi_n) \geq \limsup_{n\to\infty} F_n(\bar{\pi}_n) = F_\infty(\bar{\pi}_\infty) \geq F_\infty(\pi_\infty) \tag{2.8}$$

giving optimality of π_∞. □

From Lemma 2.15 we obtain another stability result. For this scope we introduce the following notation:

$$D := \Big\{ \nu \in \mathcal{P}_c(X) \colon \nu = \sum_{i \le k} \alpha_i \delta_{x_i}, \text{ for some } k \in \mathbb{N} \Big\}.$$

Theorem 2.16 (Stability of ℓ_p-optimal couplings III). *Let $(X, \mathsf{d}, \ll, \le, \tau)$ be a locally causally closed, globally hyperbolic Lorentzian geodesic space and fix $p \in (0,1]$. Let also $\mu_0, \mu_1 \in \mathcal{P}_c(X)$ be given and assume the existence of $\pi \in \Pi_{\le}^{p\text{-}opt}(\mu_0, \mu_1)$ with $\operatorname{supp} \pi \subset X^2_{\ll}$. Let $(\mu_{1,n}) \subset D$ with $\operatorname{supp} \mu_{1,n} \subset \operatorname{supp} \mu_1$ be a sequence narrowly convergent to μ_1 with $\ell_p(\mu_0, \mu_{1,n}) \in [0, \infty)$. Then there exists another sequence $(\bar\mu_{1,n}) \subset D$ such that the following holds true. The sequence $(\bar\mu_{1,n})$ still narrowly converges to μ_1 and $\bar\mu_{1,n}$ is absolutely continuous with respect to $\mu_{1,n}$. Moreover, for any sequence $\pi_n \in \Pi_{\le}^{p\text{-}opt}(\mu_0, \bar\mu_{1,n})$, any limit measure π_∞ in the narrow topology is ℓ_p-optimal, i.e. $\pi_\infty \in \Pi_{\le}^{p\text{-}opt}(\mu_0, \mu_1)$, and*

$$\ell_p(\mu_0, \bar\mu_{1,n}) \to \ell_p(\mu_0, \mu_1).$$

Proof. **Step 1.** Restricting μ_0.
Since $\operatorname{supp} \pi$ is compact and $X^2_{\ll}$ is an open set, for any $\varepsilon > 0$ we find finitely many points $(x_{i,\varepsilon}, y_{i,\varepsilon})$ with $i = 1, \dots, k_\varepsilon$ such that $\operatorname{supp} \pi \subset \cup_{i \le k_\varepsilon} B_\varepsilon(x_{i,\varepsilon}) \times B_\varepsilon(y_{i,\varepsilon}) \subset X^2_{\ll}$. In particular, for any $\varepsilon > 0$, $\mu_1(\cup_{i \le k_\varepsilon} B_\varepsilon(y_{i,\varepsilon})) = 1$. Then narrow convergence implies that

$$\liminf_{n \to \infty} \mu_{1,n}(A_\varepsilon) \ge \mu_1(A_\varepsilon) = 1, \qquad A_\varepsilon := \cup_{i \le k_\varepsilon} B_\varepsilon(y_{i,\varepsilon}). \tag{2.9}$$

Since we are interested in obtaining a sequence $\{\bar\mu_{1,n}\}$ absolutely continuous with respect to $\mu_{1,n}$, we can restrict and normalize $\mu_{1,n}$ to A_ε obtaining (thanks to (2.9)) a new sequence still converging narrowly to μ_1. Hence, without loss of generality, we will assume $\mu_{1,n}(A_\varepsilon) = 1$ for every $n \in \mathbb{N}$.

Step 2. Construction of the approximations.
By assumption $\mu_{1,n} = \sum_{i \le h_n} \alpha_{i,n} \delta_{y_{i,n}}$, with $\sum_{i \le h_n} \alpha_{i,n} = 1$, $\alpha_{i,n} \ge 0$ and from **Step 1** we have $\mu_{1,n}(A_\varepsilon) = 1$. Let $\{B_i\}_{i=1}^{k_\varepsilon}$ be a pairwise disjoint covering of $\operatorname{supp} \pi$, where each B_i is a Borel subset of $B_\varepsilon(x_{i,\varepsilon}) \times B_\varepsilon(y_{i,\varepsilon}) \subset X^2_{\ll}$. We define the following approximations:

$$\pi_{i,\varepsilon} := \pi \llcorner_{B_{i,\varepsilon}}, \quad \pi_{i,\varepsilon,n} := (P_1)_\sharp \pi_{i,\varepsilon} \otimes \left(\sum_{y_{i,n} \in P_2(B_{i,\varepsilon})} \alpha_{i,n} \delta_{y_{i,n}} \right) \Big/ \left(\sum_{y_{i,n} \in P_2(B_{i,\varepsilon})} \alpha_{i,n} \right), \tag{2.10}$$

and set $\pi_{\varepsilon,n} := \sum_{i \le k_\varepsilon} \pi_{i,\varepsilon,n}$. Observe that:

$$\begin{aligned}
(P_2)_\sharp \pi_{\varepsilon,n} &= \sum_{i \le k_\varepsilon} (P_2)_\sharp \pi_{i,\varepsilon,n} \\
&= \sum_{i \le k_\varepsilon} \left(\frac{\pi(B_{i,\varepsilon})}{\sum_{y_{i,n} \in P_2(B_{i,\varepsilon})} \alpha_{i,n}} \sum_{y_{i,n} \in P_2(B_{i,\varepsilon})} \alpha_{i,n} \delta_{y_{i,n}} \right) \\
&= \sum_{i \le k_\varepsilon} \left(\frac{\pi(B_{i,\varepsilon})}{\mu_{1,n}(P_2(B_{i,\varepsilon}))} \sum_{y_{i,n} \in P_2(B_{i,\varepsilon})} \alpha_{i,n} \delta_{y_{i,n}} \right) =: \bar\mu_{1,n}.
\end{aligned} \tag{2.11}$$

Moreover

$$(P_1)_\sharp \pi_{\varepsilon,n} = \sum_{i \le k_\varepsilon} (P_1)_\sharp \pi_{i,\varepsilon} = (P_1)_\sharp \pi = \mu_0. \tag{2.12}$$

It is clear from (2.11) and (2.12) that $\pi_{\varepsilon,n} \in \Pi_{\le}(\mu_0, \bar\mu_{1,n})$ and that $\bar\mu_{1,n} \ll \mu_{1,n}$. Notice indeed that $\pi_{i,\varepsilon,n}$ is concentrated over $P_1(B_{i,\varepsilon}) \times P_2(B_{i,\varepsilon})$. Being $B_{i,\varepsilon}$ a subset of the product of two balls inside $X^2_{\ll}$, causality of $\pi_{i,\varepsilon,n}$ and of $\pi_{\varepsilon,n}$ then follow.

Step 3. Convergence of the approximations.
We now estimate the difference between $\pi_{\varepsilon,n}$ and π by checking first the difference between $\pi_{i,\varepsilon,n}$ and $\pi_{i,\varepsilon}$ in duality with $(f_1, f_2) \in C_b(X)^2$: from (2.10) we deduce that

$$\int f_1(x)f_2(y)\pi_{i,\varepsilon,n}(dxdy) = \int_X f_1(x)\pi_{i,\varepsilon}(dxdy)\frac{\sum_{y_{i,n}\in P_2(B_{i,\varepsilon})} \alpha_{i,n} f_2(y_{i,n})}{\sum_{y_{i,n}\in P_2(B_{i,\varepsilon})} \alpha_{i,n}}.$$

Since $\operatorname{supp}\mu_1$ is compact, we have that $f_2|_{\operatorname{supp}\mu_1}$ is uniformly continuous. Denoting with $\omega_{f_2}(\varepsilon)$ the modulus of continuity of $f_2|_{\operatorname{supp}\mu_1}$ at distance ε, and recalling that $P_2(B_{i,\varepsilon}) \subset B_\varepsilon(y_{i,\varepsilon})$ we estimate

$$\left|\int f_1(x)f_2(y)\pi_{i,\varepsilon,n}(dxdy) - \int_X f_1(x)\pi_{i,\varepsilon}(dxdy)f_2(y_{i,\varepsilon})\right| \le \omega_{f_2}(\varepsilon)\int_X |f_1(x)|\pi_{i,\varepsilon}(dxdy).$$

In the same way:

$$\left|\int f_1(x)f_2(y)\pi_{i,\varepsilon}(dxdy) - \int f_1(x)\pi_{i,\varepsilon}(dxdy)f_2(y_{i,\varepsilon})\right| \le \omega_{f_2}(\varepsilon)\int_X |f_1(x)|\pi_{i,\varepsilon}(dxdy).$$

Hence, summing over all $i \le k_\varepsilon$, we obtain

$$\left|\int_{X\times X} f_1(x)f_2(y)\pi_{\varepsilon,n}(dxdy) - \int_{X\times X} f_1(x)f_2(y)\pi(dxdy)\right| \le 2\omega_{f_2}(\varepsilon)\|f_1\|_\infty.$$

Recall that every $\varphi \in C(\operatorname{supp}\mu_0 \times \operatorname{supp}\mu_1;\mathbb{R})$ can be approximated in C^0-norm by finite linear combinations of product functions $f_{i,1} \otimes f_{i,2}$ with $f_{i,1} \in C(\operatorname{supp}\mu_0;\mathbb{R})$, $f_{i,2} \in C(\operatorname{supp}\mu_1;\mathbb{R})$. Thus, letting $\varepsilon_n \downarrow 0$ be such that $\liminf_{n\to\infty}\mu_{1,n}(A_{\varepsilon_n}) = 1$ (the existence of the sequence (ε_n) is granted by (2.9)) and defining $\pi_n := \pi_{\varepsilon_n,n}$ for every $n \in \mathbb{N}$, we have

$$\pi_n \in \Pi_\le(\mu_0, \bar\mu_{1,n}), \quad \bar\mu_{1,n} \ll \mu_{1,n}, \quad \pi_n \to \pi \text{ narrowly.}$$

In particular the last convergence implies that $\bar\mu_{1,n}$ converges narrowly to μ_1, applying $(P_2)_\sharp$. Since for large n the construction gives $\operatorname{supp}\pi_n \subset (\operatorname{supp}\pi)^\varepsilon \Subset X^2$ (here $(\operatorname{supp}\pi)^\varepsilon$ denotes an ε-enlargement of $\operatorname{supp}\pi$ with respect to d), we have that (π_n) is τ^p-uniformly integrable and thus $F_n(\pi_n) \to F_\infty(\pi)$ by Lemma 2.12. The conclusion then follows from Lemma 2.15. □

Remark 2.17. The previous stability results can be seen as the Lorentzian counterpart of the metric fact that $W_p(\mu_n, \mu_\infty) \to 0$ if and only if $\mu_n \to \mu_\infty$ narrowly and (μ_n) has uniformly integrable p-moments. The remarkable differences in the Lorentzian setting are first that the cost is not continuous implying the ℓ_p-optimality does not pass to the limit automatically, and second that $\ell_p(\mu_n, \mu_\infty) \to 0$ *does not* imply $\mu_n \to \mu_\infty$ narrowly: it is easy to construct a counterexample (e.g. already in 1+1 dimensional Minkowski space-time) using that if $\operatorname{supp}\mu_1$ and $\operatorname{supp}\mu_2$ are contained in the light cone of a given common point then $\ell_p(\mu_1, \mu_2) = 0$.

2.4 Kantorovich duality

In the smooth Lorentzian setting, Kantorovich duality has been studied in [74, 46] in case $p = 1$ and in [58] for $p \in (0, 1)$, see also [7, 8] for relativistic cost functions in $\mathbb{R}^n$. In this section we study Kantoroch duality in the Lorentzian synthetic setting. The following definition, relaxing the notion of q-separated introduced by McCann [58, Definition 4.1] in the smooth Lorentzian setting will turn out to be very useful. Recall the definition (2.2) of the cost function ℓ^p.

Definition 2.18 (Timelike p-dualisable)**.** Let $(X, \mathsf{d}, \ll, \le, \tau)$ be a Lorentzian pre-length space and let $p \in (0, 1]$. We say that $(\mu, \nu) \in \mathcal{P}(X)^2$ is *timelike p-dualisable (by $\pi \in \Pi_\ll(\mu, \nu)$)* if

1. $\ell_p(\mu, \nu) \in (0, \infty)$;
2. $\pi \in \Pi_\le^{p\text{-opt}}(\mu, \nu)$ and $\pi(X^2_\ll) = 1$;

3. there exist measurable functions $a, b : X \to \mathbb{R}$, with $a \oplus b \in L^1(\mu \otimes \nu)$ such that $\ell^p \le a \oplus b$ on $\operatorname{supp}\mu \times \operatorname{supp}\nu$.

The motivation for considering timelike p-dualisable pairs of measures is twofold: firstly the p-optimal coupling $\pi(dxdy)$ matches events described by $\mu(dx)$ with events described by $\nu(dy)$ so that $x \ll y$, secondly Kantorovich duality holds (cf. [74, Proposition 2.7] in smooth Lorentzian setting and in case $p = 1$):

Proposition 2.19 (Weak Kantorovich duality I)**.** *Fix $p \in (0, 1]$. Let $(X, \mathsf{d}, \ll, \le, \tau)$ be a (resp. locally) causally closed globally hyperbolic Lorentz geodesic space. If $(\mu, \nu) \in \mathcal{P}(X)^2$ (resp. $\mathcal{P}_c(X)^2$) is timelike p-dualisable, then (weak) Kantorovich duality holds:*

$$\ell_p(\mu,\nu)^p = \inf \left\{ \int_X u\,\mu + \int_X v\,\nu \right\}, \tag{2.13}$$

where the inf *is taken over all measurable functions $u : \operatorname{supp}\mu \to \mathbb{R}\cup\{+\infty\}$ and $v : \operatorname{supp}\nu \to \mathbb{R}\cup\{+\infty\}$ with $u \oplus v \ge \ell^p$ on $\operatorname{supp}\mu \times \operatorname{supp}\nu$ and $u \oplus v \in L^1(\mu \otimes \nu)$. Furthermore, the value of the right hand side does not change if one restricts the* inf *to bounded and continuous functions.*

Proof. The claim follows from Remark 2.2 combined with [77, Theorem 1.3]. □

Remark 2.20. The notion of timelike p-dualisabily is not empty, indeed for instance if X is globally hyperbolic and $\mu, \nu \in \mathcal{P}_c(X)$ admit an optimal coupling $\pi \in \Pi_{\le}^{p\text{-opt}}(\mu, \nu)$ concentrated on $X^2_{\ll}$, then all the three conditions are satisfied. The only one requiring a comment is the last one: since by global hyperbolicity $\tau : X^2 \to \mathbb{R}$ is continuous then it is bounded on the compact set $\operatorname{supp}\mu \times \operatorname{supp}\nu \Subset X^2$ and we can choose a and b to be constant functions.

Under stronger assumptions on the causality relation on (μ, ν), (weak) Kantorovich duality holds for general Lorentzian pre-length spaces:

Proposition 2.21 (Weak Kantorovich duality II)**.** *Fix $p \in (0, 1]$. Let $(X, \mathsf{d}, \ll, \le, \tau)$ be a Lorentzian pre-length space and let $(\mu, \nu) \in \mathcal{P}(X)^2$ such that $(\mu\otimes\nu)(X^2_{\le}) = 1$. Assume that there exist measurable functions $a, b : X \to \mathbb{R}$, with $a \oplus b \in L^1(\mu \otimes \nu)$ such that $\tau^p \le a \oplus b$, $\mu \otimes \nu$-a.e..*
Then (weak) Kantorovich duality (2.13) *holds.*

Proof. The result follows from [5, Theorem 1] where (weak) Kantorovich duality (for the minimum optimal transport problem) is proved to hold for general $\mu \otimes \nu$-a.e. finite Borel costs with values in $[0, \infty]$, observing that the cost $(a \oplus b) - \ell^p$ takes values in $[0, \infty]$, $\mu \otimes \nu$-a.e.. □

We next discuss the validity of the *strong* Kantorovich duality, i.e. the existence of optimal functions (called Kantorovich potentials) achieving the infimum in the right hand side of (2.13). To that aim the next definition is key.

Definition 2.22 (ℓ^p-concave functions, ℓ^p-transform and ℓ^p-subdifferential)**.** Fix $p \in (0, 1]$ and let $U, V \subset X$. A measurable function $\varphi : U \to \mathbb{R}$ is *ℓ^p-concave relatively to* (U, V) if there exists a function $\psi : V \to \mathbb{R}$ such that

$$\varphi(x) = \inf_{y \in V} \psi(y) - \ell^p(x, y), \quad \forall x \in U.$$

The function

$$\varphi^{(\ell^p)} : V \to \mathbb{R} \cup \{-\infty\}, \quad \varphi^{(\ell^p)}(y) := \sup_{x \in U} \varphi(x) + \ell^p(x, y) \tag{2.14}$$

is called *ℓ^p-transform of* φ. The *ℓ^p-subdifferential* $\partial_{\ell^p}\varphi \subset (U \times V) \cap X^2_{\le}$ is defined by

$$\partial_{\ell^p}\varphi := \{(x, y) \in (U \times V) \cap X^2_{\le} \,:\, \varphi^{(\ell^p)}(y) - \varphi(x) = \ell^p(x, y)\}.$$

Replacing ℓ^p with τ^p in all the definitions above, one obtains the notions of τ^p-concave functions, τ^p-transform and τ^p-subdifferential.

Let us explicitely observe that, by the very definition (2.14) of ℓ^p-transform it holds

$$\varphi^{(\ell^p)}(y) - \varphi(x) \geq \ell^p(x,y), \quad \forall (x,y) \in U \times V, \tag{2.15}$$

and analogous inequality replacing ℓ^p with τ^p.

Definition 2.23 (Strong Kantorovich duality)**.** Fix $p \in (0,1]$. We say that $(\mu,\nu) \in \mathcal{P}(X)^2$ satisfies *strong ℓ^p-Kantorovich duality* if

1. $\ell_p(\mu,\nu) \in (0,\infty)$;
2. there exists Borel subsets $A_1 \subset \operatorname{supp}\mu, A_2 \subset \operatorname{supp}\nu$ with $\mu(A_1) = \nu(A_2) = 1$, and there exists $\varphi : A_1 \to \mathbb{R}$ which is ℓ^p-concave relatively to (A_1, A_2) and satisfying

$$\ell_p(\mu,\nu)^p = \int_X \varphi^{(\ell^p)}(y)\,\nu(dy) - \int_X \varphi(x)\,\mu(dx).$$

Replacing ℓ^p with τ^p in condition 2 above, one obtains the notion of *strong τ^p-Kantorovich duality.*

Remark 2.24. Using (2.15), it is immediate to check that if $(\mu,\nu) \in \mathcal{P}(X)^2$ satisfies strong ℓ^p-Kantorovich duality then the following holds. A coupling $\pi \in \Pi_{\leq}(\mu,\nu)$ is ℓ_p-optimal if and only if

$$\varphi^{(\ell^p)}(y) - \varphi(x) = \ell^p(x,y) = \tau(x,y)^p, \quad \text{for } \pi\text{-a.e. } (x,y),$$

i.e. if and only if $\pi(\partial_{\ell^p}\varphi) = 1$. Analogously, if $(\mu,\nu) \in \mathcal{P}(X)^2$ satisfies strong τ^p-Kantorovich duality then $\pi \in \Pi_{\leq}(\mu,\nu)$ is ℓ_p-optimal if and only if $\pi(\partial_{\tau^p}\varphi) = 1$.

Remark 2.25 (The case $\operatorname{supp}\mu \times \operatorname{supp}\nu \subset X^2_{\leq}$)**.** In case $\operatorname{supp}\mu \times \operatorname{supp}\nu \subset X^2_{\leq}$, it is readily seen from the definitions above that $\varphi : \operatorname{supp}\mu \to \mathbb{R}$ is ℓ^p-concave (relatively to $\operatorname{supp}\mu \times \operatorname{supp}\nu$) if and only if it is τ^p-concave, moreover $\varphi^{(\ell^p)} = \varphi^{(\tau^p)}$, and $\partial_{\ell^p}\varphi = \partial_{\tau^p}\varphi$. It follows that also the notions of strong ℓ^p-Kantorovich duality and strong τ^p-Kantorovich duality coincide in this case.

We next relate τ^p-cyclical monotonicity with strong τ^p-Kantorovich duality.

Theorem 2.26 (τ^p-cyclical monotonicity $\Rightarrow$ strong τ^p-Kantorovich duality)**.** *Fix $p \in (0,1]$. Let $(X, \mathsf{d}, \ll, \leq, \tau)$ be a Lorentzian pre-length space and let $\mu,\nu \in \mathcal{P}(X)$ be with $\ell_p(\mu,\nu) \in (0,\infty)$. and that $\ell_p(\mu,\nu) \in (0,\infty)$. For any $\pi \in \Pi_{\leq}(\mu,\nu)$ the following holds.*

If π is τ^p-cyclically monotone then π is ℓ_p-optimal. Moreover, (μ,ν) satisfies strong τ^p-Kantorovich duality and $\pi(\partial_{\tau^p}\varphi) = 1$.

Proof. The proof consists in constructing a τ^p-concave function φ such that $\pi(\partial_{\tau^p}\varphi) = 1$.
Let $\Gamma \subset X^2_{\leq}$ be a Borel τ^p-cyclically monotone set such that $\pi(\Gamma) = 1$, and $\tau|_\Gamma$ is real valued. It follows that τ^p is real valued on $P_1(\Gamma) \times P_2(\Gamma)$. Notice that $P_i(\Gamma) \subset X$ is a Suslin set, for $i = 1,2$.

Step 1. Definition of $\varphi_{(x_0,y_0)} = \varphi$, and proof that $\varphi(x_0) = 0$.
Fix $(x_0,y_0) \in \Gamma$. Define $\varphi_{(x_0,y_0)} = \varphi : P_1(\Gamma) \to \mathbb{R} \cup \{\pm\infty\}$ by

$$\varphi_{(x_0,y_0)}(x) = \varphi(x) := \inf \left\{ \sum_{i=0}^{k} \left[\tau(x'_i, y'_i)^p - \tau(x'_{i+1}, y'_i)^p \right] \right\} \tag{2.16}$$

where the inf is taken over all $k \in \mathbb{N}$ and all “chains”

$$\{(x'_i, y'_i)\}_{0 \leq i \leq k+1} \subset \Gamma \text{ with } x'_{k+1} = x, \ (x'_0, y'_0) = (x_0, y_0).$$

Let us stress that y'_{k+1} does not enter in the expression of the right hand side of (2.16) (this will be useful in step 3). It is readily seen that

$$\varphi(x_0) = 0. \tag{2.17}$$

Indeed on the one hand $\varphi(x_0) \leq \tau(x_0,y_0)^p - \tau(x_0,y_0)^p = 0$. On the other hand, since by assumption Γ is τ^p-cyclically monotone, then the right hand side of (2.16) is non-negative. Thus (2.17) is proved.

Step 2. We show that φ is real-valued on $P_1(\Gamma)$ and measurable.
Fix $x \in P_1(\Gamma)$. The very definition (2.16) of $\varphi = \varphi_{(x_0,y_0)}$ gives

$$\varphi(x) + [\tau(x,y)^p - \tau(x_0,y)^p] \geq \varphi(x_0) \overset{(2.17)}{=} 0,$$

where y is such that $(x,y) \in \Gamma$. In particular, $\varphi(x) > -\infty$. Analogously,

$$\varphi(x_0) + [\tau(x_0,y_0)^p - \tau(x,y_0)^p] \geq \varphi(x)$$

and thus $\varphi(x) < +\infty$. Notice that, under the stronger assumption that X is a globally hyperbolic Lorentzian geodesic space (so that τ is continuous), then φ would be upper semi-continuous (as infimum of a family of continuous functions) and thus measurable.
We now prove that φ is measurable also in the general setting. Since τ^p is lower semicontinuous, there exists compact subset $\Gamma_j \Subset \Gamma$ such that $\Gamma_j \subset \Gamma_{j+1}$, $\Gamma = \bigcup_{j\in\mathbb{N}} \Gamma_j$ and $\tau^p|_{\Gamma_j}$ is continuous and real valued. We choose continuous functions c_l such that $c_l \uparrow \tau^p$. Notice that $c_l|_{\Gamma_j} \to \tau^p|_{\Gamma_j}$ uniformly as $l \to \infty$, for every $j \in \mathbb{N}$, by Dini's Theorem. Define the auxiliary functions

$$\varphi_{k,j,l}(x) := \inf\left\{\sum_{i=0}^{k}\left[c_l(x'_i,y'_i) - c_l(x'_{i+1},y'_i)\right]\right\}, \quad \varphi_{k,j}(x) := \inf\left\{\sum_{i=0}^{k}\left[\tau(x'_i,y'_i)^p - \tau(x'_{i+1},y'_i)^p\right]\right\},$$

where the infimum is taken over all "chains"

$$\{(x'_i,y'_i)\}_{0\leq i\leq k+1} \subset \Gamma_j \text{ with } x'_{k+1} = x,\ (x'_0,y'_0) = (x_0,y_0) \in \Gamma_1.$$

The uniform convergence $c_l|_{\Gamma_j} \to \tau^p|_{\Gamma_j}$ ensures that $\varphi_{k,j,l}|_{\Gamma_j} \to \varphi_{k,j}|_{\Gamma_j}$ pointwise. The monotonicity of the quantities in j and k gives

$$\varphi(x) = \lim_{k\to\infty}\lim_{j\to\infty}\varphi_{k,j}(x) = \lim_{k\to\infty}\lim_{j\to\infty}\lim_{l\to\infty}\varphi_{k,j,l}(x), \quad \forall x \in P_1(\Gamma).$$

As each $\varphi_{k,j,l}$ is upper semi continuous, we get that φ is measurable.

Step 3. We show that φ is τ^p-concave relatively to $(P_1(\Gamma), P_2(\Gamma))$.
Define $\psi_{(x_0,y_0)} = \psi : P_2(\Gamma) \to \mathbb{R} \cup \{-\infty\}$ by

$$\psi_{(x_0,y_0)}(y) = \psi(y) := \inf\left\{\sum_{i=0}^{k}\left[\tau(x'_i,y'_i)^p - \tau(x'_{i+1},y'_i)^p\right] + \tau(x'_{k+1},y)^p\right\}, \tag{2.18}$$

where the inf is taken over all $k \in \mathbb{N}$ and all chains

$$\{(x'_i,y'_i)\}_{0\leq i\leq k+1} \subset \Gamma \text{ with } y'_{k+1} = y,\ (x'_0,y'_0) = (x_0,y_0).$$

Notice that, for every $x \in P_1(\Gamma)$ there exists $y \in P_2(\Gamma)$ such that $(x,y) \in \Gamma$; thus, any chain in the definition (2.16) of $\varphi(x)$ can be concatenated with (x,y), giving an admissible chain for the definition (2.18) of $\psi(y)$. It follows that $\varphi(x) + \tau(x,y)^p \geq \psi(y)$ and thus

$$\varphi(x) \geq \inf_{y\in P_2(\Gamma)} \psi(y) - \tau(x,y)^p, \quad \forall x \in P_1(\Gamma).$$

Conversely, it is readily seen from the definition (2.16) (resp. (2.18)) of $\varphi(x)$ (resp. $\psi(y)$) that (recall that y'_{k+1} does not play any role in (2.16))

$$\varphi(x) \leq \psi(y) - \tau(x,y)^p, \quad \forall (x,y) \in P_1(\Gamma) \times P_2(\Gamma).$$

We conclude that

$$\varphi(x) = \inf_{y\in P_2(\Gamma)} \psi(y) - \tau(x,y)^p, \quad \forall x \in P_1(\Gamma).$$

It follows that ψ is real valued on $P_2(\Gamma)$ and φ is τ^p-concave relatively to $(P_1(\Gamma), P_2(\Gamma))$.

Step 4. We show that $\Gamma \subset \partial_{\tau^p}\varphi$.
Let $(\bar{x}, \bar{y}) \in \Gamma$. From the definition (2.16) of $\varphi(x)$ we have

$$\varphi(\bar{x}) + [\tau(\bar{x}, \bar{y})^p - \tau(x, \bar{y})^p] \geq \varphi(x), \quad \forall x \in P_1(\Gamma),$$

which can be rewritten as

$$\varphi(\bar{x}) + \tau(\bar{x}, \bar{y})^p \geq \sup_{x \in P_1(\Gamma)} \varphi(x) + \tau(x, \bar{y})^p = \varphi^{(\tau^p)}(\bar{y}).$$

Since the inequality $\varphi(\bar{x}) + \tau(\bar{x}, \bar{y})^p \leq \varphi^{(\tau^p)}(\bar{y})$ is trivial from the definition of τ^p-transform, we conclude that equality holds and thus $(\bar{x}, \bar{y}) \in \partial_{\tau^p}\varphi$.

Step 5. Conclusion: we claim that

$$\ell_p(\mu, \nu)^p = \int_X \varphi^{(\tau^p)}(y)\, \nu(dy) - \int_X \varphi(x)\, \mu(dx) = \int_{X^2} \tau(x, y)^p\, \pi(dxdy). \tag{2.19}$$

From Step 4, we know that

$$\varphi^{(\tau^p)}(y) - \varphi(x) = \tau(x, y)^p, \quad \text{for all } (x, y) \in \Gamma,$$

which integrated with respect to π gives the second identity of (2.19).
On the other hand, integrating the inequality

$$\varphi^{(\tau^p)}(y) - \varphi(x) \geq \tau(x, y)^p, \qquad \mu \otimes \nu\text{-a.e. } (x, y),$$

with respect to any $\pi' \in \Pi_{\leq}(\mu, \nu)$ gives that

$$\ell_p(\mu, \nu)^p = \sup_{\pi' \in \Pi_{\leq}(\mu, \nu)} \int_{X^2} \tau(x, y)^p\, \pi'(dxdy) \leq \int_X \varphi^{(\tau^p)}(y)\, \nu(dy) - \int_X \varphi(x)\, \mu(dx).$$

The claimed (2.19) follows. □

Definition 2.27 (Strongly timelike p-dualisability)**.** A pair $(\mu, \nu) \in (\mathcal{P}(X))^2$ is said to be *strongly timelike p-dualisable* if

1. (μ, ν) is timelike p-dualisable;
2. there exists a measurable ℓ^p-cyclically monotone set $\Gamma \subset X^2_{\ll} \cap (\operatorname{supp}\mu \times \operatorname{supp}\nu)$ such that a coupling $\pi \in \Pi_{\leq}(\mu, \nu)$ is ℓ_p-optimal if and only if π is concentrated on Γ, i.e. $\pi(\Gamma) = 1$.

Remark 2.28. Let (μ, ν) be timelike p-dualisable and satisfying strong ℓ^p-Kantorovich duality (resp. strong τ^p-Kantorovich duality). It follows from Remark 2.24 that if $\Gamma := \partial_{\ell^p}\varphi \subset X^2_{\ll}$ (resp. $\Gamma := \partial_{\tau^p}\varphi \subset X^2_{\ll}$) then (μ, ν) is strongly timelike p-dualisable.

In the next two corollaries we show that the notion of strongly timelike p-dualisability is non-empty:

Corollary 2.29. *Fix $p \in (0, 1]$. Let $(X, \mathsf{d}, \ll, \leq, \tau)$ be a causally closed (resp. locally causally closed) globally hyperbolic Lorentzian geodesic space and assume that $\mu, \nu \in \mathcal{P}(X)$ (resp. $\mathcal{P}_c(X)$) satisfy:*

1. *there exist measurable functions $a, b : X \to \mathbb{R}$ with $a \oplus b \in L^1(\mu \otimes \nu)$ such that $\tau^p \leq a \oplus b$ on $\operatorname{supp}\mu \times \operatorname{supp}\nu$;*
2. $\operatorname{supp}\mu \times \operatorname{supp}\nu \subset X^2_{\ll}$.

Then (μ, ν) satisfies strong τ^p-Kantorovich duality and is strongly timelike p-dualisable.

Proof. The fact that there exists $\pi \in \Pi_{\leq}^{p\text{-opt}}(\mu,\nu)$ follows from Proposition 2.3; morover, since $\operatorname{supp}\pi \subset \operatorname{supp}\mu \times \operatorname{supp}\nu \subset X^2_{\ll}$, we infer that (μ,ν) is timelike p-dualisable.
From part 1 of Proposition 2.8 we have π is ℓ^p-cyclically monotone and thus, from Remark 2.7, also τ^p-cyclically monotone since $\operatorname{supp}\mu \times \operatorname{supp}\nu \subset X^2_{\leq}$.
Using now Theorem 2.26 we infer that (μ,ν) satisfies strong τ^p-Kantorovich duality. Setting $\Gamma := \partial_{\tau^p}\varphi \subset \operatorname{supp}\mu \times \operatorname{supp}\nu$, it is a direct consequence of the assumptions that $\Gamma \subset X^2_{\ll}$ and thus Remark 2.24 yields that condition 2 of Definition 2.27 is satisfied. □

In the next corollary we show that, in case ν is a Dirac measure, the strongly timelike p-dualisability is equivalent to the timelike p-dualisability.

Corollary 2.30. *Let $(X,\mathsf{d},\ll,\leq,\tau)$ be a Lorentzian pre-length space and let $p \in (0,1]$. Fix $\bar{x} \in X$ and let $\nu := \delta_{\bar{x}}$. Assume that $\mu \in \mathcal{P}(X)$ satisfies:*

$$\tau(\cdot,\bar{x})^p \in L^1(X,\mu) \quad \text{and} \quad \tau(\cdot,\bar{x}) > 0 \ \mu\text{-a.e.} \ . \tag{2.20}$$

Then (μ,ν) is strongly timelike p-dualisable. In other terms, in case ν is a Dirac measure, the strongly timelike p-dualisability is equivalent to the timelike p-dualisability.

Proof. Let $\pi := \mu \otimes \delta_{\bar{x}}$ and choose $b \equiv 0, a(x) := \tau(x,\bar{x})^p$. Noticing that $\Pi(\mu,\delta_{\bar{x}}) = \{\pi\}$ we get that (2.20) implies: $\ell_p(\mu,\delta_{\bar{x}}) \in (0,\infty)$, π is the unique ℓ_p-optimal coupling for $(\mu,\delta_{\bar{x}})$ and $\pi(X^2_{\ll}) = 1$. It follows that (μ,ν) is strongly timelike p-dualisable. □

2.5 ℓ_p-geodesics of probability measures

Continuing the analogy with the metric setting, the next definition is natural (cf. [58, Definition 1.1]).

Definition 2.31 (Geodesics of probability measures in a Lorentz pre-length space)**.** Let $(X,\mathsf{d},\ll,\leq,\tau)$ be a Lorentzian pre-length space and let $p \in (0,1]$. We say that $(\mu_s)_{s\in[0,1]} \subset \mathcal{P}(X)$ is an ℓ_p-geodesic if and only if

$$\ell_p(\mu_s,\mu_t) = (t-s)\ell_p(\mu_0,\mu_1) \in (0,\infty), \quad \text{for all } 0 \leq s < t \leq 1. \tag{2.21}$$

Note that, with this convention, ℓ_p-geodesics are implicitly future-directed and timelike.
In the next proposition we collect some useful properties of ℓ_p-geodesics. Before stating it, we introduce the evaluation map

$$\mathrm{e}_t : C([0,1],X) \to X, \quad \gamma \mapsto \mathrm{e}_t(\gamma) := \gamma_t, \quad \forall t \in [0,1], \tag{2.22}$$

and the stretching/restriction operator $\mathrm{restr}_{s_1}^{s_2} : C([0,1],X) \to C([0,1],X)$

$$(\mathrm{restr}_{s_1}^{s_2}\gamma)_t := \gamma_{(1-t)s_1+ts_2}, \quad \forall s_1,s_2 \in [0,1], s_1 < s_2, \forall t \in [0,1]. \tag{2.23}$$

Proposition 2.32. *Fix $p \in (0,1)$ and let $(X,\mathsf{d},\ll,\leq,\tau)$ be a $\mathcal{K}$-globally hyperbolic, Lorentzian geodesic space. Let $\mu_0,\mu_1 \in \mathcal{P}_c(X)$ such that there exists $\pi \in \Pi_{\leq}^{p\text{-}opt}(\mu_0,\mu_1)$ with $\operatorname{supp}\pi \Subset \{\tau > 0\}$ (in particular, if $\operatorname{supp}(\mu_0 \otimes \mu_1) \Subset \{\tau > 0\}$). Then*

1. *There always exists an ℓ_p-geodesic from μ_0 to μ_1.*

2. *For every ℓ_p-geodesic $(\mu_t)_{t\in[0,1]}$ from μ_0 to μ_1 there exists a probability measure $\eta \in \mathcal{P}(C([0,1],X)$ such that $(\mathrm{e}_t)_\sharp\eta = \mu_t$ for every $t \in [0,1]$ and η-a.e. γ is a maximal causal curve from $\gamma_0 \in \operatorname{supp}\mu_0$ to $\gamma_1 \in \operatorname{supp} = \mu_1$, $\tau(\gamma_0,\gamma_1) > 0$. Such an η is called* ℓ_p-dynamical optimal plan *and the set of dynamical optimal plans from μ_0 to μ_1 is denoted by* $\mathrm{OptGeo}_{\ell_p}(\mu_0,\mu_1)$.

3. *If $\eta \in \mathrm{OptGeo}_{\ell_p}(\mu_0,\mu_1)$ then $\eta^{s_1,s_2} := (\mathrm{restr}_{s_1}^{s_2})_\sharp\eta \in \mathrm{OptGeo}_{\ell_p}((\mathrm{e}_{s_1})_\sharp\eta,(\mathrm{e}_{s_2})_\sharp\eta)$, for all $s_1 < s_2$, $s_1,s^2 \in [0,1]$.*

4. *Let $\eta \in \mathrm{OptGeo}_{\ell_p}(\mu_0,\mu_1)$ and let $\tilde{\eta}$ be a measure on $C([0,1],X)$ such that $\tilde{\eta} \leq \eta^{s_1,s_2}$ and $\tilde{\eta}(C([0,1],X)) > 0$. Then $\eta' := \frac{1}{\tilde{\eta}(C([0,1],X))}\,\tilde{\eta}$ is an ℓ_p-dynamical optimal plan.*

5. *If X is timelike non-branching then*

   (a) *If $(s_1, s_2) \neq (0,1)$ then η' as in 4. is the unique element of $\mathrm{OptGeo}_{\ell_p}((\mathrm{e}_0)_\sharp\eta', (\mathrm{e}_1)_\sharp\eta')$, and $(\mu'_t := (\mathrm{e}_t)_\sharp\eta')_{t\in[0,1]}$ is the unique ℓ_p-geodesic joining its endpoints.*

   (b) *If $\gamma^1, \gamma^2 \in \operatorname{supp}\eta$ cross at some intermediate time $t_0 \in (0,1)$, i.e. there exists $t_0 \in (0,1)$ such that $\gamma^1_{t_0} = \gamma^2_{t_0}$, then $\gamma^1_t = \gamma^2_t$ for all $t \in [0,1]$.*

6. *Assume X is timelike non-branching and let $\eta \in \mathrm{OptGeo}_{\ell_p}(\mu_0, \mu_1)$. Assume that η can be written as $\eta = \lambda_1\eta^1 + \lambda_2\eta^2$, for some $\eta^i \in \mathcal{P}(C([0,1], X))$, $\lambda_i \in (0,1)$ for $i = 1,2$, $\lambda_1 + \lambda_2 = 1$, with $\operatorname{supp}\eta^1 \cap \operatorname{supp}\eta^2 = \emptyset$. Then η^1, η^2 are ℓ_p-optimal dynamical plans and they satisfy $(\mathrm{e}_t)_\sharp\eta^1 \perp (\mathrm{e}_t)_\sharp\eta^2$ for all $t \in (0,1)$.*

7. *Every ℓ_p-geodesic $(\mu_t = (\mathrm{e}_t)_\sharp\eta)_{t\in[0,1]}$ from μ_0 to μ_1 is an absolutely continuous curve in the W_1-Kantorovich Wasserstein space $(\mathcal{P}(X), W_1)$ w.r.t. d, with length*

$$\mathrm{L}_{W_1}\big((\mu_t)_{t\in[0,1]}\big) \leq \int \mathrm{L}_{\mathsf{d}}(\gamma)\,\eta(d\gamma) \leq \bar{C} < \infty \tag{2.24}$$

   where $\bar{C} > 0$ depends only on the compact subset $J^+(\operatorname{supp}\mu_0) \cap J^-(\operatorname{supp}\mu_1) \Subset X$.

Proof. First of all notice that by $\mathcal{K}$-global hyperbolicity

$$\bigcup_{t\in[0,1]} \operatorname{supp}\mu_t \subset J^+(\operatorname{supp}\mu_0) \cap J^-(\operatorname{supp}\mu_1) \Subset X,$$

with $J^+(\operatorname{supp}\mu_0) \cap J^-(\operatorname{supp}\mu_1)$ compact subset.

1. and 2. (resp. 3. 4. and 5.) follow by applying [78, Theorem 7.21] (resp. [78, Theorem 7.30]) with $\mathcal{X} = J^+(\operatorname{supp}\mu_0) \cap J^-(\operatorname{supp}\mu_1) \Subset X$.

6. The optimality of η^1 and η^2 follows directly from 4. Call $\mu_t := (\mathrm{e}_t)_\sharp\eta$ and $\mu^i_t := (\mathrm{e}_t)_\sharp\eta^i$, for $i = 1,2$, $t \in [0,1]$. Assume by contradiction that for some $t_0 \in (0,1)$ there exists

$$A \Subset X \text{ compact subset s. t. } A \subset \operatorname{supp}\mu^1_{t_0} \cap \operatorname{supp}\mu^2_{t_0},\ \mu^1_{t_0}(A) > 0,\ \mu^1_{t_0}\llcorner A \ll \mu^2_{t_0}. \tag{2.25}$$

From 4. we know that

$$\bar{\eta} := \frac{1}{\mu_{t_0}(A)}\eta\llcorner \mathrm{e}_{t_0}^{-1}(A), \quad \bar{\eta}^i := \frac{1}{\mu^i_{t_0}(A)}\eta^i\llcorner \mathrm{e}_{t_0}^{-1}(A) \quad \text{for } i = 1,2,$$

are all ℓ_p-optimal dynamical plans. Clearly $\operatorname{supp}\bar{\eta}^1 \cup \operatorname{supp}\bar{\eta}^2 \subset \operatorname{supp}\bar{\eta}$ and thus 5(b) implies that

$$\gamma^1_{t_0} = \gamma^2_{t_0} \text{ for some } \gamma^i \subset \operatorname{supp}\bar{\eta}^i \implies \gamma^1 = \gamma^2. \tag{2.26}$$

The combination of (2.25) with (2.26) gives that $\operatorname{supp}\bar{\eta}^1 = \operatorname{supp}\bar{\eta}^2$. Since $\operatorname{supp}\bar{\eta}^i \subset \operatorname{supp}\eta^i$, $i = 1,2$, we arrive to a contradiction with the assumption $\operatorname{supp}\eta^1 \cap \operatorname{supp}\eta^2 = \emptyset$.

7. From the non-totally imprisoning property, it follows that

$$\begin{aligned} &\sup\{\mathrm{L}_{\mathsf{d}}(\gamma)\,:\,\gamma \in \operatorname{supp}\eta\} \\ &\quad\leq \sup\big\{\mathrm{L}_{\mathsf{d}}(\gamma)\,:\,\gamma(I) \subset J^+(\operatorname{supp}\mu_0) \cap J^-(\operatorname{supp}\mu_1),\ \gamma : I \to X \text{ casual }\big\} =: \bar{C} < \infty. \end{aligned} \tag{2.27}$$

In particular $\int \mathrm{L}_{\mathsf{d}}(\gamma)\,\eta(d\gamma) \leq \bar{C} < \infty$. The claim (2.24) follows from [53, Theorem 4]. □

Remark 2.33. If for every $x \in \operatorname{supp}\mu$ and every $y \in \operatorname{supp}\nu$ it holds $I^-(x) \neq \emptyset$, $I^+(y) \neq \emptyset$, then the assumption of $\mathcal{K}$-global hyperbolicity in Proposition 2.32 can be relaxed to global hyperbolicity (thanks to Lemma 1.5).

3 Synthetic Ricci curvature lower bounds

3.1 Timelike Curvature Dimension condition

The goal of this section to give a synthetic formulation of the strong energy condition (and more generally a synthetic formulation of $\mathrm{Ric}_g \geq -Kg$ in the timelike directions and $\dim \leq N$) for a measured pre-length space $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$. Let us recall the characterization of Ricci curvature bounded below and dimension bounded above in the smooth Lorentzian globally hyperbolic setting proved by McCann [58, Corollary 6.6, Corollary 7.5] (see also [61, Corollary 4.4]).

Theorem 3.1. *Let (M^n, g) be a globally hyperbolic spacetime and $0 < p < 1$. Then the following are equivalent:*

1. $\mathrm{Ric}_g(v, v) \geq -Kg(v, v)$, *for every timelike $v \in TM$.*
2. *For any couple $(\mu_0, \mu_1) \in (\mathrm{Dom}(\mathrm{Ent}(\cdot|\mathfrak{m})))^2$ which is timelike p-dualisable (in the sense of Definition 2.18) there exists a (unique) ℓ_p-geodesic $(\mu_t)_{t\in[0,1]}$ joining them such that the function $[0, 1] \ni t \mapsto e(t) := \mathrm{Ent}(\mu_t|\mathrm{Vol}_g)$ is semi-convex (and thus in particular it is locally Lipschitz in $(0, 1)$) and it satisfies:*
$$e''(t) - \frac{1}{n} e'(t)^2 \geq K \int_{M\times M} \tau(x, y)^2 \, \pi(dxdy), \tag{3.1}$$
in the distributional sense on $[0, 1]$.
3. *For any couple $(\mu_0, \mu_1) \in (\mathrm{Dom}(\mathrm{Ent}(\cdot|\mathfrak{m}))\cap\mathcal{P}_c(X))^2$ which is strongly timelike p-dualisable (in the sense of Definition 2.27) there exists a (unique) ℓ_p-geodesic $(\mu_t)_{t\in[0,1]}$ joining them and satisfying* (3.1).

Proof. The equivalence of 1 and 2 was proved in McCann [58, Corollary 6.6, Corollary 7.5] (see also [61, Corollary 4.4]). Trivially $2 \Longrightarrow 3$. The implication $3 \Longrightarrow 1$ can be proved along the lines of [61, Corollary 4.4] using Corollary 2.29. □

The following definition is thus natural.

Definition 3.2 ($\mathsf{TCD}^e_p(K, N)$ and $\mathsf{wTCD}^e_p(K, N)$ conditions)**.** Fix $p \in (0, 1)$, $K \in \mathbb{R}$, $N \in (0, \infty)$. We say that a measured pre-length space $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ satisfies $\mathsf{TCD}^e_p(K, N)$ (resp. $\mathsf{wTCD}^e_p(K, N)$) if the following holds. For any couple $(\mu_0, \mu_1) \in (\mathrm{Dom}(\mathrm{Ent}(\cdot|\mathfrak{m})))^2$ which is timelike p-dualisable (resp. $(\mu_0, \mu_1) \in [\mathrm{Dom}(\mathrm{Ent}(\cdot|\mathfrak{m}))\cap\mathcal{P}_c(X)]^2$ which is strongly timelike p-dualisable) by some $\pi \in \Pi^{p\text{-opt}}_{\ll}(\mu_0, \mu_1)$, there exists an ℓ_p-geodesic $(\mu_t)_{t\in[0,1]}$ such that the function $[0, 1] \ni t \mapsto e(t) := \mathrm{Ent}(\mu_t|\mathrm{Vol}_g)$ is semi-convex (and thus in particular it is locally Lipschitz in $(0, 1)$) and it satisfies
$$e''(t) - \frac{1}{N} e'(t)^2 \geq K \int_{X\times X} \tau(x, y)^2 \, \pi(dxdy), \tag{3.2}$$
in the distributional sense on $[0, 1]$.

Definition 3.2 corresponds to a differential/infinitesimal formulation of the $\mathsf{TCD}^e_p(K, N)$ condition. In order to have also an integral/global formulation it is convenient to introduce the following entropy (cf. [28])
$$U_N(\mu|\mathfrak{m}) := \exp\left(-\frac{\mathrm{Ent}(\mu|\mathfrak{m})}{N}\right). \tag{3.3}$$
It is clear that (1.15) implies the upper-semicontinuity of U_N under narrow convergence:
$$\mu_n \to \mu_\infty \text{ narrowly and } \mathfrak{m}\Big(\bigcup_{n\in\mathbb{N}} \operatorname{supp} \mu_n\Big) < \infty \quad \Longrightarrow \quad \limsup_{n\to\infty} U_N(\mu_n|\mathfrak{m}) \leq U_N(\mu_\infty|\mathfrak{m}). \tag{3.4}$$

It is straightforward to check that $[0,1] \ni t \mapsto e(t)$ is semi-convex and satisfies (3.1) if and only if $[0,1] \ni t \mapsto u_N(t) := \exp(-e(t)/N)$ is semi-convex and satisfies

$$u_N'' \le -\frac{K}{N} \|\tau\|^2_{L^2(\pi)} \, u_N. \tag{3.5}$$

Set

$$\mathfrak{s}_\kappa(\vartheta) := \begin{cases} \frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa}\vartheta), & \kappa > 0 \\ \vartheta, & \kappa = 0 \\ \frac{1}{\sqrt{-\kappa}} \sinh(\sqrt{-\kappa}\vartheta), & \kappa < 0 \end{cases}, \qquad \mathfrak{c}_\kappa(\vartheta) := \begin{cases} \cos(\sqrt{\kappa}\vartheta), & \kappa \ge 0 \\ \cosh(\sqrt{-\kappa}\vartheta) & \kappa < 0 \end{cases}, \tag{3.6}$$

and

$$\sigma_\kappa^{(t)}(\vartheta) := \begin{cases} \frac{\mathfrak{s}_\kappa(t\vartheta)}{\mathfrak{s}_\kappa(\vartheta)}, & \kappa\vartheta^2 \neq 0 \text{ and } \kappa\vartheta^2 < \pi^2 \\ t, & \kappa\vartheta^2 = 0 \\ +\infty & \kappa\vartheta^2 \ge \pi^2 \end{cases}. \tag{3.7}$$

Note that the function $\kappa \mapsto \sigma_\kappa^{(t)}(\vartheta)$ is non-decreasing for every fixed ϑ, t. With the above notation, the differential inequality (3.5) is equivalent to the integrated version (cf. [28, Lemma 2.2]):

$$u_N(t) \ge \sigma_{K/N}^{(1-t)} \left(\|\tau\|_{L^2(\pi)} \right) u_N(0) + \sigma_{K/N}^{(t)} \left(\|\tau\|_{L^2(\pi)} \right) u_N(1). \tag{3.8}$$

We thus proved the following proposition.

Proposition 3.3. *Fix $p \in (0,1)$, $K \in \mathbb{R}$ and $N \in (0,\infty)$. The measured Lorentzian pre-length space $(X, \mathsf{d}, \mathfrak{m}, \ll, \le, \tau)$ satisfies (resp. weak) $\mathsf{TCD}^e_p(K,N)$ if and only if for any couple $(\mu_0,\mu_1) \in \big(\mathrm{Dom}(\mathrm{Ent}(\cdot|\mathfrak{m}))\big)^2$ which is timelike p-dualisable (resp. $(\mu_0,\mu_1) \in [\mathrm{Dom}(\mathrm{Ent}(\cdot|\mathfrak{m})) \cap \mathcal{P}_c(X)]^2$ which is strongly timelike p-dualisable) by some $\pi \in \Pi^{p\text{-}opt}_{\ll}(\mu_0,\mu_1)$, there exists an ℓ_p-geodesic $(\mu_t)_{t\in[0,1]}$ such that the function $[0,1] \ni t \mapsto u_N(t) := U_N(\mu_t|\mathfrak{m})$ satisfies (3.8).*

As an example of geometric application of the $\mathsf{TCD}^e_p(K,N)$ we next show a timelike Brunn-Minkowski inequality (for the Riemannian/metric counterparts see [73, 28, 17]).

Proposition 3.4 (A timelike Brunn-Minkowski inequality)**.** *Let $(X, \mathsf{d}, \mathfrak{m}, \ll, \le, \tau)$ be a measured Lorentzian pre-length space satisfying (resp. weak) $\mathsf{TCD}^e_p(K,N)$, for some $K \in \mathbb{R}, N \in [1,\infty), p \in (0,1)$. Let $A_0, A_1 \subset X$ be measurable subsets with $\mathfrak{m}(A_0), \mathfrak{m}(A_1) \in (0,\infty)$. Calling $\mu_i := 1/\mathfrak{m}(A_i)\, \mathfrak{m}\llcorner_{A_i}$, $i = 1,2$, assume that (μ_0,μ_1) is (resp. strongly) timelike p-dualisable. Then*

$$\mathfrak{m}(A_t)^{1/N} \ge \sigma_{K/N}^{(1-t)}(\Theta)\; \mathfrak{m}(A_0)^{1/N} + \sigma_{K/N}^{(t)}(\Theta)\; \mathfrak{m}(A_1)^{1/N} \tag{3.9}$$

where $A_t := \mathcal{I}(A_0, A_1, t)$ defined in (1.6) is the set of t-intermediate points of geodesics from A_0 to A_1, and Θ is the maximal/minimal time-separation between points in A_0 and A_1, i.e.:

$$\Theta := \begin{cases} \sup\{\tau(x_0,x_1) : x_0 \in A_0, x_1 \in A_1\} & \text{if } K < 0, \\ \inf\{\tau(x_0,x_1) : x_0 \in A_0, x_1 \in A_1\} & \text{if } K \ge 0. \end{cases}$$

In particular, if $K \ge 0$ it holds:

$$\mathfrak{m}(A_t)^{1/N} \ge (1-t)\, \mathfrak{m}(A_0)^{1/N} + t\, \mathfrak{m}(A_1)^{1/N}.$$

Proof. Let $(\mu_t)_{t\in[0,1]}$ be the ℓ_p-geodesic given by Proposition 3.3, satisfying

$$U_N(\mu_t|\mathfrak{m}) \ge \sigma_{K/N}^{(1-t)} \left(\|\tau\|_{L^2(\pi)} \right) \mathfrak{m}(A_0)^{1/N} + \sigma_{K/N}^{(t)} \left(\|\tau\|_{L^2(\pi)} \right) \mathfrak{m}(A_1)^{1/N}.$$

Since $\mu_t = \rho_t \mathfrak{m}$ is concentrated on A_t, which is Suslin, applying Jensen's inequality twice gives:

$$U_N(\mu_t|\mathfrak{m}) = \exp\left(-\frac{1}{N} \int \log \rho_t \, \mu_t \right) \le \int \rho_t^{-1/N} \, \mu_t = \int_{A_t} \rho_t^{1-\frac{1}{N}} \mathfrak{m} \le \mathfrak{m}(A_t)^{1/N}. \tag{3.10}$$

The claim follows observing that $\vartheta \mapsto \sigma_{K/N}(\vartheta)$ is non-increasing for $K \le 0$ (resp. non-decreasing for $K > 0$) and that $\|\tau\|_{L^2(\pi)} \le \Theta$ (resp. $\|\tau\|_{L^2(\pi)} \ge \Theta$). Notice that in the case of wTCD, we first assume A_0, A_1 to be compact and then obtain the full claim arguing by inner regularity of $\mathfrak{m}$ with respect to compact sets. □

The Brunn–Minkowski inequality implies further geometric consequences like a timelike Bishop-Gromov volume growth estimate and a timelike Bonnet-Myers theorem. In order to state them, let us introduce some notation. Fix $x_0 \in X$ and let

$$B^\tau(x_0, r) := \{x \in I^+(x_0) \cup \{x_0\} : \tau(x_0, x) < r\}$$

be the τ-ball of radius r and center x_0. Since typically the volume of a τ-ball is infinite (e.g. in Minkowski space it is the region below an hyperboloid), it is useful to localise volume estimates using star-shaped sets. To this aim, we say that $E \subset I^+(x_0) \cup \{x_0\}$ is τ-star-shaped with respect to x_0 if $\mathfrak{I}(x_0, x, t) \subset E$ for every $x \in E$ and $t \in (0, 1]$. Define

$$v(E, r) := \mathfrak{m}(\overline{B^\tau(x_0, r)} \cap E), \quad s(E, r) := \limsup_{\delta \downarrow 0} \frac{1}{\delta} \mathfrak{m}\big((\overline{B^\tau(x_0, r+\delta)} \setminus B^\tau(x_0, r)) \cap E\big)$$

the volume of τ-ball of radius r (respectively of the τ-sphere of radius r) intersected with a compact subset $E \subset I^+(x_0) \cup \{x_0\}$, τ-star-shaped with respect to x_0.

Proposition 3.5 (A timelike Bishop-Gromov inequality)**.** *Let $(X, \mathsf{d}, \mathfrak{m}, \ll, \le, \tau)$ be a measured globally hyperbolic, locally causally Lorentzian geodesic space satisfying $\mathsf{wTCD}^e_p(K, N)$, for some $K \in \mathbb{R}, N \in [1, \infty), p \in (0, 1)$. Then, for each $x_0 \in X$, each compact subset $E \subset I^+(x_0) \cup \{x_0\}$ τ-star-shaped with respect to x_0, and each $0 < r < R \le \pi\sqrt{N/(K \vee 0)}$, it holds:*

$$\frac{s(E, r)}{s(E, R)} \ge \left(\frac{\mathfrak{s}_{K/N}(r)}{\mathfrak{s}_{K/N}(R)}\right)^N, \quad \frac{v(E, r)}{v(E, R)} \ge \frac{\int_0^r \mathfrak{s}_{K/N}(t)^N \, dt}{\int_0^R \mathfrak{s}_{K/N}(t)^N \, dt}. \tag{3.11}$$

Proof. We briefly sketch the argument. The basic idea is to apply Proposition 3.4 to $A_0 := B^\tau(x_0, \varepsilon) \cap E$ and $A_1 := \big(\overline{B^\tau(x_0, R+\delta R)} \setminus B^\tau(x_0, R)\big) \cap E$. Observe that, for $\varepsilon > 0$ small enough, it holds $A_0 \times A_1 \subset X^2_{\ll}$ and thus the measures (μ_0, μ_1) in the statement of Proposition 3.4 are strongly timelike p-dualisable thanks to Corollary 2.29. Thus we can apply Proposition 3.4 and follow verbatim the proof of [73, Theorem 2.3] replacing the coefficients $\tau^{(t)}_{K/N}(\vartheta)$ with $\sigma^{(t)}_{K/N}(\vartheta)$. □

Proposition 3.6 (A timelike Bonnet-Myers inequality)**.** *Let $(X, \mathsf{d}, \mathfrak{m}, \ll, \le, \tau)$ be a measured globally hyperbolic, locally causally closed Lorentzian geodesic space satisfying $\mathsf{wTCD}^e_p(K, N)$, for some $K > 0, N \in [1, \infty), p \in (0, 1)$. Then*

$$\sup_{x, y \in X} \tau(x, y) \le \pi\sqrt{\frac{N}{K}}. \tag{3.12}$$

In particular, for any causal curve γ it holds $\mathrm{L}_\tau(\gamma) \le \pi\sqrt{\frac{N}{K}}$.

Proof. Assume by contradiction that there exist $x_0', x_1' \in X$ with $\tau(x_0', x_1') \ge \pi\sqrt{N/K} + 4\varepsilon$, for some $\varepsilon > 0$. Let $\delta > 0$ and $x_0, y_0 \in X$ be such that

$$B^{\mathsf{d}}(x_0, \delta) \subset I^+(x_0'),\ B^{\mathsf{d}}(x_1, \delta) \subset I^-(x_1'),\ \inf\{\tau(x, y) : x \in B^{\mathsf{d}}(x_0, \delta),\ y \in B^{\mathsf{d}}(x_1, \delta)\} \ge \pi\sqrt{N/K} + \varepsilon,$$

where $B^{\mathsf{d}}(x, r)$ denotes the d-metric ball of radius r centred at x. From Corollary 2.29 it follows that $A_0 := B^{\mathsf{d}}(x_0, \delta), A_1 := B^{\mathsf{d}}(x_0, \delta)$ satisfy the assumptions of Proposition 3.4. Note that, for this choice of sets, $\Theta \ge \pi\sqrt{N/K} + \varepsilon$ and thus $\mathfrak{m}(A_{1/2}) = +\infty$. However, $A_{1/2} \subset J^+(x_0') \cap J^-(x_1')$ is relatively compact by global hyperbolicity and thus it has finite $\mathfrak{m}$-measure. Contradiction. □

3.2 Timelike Measure Contraction Property

A weaker variant of the $\mathsf{TCD}^e_p(K,N)$ condition is obtained by considering (K,N)-convexity properties only for those ℓ_p-geodesics $(\mu_t)_{t\in[0,1]}$ where μ_1 is a Dirac delta. In the metric measure setting, such a variant goes under the name of Measure Contraction Property (MCP for short) and was developed independently by Sturm [73] and Ohta [62].

Definition 3.7. Fix $p \in (0,1)$, $K \in \mathbb{R}$, $N \in (0,\infty)$. The measured Lorentzian pre-lengh space $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ satisfies $\mathsf{TMCP}^e_p(K,N)$ if and only if for any $\mu_0 \in \mathcal{P}_c(X) \cap \mathrm{Dom}(\mathrm{Ent}(\cdot|\mathfrak{m}))$ and for any $x_1 \in X$ such that $x \ll x_1$ for μ_0-a.e. $x \in X$, there exists an ℓ_p-geodesic $(\mu_t)_{t\in[0,1]}$ from μ_0 to $\mu_1 = \delta_{x_1}$ such that

$$U_N(\mu_t|\mathfrak{m}) \geq \sigma^{(1-t)}_{K/N}\left(\|\tau(\cdot, x_1)\|_{L^2(\mu_0)}\right) U_N(\mu_0|\mathfrak{m}), \quad \forall t \in [0,1). \tag{3.13}$$

Remark 3.8 (Geometric Properties)**.** As in the Riemannian/metric case [73], many properties valid for $\mathsf{TCD}^e_p(K,N)$ remain true also for $\mathsf{TMCP}^e_p(K,N)$. More precisely, this is the case for:

- Timelike Bishop-Gromov inequality, Proposition 3.5;
- Timelike Bonnet-Myers inequality, Proposition 3.6.

Actually, in Section 5.3, the above results will be improved to sharp forms in case of timelike non-branching $\mathsf{TMCP}^e_p(K,N)$ spaces. Such an improvement will be a product of the techniques developed in Section 3.4 and Section 4.

Remark 3.9. If a Lorentzian pre-lengh space $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ satisfies $\mathsf{TMCP}^e_p(K,N)$, then for any $x_1 \in X$ and $\mathfrak{m}$-a.e. $x \ll x_1$ there exists $\gamma \in \mathrm{TGeo}(X)$ such that $\gamma_0 = x$ and $\gamma_1 = x$. If in addition X is $\mathcal{K}$-globally hyperbolic, it follows that X is time-like geodesic. Indeed, given any $x_1 \in X$ and $x \ll x_1$ by $\mathsf{TMCP}^e_p(K,N)$ there is a sequence $x_n \to x$ and $\gamma^n \in \mathrm{TGeo}(X)$ with $\gamma^n_0 = x_n$ and $\gamma^n_1 = x_1$. Since X is $\mathcal{K}$-globally hyperbolic, it follows the existence of a limit $\gamma^\infty \in \mathrm{TGeo}(X)$ with $\gamma^\infty_0 = x$ and $\gamma^\infty_1 = x_1$ giving that X is timelike geodesic.

If instead $x \leq y$ one needs to further assume X to be causally path connected, i.e. for any $x, y \in X$ such that $x \leq y$ there exists a causal curve γ with $\gamma_0 = x$ and $\gamma_1 = y$. Hence if a Lorentzian pre-lengh space $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ satisfies $\mathsf{TMCP}^e_p(K,N)$, it is $\mathcal{K}$-globally hyperbolic and causally path connected, then it is geodesic.

Lemma 3.10. *Fix $p \in (0,1)$, $K \in \mathbb{R}$, $N \in (0,\infty)$. Let the measured Lorentzian pre-length space $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ satisfy $\mathsf{TCD}^e_p(K,N)$ (resp. $\mathsf{wTCD}^e_p(K,N), \mathsf{TMCP}^e_p(K,N)$). Then*

1. Consistency*: $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ satisfy $\mathsf{TCD}^e_p(K',N')$ (resp. $\mathsf{wTCD}^e_p(K',N'), \mathsf{TMCP}^e_p(K',N')$) for every $K' \leq K$ and $N' \geq N$.*
2. Scaling*: The rescaled space $(X, a\cdot\mathsf{d}, b\cdot\mathfrak{m}, \ll, \leq, r\cdot\tau)$, for $a, b, r > 0$ satisfies $\mathsf{TCD}^e_p(K/r^2, N)$ (resp. $\mathsf{wTCD}^e_p(K/r^2,N), \mathsf{TMCP}^e_p(K/r^2,N)$).*

Proof. 1. Consistency for $\mathsf{TCD}^e_p(K,N)$ follows directly by the definition (3.2).

Regarding TMCP^e_p: the consistency in K follows by the fact that the map $\kappa \mapsto \sigma^{(t)}_\kappa(\vartheta)$ is monotone increasing. For the consistency in N, observe that taking the logarithm of (3.13) one obtains the equivalent condition

$$\mathrm{Ent}(\mu_t|\mathfrak{m}) \leq \mathrm{Ent}(\mu_0|\mathfrak{m}) - N\log\left(\sigma^{(1-t)}_{K/N}\left(\|\tau(\cdot,x_1)\|_{L^2(\mu_0)}\right)\right). \tag{3.14}$$

It follows from [73, Lemma 1.2] that

$$\left(\sigma^{(t)}_{K/N'}(\vartheta)\right)^{N'} \leq t^{N'-N}\left(\sigma^{(t)}_{K/N}(\vartheta)\right)^N \leq \left(\sigma^{(t)}_{K/N}(\vartheta)\right)^N \quad \forall t \in [0,1],\ K \in \mathbb{R},\ N' \geq N,$$

giving that the function $N \mapsto -N\log\left(\sigma^{(1-t)}_{K/N}(\vartheta)\right)$ is non-decreasing for every fixed K, t, ϑ.

2. Follows by the very definitions, observing that $\mathrm{Ent}(\mu|b\cdot\mathfrak{m}) = \mathrm{Ent}(\mu|\mathfrak{m}) - \log(b)$, $\|r\cdot\tau\|_{L^2(\pi)} = r\|\tau\|_{L^2(\pi)}$ and that $\sigma^{(t)}_{\kappa/r^2}(r\cdot\vartheta) = \sigma^{(t)}_\kappa(\vartheta)$. □

We refer to Appendix A for a discussion of $\mathsf{TMCP}^e_p(K,N)$ in case of smooth Lorentzian manifolds.

Proposition 3.11 ($\mathsf{wTCD}^e_p(K,N) \Rightarrow \mathsf{TMCP}^e_p(K,N)$)**.** *Fix $p \in (0,1)$, $K \in \mathbb{R}$, $N \in (0,\infty)$. The $\mathsf{wTCD}^e_p(K,N)$ condition implies $\mathsf{TMCP}^e_p(K,N)$ for locally causally closed, $\mathcal{K}$-globally hyperbolic Lorentzian geodesic spaces.*

Proof. **Step 1.**
Let $\mu_0 = \rho_0\,\mathfrak{m} \in \mathrm{Dom}(\mathrm{Ent}(\cdot|\mathfrak{m})) \cap \mathcal{P}_c(X)$ and $x_1 \in X$ be such that $x \ll x_1$ for μ_0-a.e. $x \in X$.
For each $\varepsilon > 0$ consider $K_\varepsilon \Subset \operatorname{supp}\mu_0 \Subset X$ compact subset such that (the last condition will be used later in step 2)

$$\int_{X\setminus K_\varepsilon} \rho_0\,|\log(\rho_0)|\,\mathfrak{m} \le \varepsilon, \qquad \mu_0(K_\varepsilon) \ge 1-\varepsilon, \qquad K_\varepsilon \times \{x_1\} \subset \{\tau > 0\} \subset X^2,$$

and consider the restricted measure $\mu_0^\varepsilon := \mu_0 \llcorner_{K_\varepsilon}/\mu_0(K_\varepsilon)$. A straightforward computation gives

$$\mathrm{Ent}(\mu_0|\mathfrak{m}) = \int_{X\setminus K_\varepsilon} \rho\log(\rho)\,\mathfrak{m} + \mathrm{Ent}(\mu_0^\varepsilon|\mathfrak{m})\,\mu_0(K_\varepsilon) + \mu_0(K_\varepsilon)\,\log(\mu_0(K_\varepsilon)). \tag{3.15}$$

Hence

$$\mathrm{Ent}(\mu_0|\mathfrak{m}) \ge \mathrm{Ent}(\mu_0^\varepsilon|\mathfrak{m})(1-\varepsilon) - 2\varepsilon,$$

giving

$$U_N(\mu_0^\varepsilon|\mathfrak{m}) \ge \exp\left(-\frac{2\varepsilon}{N(1-\varepsilon)}\right) U_N(\mu_0|\mathfrak{m})^{1/(1-\varepsilon)}. \tag{3.16}$$

Step 2.
Fix $\varepsilon \ll 1$. Since the set $\{\tau > 0\} \subset X \times X$ is open and by construction it contains $K_\varepsilon \times \{x_1\}$, for $\eta > 0$ small enough it holds

$$K_\varepsilon \times B_\eta(x_1) \subset \{\tau > 0\}. \tag{3.17}$$

Define $\mu_1^\eta := \mathfrak{m}\llcorner B_\eta(x_1)/\mathfrak{m}(B_\eta(x_1))$. By Corollary 2.29, we know that $(\mu_0^\varepsilon, \mu_1^\eta)$ is strongly timelike p-dualisable. It also clear that $\mu_0^\varepsilon, \mu_1^\eta \in \mathrm{Dom}(\mathrm{Ent}(\cdot|\mathfrak{m})) \cap \mathcal{P}_c(X)$.

The $\mathsf{wTCD}^e_p(K,N)$ condition thus implies that for each $\varepsilon, \eta > 0$ small enough there exists an ℓ_p-optimal coupling $\pi_{\varepsilon,\eta} \in \Pi_\le(\mu_0^\varepsilon, \mu_1^\eta)$ and an ℓ_p-geodesic $(\mu_t^{\varepsilon,\eta})_{t\in[0,1]}$ joining μ_0^ε and μ_1^η verifying for all $t \in [0,1]$:

$$\begin{aligned} U_N(\mu_t^{\varepsilon,\eta}|\mathfrak{m}) &\ge \sigma_{K/N}^{(1-t)}\left(\|\tau\|_{L^2(\pi_{\varepsilon,\eta})}\right) U_N(\mu_0^\varepsilon|\mathfrak{m}) + \sigma_{K/N}^{(t)}\left(\|\tau\|_{L^2(\pi_{\varepsilon,\eta})}\right) U_N(\mu_1^\eta|\mathfrak{m}) \\ &\ge \sigma_{K/N}^{(1-t)}\left(\|\tau\|_{L^2(\pi_{\varepsilon,\eta})}\right) U_N(\mu_0^\varepsilon|\mathfrak{m}) \\ &\ge \sigma_{K/N}^{(1-t)}\left(\|\tau\|_{L^2(\pi_{\varepsilon,\eta})}\right) \exp\left(-\frac{2\varepsilon}{N(1-\varepsilon)}\right) U_N(\mu_0|\mathfrak{m})^{1/(1-\varepsilon)}, \end{aligned} \tag{3.18}$$

where in the last inequality we used (3.16).

Step 3.
In this last step we pass into the limit, first as $\eta \to 0$, then as $\varepsilon \to 0$.
First of all it is clear that $\mu_0^\varepsilon \to \mu_0$ and $\mu_1^\eta \to \mu_1$ narrowly. $\mathcal{K}$-global hyperbolicity implies that

$$\bar{K} := \bigcup_{s\in[0,1]} \mathfrak{I}(K_{\varepsilon_0}, B_{\eta_0}(x_1), s) \Subset X$$

is a compact subset, see (1.6),(1.7). It is easily seen that

$$\operatorname{supp}\mu_t^{\varepsilon,\eta} \subset \mathfrak{I}(K_\varepsilon, B_\eta(x_1), t) \subset \bar{K}, \quad \forall t \in [0,1], \eta \in [0,\eta_0]. \tag{3.19}$$

Fix $\varepsilon \in (0, \varepsilon_0)$ and a sequence (η_n) with $\eta_n \downarrow 0$. We aim to construct a limit ℓ_p-geodesic $(\mu_t^\varepsilon)_{t\in[0,1]}$ from μ_0^ε to $\mu_1 = \delta_{x_1}$. From (2.24) we get that

$$\sup_{n\in\mathbb{N}} \mathrm{L}_{W_1}\left((\mu_t^{\varepsilon,\eta_n})_{t\in[0,1]}\right) \le \bar{C} < \infty.$$

By the metric Arzelá-Ascoli Theorem we deduce that there exists a limit continuous curve $(\mu_t^\varepsilon)_{t\in[0,1]} \subset (\mathcal{P}(\bar{K}), W_1)$ such that (up to a sub-sequence) $W_1(\mu_t^{\varepsilon,\eta_n}, \mu_t^\varepsilon) \to 0$ and thus $\mu_t^{\varepsilon,\eta_n} \to \mu_t^\varepsilon$ narrowly, as $n \to \infty$. Lemma 2.11 yields that

$$\ell_p(\mu_0^\varepsilon, \mu_t^\varepsilon) = \lim_{n\to\infty} \ell_p(\mu_0^\varepsilon, \mu_t^{\varepsilon,\eta_n}) = t \lim_{n\to\infty} \ell_p(\mu_0^\varepsilon, \mu_1^{\eta_n}) = t\, \ell_p(\mu_0^\varepsilon, \mu_1). \tag{3.20}$$

In other terms, the curve $(\mu_t^\varepsilon)_{t\in[0,1]}$ is an ℓ_p-geodesic from μ_0^ε to $\mu_1 = \delta_{x_1}$. The upper-semicontinuity of $U_N(\cdot|\mathfrak{m})$ under narrow convergence (3.4) yields

$$\limsup_{i\to\infty} U_N(\mu_t^{\varepsilon,\eta_{n_i}}|\mathfrak{m}) \le U_N(\mu_t^\varepsilon), \quad \forall t \in [0,1]. \tag{3.21}$$

Moreover, it is readily seen that $\pi^{\varepsilon,\eta_{n_i}} \to \mu_0^\varepsilon \otimes \delta_{x_1}$ narrowly and

$$\lim_{i\to\infty} \sigma_{K/N}^{(1-t)}\left(\|\tau\|_{L^2(\pi_{\varepsilon,\eta_{n_i}})}\right) = \sigma_{K/N}^{(1-t)}\left(\|\tau(\cdot, x_1)\|_{L^2(\mu_0^\varepsilon)}\right). \tag{3.22}$$

Combining (3.18),(3.21) and (3.22) gives

$$U_N(\mu_t^\varepsilon|\mathfrak{m}) \ge \sigma_{K/N}^{(1-t)}\left(\|\tau(\cdot, x_1)\|_{L^2(\mu_0^\varepsilon)}\right) \exp\left(-\frac{2\varepsilon}{N(1-\varepsilon)}\right) U_N(\mu_0|\mathfrak{m})^{1/(1-\varepsilon)}, \quad \forall t \in [0,1]. \tag{3.23}$$

In order to conclude the proof we now pass to the limit as $\varepsilon \downarrow 0$ in (3.23). Observe that

$$\bar{K}' := \bigcup_{s\in[0,1]} \mathfrak{I}(\operatorname{supp}\mu_0, x_1, s) \Subset X$$

is a compact subset by $\mathcal{K}$-hyperbolicity and

$$\operatorname{supp}\mu_t^\varepsilon \subset \mathfrak{I}(\operatorname{supp}\mu_0, x_1, t) \subset \bar{K}', \quad \forall t \in [0,1], \varepsilon \in [0, \varepsilon_0].$$

The argument from (3.19) to (3.23) can be adapted to show that there exists an ℓ_p-geodesic $(\mu_t)_{t\in[0,1]}$ satisfying (3.13). □

3.3 Stability of $\mathsf{TCD}_p^e(K,N)$ and $\mathsf{TMCP}_p^e(K,N)$ conditions

This section is of independent interest and will not be used in the rest of the paper. In the next theorem we show the stability of the $\mathsf{TMCP}_p^e(K,N)$ condition under convergence of Lorentzian spaces. Throughout this part we will make use of topological embeddings to identify spaces with their image inside a larger space. Recall that a topological embedding is a map $f : X \to Y$ between two topological spaces X and Y such that f is continuous, injective and with continuous inverse between X and $f(X)$.

Theorem 3.12 (Stability of $\mathsf{TMCP}_p^e(K,N)$). *Let $\{(X_j, \mathsf{d}_j, \mathfrak{m}_j, \ll_j, \le_j, \tau_j)\}_{j\in\mathbb{N}\cup\{\infty\}}$ be a sequence of measured Lorentzian geodesic spaces satisfying the following properties:*

1. *There exists a locally causally closed, $\mathcal{K}$-globally hyperbolic Lorentzian geodesic space $(\bar{X}, \bar{\mathsf{d}}, \bar{\ll}, \bar{\le}, \bar{\tau})$ such that each $(X_j, \mathsf{d}_j, \mathfrak{m}_j, \ll_j, \le_j, \tau_j)$, $j \in \mathbb{N}\cup\{\infty\}$, is isomorphically embedded in it, i.e. there exist topological embedding maps $\iota_j : X_j \to \bar{X}$ such that*
   - *$x_j^1 \le_j x_j^2$ if and only if $\iota_j(x_j^1) \bar{\le} \iota_j(x_j^2)$, for every $j \in \mathbb{N}\cup\{\infty\}$, for every $x_j^1, x_j^2 \in X_j$;*
   - *$\bar{\tau}(\iota_j(x_j^1), \iota_j(x_j^2)) = \tau_j(x_j^1, x_j^2)$ for every $x_j^1, x_j^2 \in X_j$, for every $j \in \mathbb{N}\cup\{\infty\}$;*

2. *The measures $(\iota_j)_\sharp \mathfrak{m}_j$ converge to $(\iota_\infty)_\sharp \mathfrak{m}_\infty$ weakly in duality with $C_c(\bar{X})$ in $\bar{X}$, i.e.*

$$\int \varphi \, (\iota_j)_\sharp \mathfrak{m}_j \to \int \varphi \, (\iota_\infty)_\sharp \mathfrak{m}_\infty \quad \forall \varphi \in C_c(\bar{X}), \tag{3.24}$$

where $C_c(\bar{X})$ denotes the set of continuous functions with compact support.

3. *There exist $p \in (0,1), K \in \mathbb{R}, N \in (0,\infty)$ such that $(X_j, \mathsf{d}_j, \mathfrak{m}_j, \ll_j, \leq_j, \tau_j)$ satisfies $\mathsf{TMCP}^e_p(K,N)$, for each $j \in \mathbb{N}$.*

Then also the limit space $(X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, \ll_\infty, \leq_\infty, \tau_\infty)$ satisfies $\mathsf{TMCP}^e_p(K,N)$.

Remark 3.13. Even though we haven't specifically list any topological assumption on the sequence of spaces X_j, they actually inherit them from X_∞ via the topological embeddings ι_j. The map ι_j preserves both the causal relations and τ_j hence $(X_j, \mathsf{d}_j, \mathfrak{m}_j, \ll_j, \leq_j, \tau_j)$ are locally causally closed and $\mathcal{K}$-globally hyperbolic Lorentzian geodesic (by assumption) spaces.

Proof. For simplicity of notation, we will identify X_j with its isomorphic image $\iota_j(X_j) \subset \bar{X}$ and the measure $\mathfrak{m}_j$ with $(\iota_j)_\sharp \mathfrak{m}_j$, for each $j \in \mathbb{N} \cup \{\infty\}$.

Fix arbitrary $\mu_0^\infty = \rho_0^\infty \mathfrak{m}_\infty \in \mathcal{P}_c(X_\infty) \cap \mathrm{Dom}(\mathrm{Ent}(\cdot|\mathfrak{m}_\infty))$ and $x_1^\infty \in X_\infty$ such that $x \ll_\infty x_1^\infty$ for μ_0^∞-a.e. $x \in X_\infty$. Since μ_0^∞ has compact support and $\bar{X}$ is $\mathcal{K}$-globally hyperbolic, we can restrict all the arguments to a large compact subset of $\bar{X}$ whose $\mathfrak{m}_\infty$-measure is the limit of its $\mathfrak{m}_j$-measures. For easy of notation, without loss of generality we can thus directly assume that $\bar{X}$ is compact and that (up to a convergent sequence of normalizations) $\mathfrak{m}_j$ are probability measures converging to $\mathfrak{m}_\infty$ in narrow topology. Since on compact metric spaces narrow convergence is equivalent to W_2 convergence, we actually assume $\mathfrak{m}_j \to \mathfrak{m}_\infty$ in $W_2^{(\bar{X},\bar{\mathsf{d}})}$. Denote with $\boldsymbol{\gamma}_j \in \Pi(\mathfrak{m}_\infty, \mathfrak{m}_j)$ an optimal coupling for $W_2^{(\bar{X},\bar{\mathsf{d}})}$.

Step 1. We show that, up to a subsequence, for every $j \in \mathbb{N}$ there exists $\mu_0^j \in \mathcal{P}_c(X_j) \cap \mathrm{Dom}(\mathrm{Ent}(\cdot|\mathfrak{m}_j))$, $x_1^j \in X_j$ such that

$$\mu_0^j\left(I^-_{\ll_j}(x_1^j)\right) = 1, \quad x_1^j \to x_1^\infty, \quad \mu_0^j \to \mu_0^\infty \text{ narrowly}, \quad U_N(\mu_0^\infty|\mathfrak{m}_\infty) \leq \liminf_{j\to\infty} U_N(\mu_0^j|\mathfrak{m}_j). \tag{3.25}$$

Step 1a. Let us first consider the case $\mu_0^\infty = \rho_0^\infty \mathfrak{m}_\infty \in \mathcal{P}_c(X_\infty)$ has density $\rho_0^\infty \in L^\infty(\mathfrak{m}_\infty)$ and $x_1^\infty \in X_\infty$ is such that $\mathrm{supp}\, \mu_0^\infty \Subset I^-_{\ll_\infty}(x_1^\infty)$.

From narrow convergence we deduce the existence of a sequence $x_1^j \in \mathrm{supp}\, \mathfrak{m}_j \subset X_j$ with $x_1^j \to x_1^\infty$ with respect to $\bar{\mathsf{d}}$. Since $\bar{\tau} : \bar{X}^2 \to \mathbb{R}$ is continuous and $\mathrm{supp}\, \mu_0^\infty$ is compact,

$$\lim_{j\to\infty} \min_{x \in \mathrm{supp}\, \mu_0^\infty} \bar{\tau}(x, x_1^j) = \min_{x \in \mathrm{supp}\, \mu_0^\infty} \bar{\tau}(x, x_1^\infty) = \min_{x \in \mathrm{supp}\, \mu_0^\infty} \tau_\infty(x, x_1^\infty) > 0.$$

Hence, for j sufficiently large, we can assume that $x \bar{\ll} x_1^j$ for μ_0^∞-a.e. $x \in X_\infty$. Then since $I^-_{\bar{\ll}}(x_1^j)$ is open, any narrow converging sequence of probability measures $\mu_0^k \to \mu_0^\infty$ satisfies

$$\liminf_{k\to\infty} \mu_0^k(I^-_{\bar{\ll}}(x_1^j)) \geq \mu_0^\infty(I^-_{\bar{\ll}}(x_1^j)) = 1. \tag{3.26}$$

Define now $\boldsymbol{\gamma}'_j \in \mathcal{P}(\bar{X}^2)$ as $\boldsymbol{\gamma}'_j(dxdy) := \rho_0^\infty(x) \boldsymbol{\gamma}_j(dxdy)$ and $\hat{\mu}_0^j := (P_2)_\sharp \boldsymbol{\gamma}'_j \in \mathcal{P}(X_j) \subset \mathcal{P}(\bar{X})$. By construction, $\boldsymbol{\gamma}'_j \ll \boldsymbol{\gamma}_j$, hence $\hat{\mu}_0^j \ll (P_2)_\sharp \boldsymbol{\gamma}_j = \mathfrak{m}_j$. Let $\hat{\mu}_0^j = \hat{\rho}_0^j \mathfrak{m}_j$. It is readily checked from the definition that it holds $\hat{\rho}_0^j(y) = \int \rho_0^\infty(x) \, (\boldsymbol{\gamma}_j)_y(dx)$, where $\{(\boldsymbol{\gamma}_j)_y\}$ is the disintegration of $\boldsymbol{\gamma}_j$ w.r.t. the projection on the second marginal. In particular, $\|\hat{\rho}_0^j\|_{L^\infty(\mathfrak{m}_j)} \leq \|\rho_0^\infty\|_{L^\infty(\mathfrak{m}_\infty)}$.

By Jensen's inequality applied to the convex function $u(z) = z\log(z)$ we have

$$\begin{aligned}
\mathrm{Ent}(\hat{\mu}_0^j|\mathfrak{m}_j) &= \int u(\hat{\rho}_0^j) \, \mathfrak{m}_j = \int u\left(\int \rho_0^\infty(x) \, (\boldsymbol{\gamma}_j)_y(dx)\right) \mathfrak{m}_j(dy) \\
&\leq \int u(\rho_0^\infty(x)) \, (\boldsymbol{\gamma}_j)_y(dx) \, \mathfrak{m}_j(dy) = \int u(\rho_0^\infty(x)) \, \boldsymbol{\gamma}_j(dxdy) \\
&= \int u(\rho_0^\infty) \, (P_1)_\sharp \boldsymbol{\gamma}_j = \int u(\rho_0^\infty) \, \mathfrak{m}_\infty = \mathrm{Ent}(\mu_0^\infty|\mathfrak{m}_\infty).
\end{aligned}$$

Since by construction we have $\gamma'_j \in \Pi(\mu_0^\infty, \hat\mu_0^j)$, it holds

$$\Big(W_2^{(\bar X,\bar{\mathsf d})}(\mu_0^\infty,\hat\mu_0^j)\Big)^2 \le \int \bar{\mathsf d}^2(x,y)\,\gamma'_j(dxdy) = \int \rho_0^\infty(x)\bar{\mathsf d}^2(x,y)\,\gamma_j(dxdy)$$
$$\le \|\rho_0^\infty\|_{L^\infty(\mathfrak m_\infty)}\Big(W_2^{(\bar X,\bar{\mathsf d})}(\mathfrak m_\infty,\mathfrak m_j)\Big)^2,$$

and therefore $W_2^{(\bar X,\bar{\mathsf d})}(\mu_0^\infty,\hat\mu_0^j)\to 0$. In particular $\hat\mu_0^j\to\mu_0^\infty$ narrowly in $\bar X$.

Moreover, reasoning like in (3.15), it will not be restrictive also to assume that $\hat\mu_0^j$ has compact support. We will also cutoff where the density $\hat\rho_0^j$ is too small in the following manner. Consider the set $K_j := \{\hat\rho_0^j \ge 1/j\}$ that is easily verified to satisfy $\hat\mu_0^j(K_j)\ge 1-1/j$ and define

$$\bar\mu_0^j := \hat\mu_0^j\llcorner_{K_j}/\hat\mu_0^j(K_j).$$

The difference between $\mathrm{Ent}(\bar\mu_0^j|\mathfrak m_j)$ and $\mathrm{Ent}(\hat\mu_0^j|\mathfrak m_j)$ is controlled (see (3.15)) by

$$\int_{\{\hat\rho_0^j\le 1/j\}} |\hat\rho_0^j\log(\hat\rho_0^j)|\mathfrak m_j \le \frac{1}{j}\log(j).$$

Hence $\bar\mu_0^j$ still verifies all the properties we have checked for $\hat\mu_0^j$. Finally it is only left to restrict $\bar\mu_0^j$ to $I_{\lll}(x_1^j)$. From (3.26), adopting a diagonal argument, we also obtain that $\bar\mu_0^j(I_{\lll}(x_1^j))\ge 1-1/j$. Hence we define

$$\mu_0^j := \bar\mu_0^j\llcorner_{I_{\lll}(x_1^j)}/\bar\mu_0^j(I_{\lll}(x_1^j)).$$

Again the difference between $\mathrm{Ent}(\bar\mu_0^j|\mathfrak m_j)$ and $\mathrm{Ent}(\mu_0^j|\mathfrak m_j)$ is controlled (see (3.15)) by

$$\int_{X\setminus I_{\lll}(x_1^j)} \hat\rho_0^j\,|\log(\hat\rho_0^j)|\,\mathfrak m_j \le \log(j)\,\bar\mu_0^j(X\setminus I_{\lll}(x_1^j)) \le \frac{1}{j}\log(j).$$

Thus (3.25) is proved in this case.

Step 1b. $\mu_0^\infty = \rho_0^\infty\mathfrak m_\infty \in \mathcal P_c(X_\infty)$ has density $\rho_0^\infty\in L^\infty(\mathfrak m_\infty)$, and $x_1^\infty\in X_\infty$ is such that $x \ll_\infty x_1^\infty$ for μ_0^∞-a.e. $x\in X_\infty$.
For $n\in\mathbb N$ define $\mu_{0,n}^\infty := \bar c_n\mu_0^\infty\llcorner\{\tau_\infty(\cdot,x_1^\infty)\ge \frac1n\}\in\mathcal P(X_\infty)$, where $\bar c_n\downarrow 1$ are the normalising constants. By the continuity of τ_∞, it is readily seen that $\operatorname{supp}\mu_{0,n}^\infty \Subset I^-_{\ll_\infty}(x_1^\infty)$. Moreover

$$\lim_{n\to\infty}\mathrm{Ent}(\mu_{0,n}^\infty|\mathfrak m_\infty) = \mathrm{Ent}(\mu_0^\infty|\mathfrak m_\infty), \qquad \lim_{n\to\infty} W_2^{(\bar X,\bar{\mathsf d})}(\mu_{0,n}^\infty,\mu_0^\infty)=0.$$

Then apply Step 1a to $\mu_{0,n}^\infty$ and conclude with a diagonal argument.

Step 1c. General case. If ρ_0^∞ is not bounded, for $k\in\mathbb N$ define $\rho_{0,k}^\infty := \bar c_k\min\{\rho_0^\infty,k\}$, $\bar c_k\downarrow 1$ being such that $\mu_{0,k}^\infty := \rho_{0,k}^\infty\,\mathfrak m_\infty\in\mathcal P(X_\infty)$. Clearly, it holds

$$\lim_{k\to\infty}\mathrm{Ent}(\mu_{0,k}^\infty|\mathfrak m_\infty) = \mathrm{Ent}(\mu_0^\infty|\mathfrak m_\infty), \qquad \lim_{k\to\infty} W_2^{(\bar X,\bar{\mathsf d})}(\mu_{0,k}^\infty,\mu_0^\infty)=0.$$

Then apply Step 1b to $\mu_{0,k}^\infty$ and conclude with a diagonal argument.

Step 2. Conclusion.
Using the assumption that $(X_j,\mathsf d_j,\mathfrak m_j,\ll_j,\le_j,\tau_j)$ satisfies $\mathsf{TMCP}^e_p(K,N)$, we obtain an ℓ_p-geodesic $(\mu_t^j)_{t\in[0,1]}$ from μ_0^j to $\mu_1^j := \delta_{x_1^j}$ such that

$$U_N(\mu_t^j|\mathfrak m_j) \ge \sigma_{K/N}^{(1-t)}\Big(\|\tau(\cdot,x_1^j)\|_{L^2(\mu_0^j)}\Big)\,U_N(\mu_0^j|\mathfrak m_j), \quad \forall t\in[0,1). \tag{3.27}$$

From (3.25), it is readily seen that $\mu_0^j \otimes \delta_{x_1^j} \to \mu_0^\infty \otimes \delta_{x_1^\infty}$ narrowly in $\bar{X}^2$. Thus, recalling that $\bar{\tau}$ is continuous and bounded, we infer

$$\int_{\bar{X}^2} \bar{\tau}(x,y)^p \, \mu_0^j \otimes \delta_{x_1^j}(dxdy) \longrightarrow \int_{\bar{X}^2} \bar{\tau}(x,y)^p \, \mu_0^\infty \otimes \delta_{x_1^\infty}(dxdy), \quad \text{as } j \to \infty. \tag{3.28}$$

Using that $\Pi_{\leq}\big(\mu_0^j, \delta_{x_1^j}\big) = \big\{\mu_0^j \otimes \delta_{x_1^j}\big\}$, the convergence (3.28) yields

$$\ell_p(\mu_0^j, \delta_{x_1^j})^p \longrightarrow \ell_p(\mu_0^\infty, \delta_{x_1^\infty})^p, \quad \text{as } j \to \infty. \tag{3.29}$$

Since by assumption $\bar{X}$ is compact and non-totally imprisoning, from (the proof of) (2.24) we deduce that

$$\sup_{j\in\mathbb{N}} \mathrm{L}_{W_1^{(\bar{X},\bar{\mathsf{d}})}} \Big((\mu_t^j)_{t\in[0,1]} \Big) \leq \bar{C} < \infty.$$

By the metric Arzelá-Ascoli Theorem (recall that the metric space $(\bar{X}, \bar{\mathsf{d}})$ is proper by definition) we deduce that there exists a limit continuous curve $(\mu_t^\infty)_{t\in[0,1]} \subset \mathcal{P}(\bar{X}, W_1^{(\bar{X},\bar{\mathsf{d}})})$ such that (up to a subsequence) $W_1^{(\bar{X},\bar{\mathsf{d}})}\Big(\mu_t^j, \mu_t^\infty\Big) \to 0$ and thus $\mu_t^j \to \mu_t^\infty$ narrowly in $\bar{X}$, as $j \to \infty$ for every $t \in [0,1]$. Using that $\bar{\tau}$ is continuous and bounded together with (3.29), it is easy to see that

$$\ell_p(\mu_0^\infty, \mu_t^\infty) \geq \lim_{j\to\infty} \ell_p(\mu_0^j, \mu_t^j) = t \lim_{j\to\infty} \ell_p(\mu_0^j, \mu_1^j) = t \, \ell_p(\mu_0^\infty, \mu_1^\infty).$$

By reverse triangle inequality, we obtain that the curve $(\mu_t^\infty)_{t\in[0,1]}$ is an ℓ_p-geodesic from μ_0^∞ to $\mu_1^\infty = \delta_{x_1^\infty}$. Finally, the joint upper semicontinuity of U_N under narrow convergence (1.14) yields:

$$U_N(\mu_t^\infty|\mathfrak{m}_\infty) \geq \limsup_{j\in\mathbb{N}} U_N(\mu_t^j|\mathfrak{m}_j), \quad \forall t \in [0,1], \tag{3.30}$$

obtaining in particular that $(\mu_t^\infty)_{t\in[0,1]} \subset \mathcal{P}(X_\infty)$. The combination of (3.25), (3.27), (3.28) and (3.30) gives that

$$U_N(\mu_t^\infty|\mathfrak{m}_\infty) \geq \sigma_{K/N}^{(1-t)} \left(\|\bar{\tau}(\cdot, x_1^\infty)\|_{L^2(\mu_0^\infty)} \right) U_N(\mu_0^\infty|\mathfrak{m}_\infty), \quad \forall t \in [0,1).$$

as desired. □

In the next theorem we show that if a sequence of $\mathsf{TCD}_p^e(K,N)$ Lorentzian spaces converge to a limit Lorentzian space, then the latter is $\mathsf{wTCD}_p^e(K,N)$. The same observation of Remark 3.13 will be valid for the next theorem.

Theorem 3.14 (Weak stability of $\mathsf{TCD}_p^e(K,N)$)**.** *Let $\{(X_j, \mathsf{d}_j, \mathfrak{m}_j, \ll_j, \leq_j, \tau_j)\}_{j\in\mathbb{N}\cup\{\infty\}}$ be a sequence of measured Lorentzian geodesic spaces satisfying the following properties:*

1. *There exists a locally causally closed, $\mathcal{K}$-globally hyperbolic Lorentzian geodesic space $(\bar{X}, \bar{\mathsf{d}}, \bar{\ll}, \bar{\leq}, \bar{\tau})$ such that each $(X_j, \mathsf{d}_j, \mathfrak{m}_j, \ll_j, \leq_j, \tau_j)$, $j \in \mathbb{N}\cup\{\infty\}$, is isomorphically embedded in it (as in 1. of Theorem 3.12).*
2. *The measures $(\iota_j)_\sharp \mathfrak{m}_j$ converge to $(\iota_\infty)_\sharp \mathfrak{m}_\infty$ weakly in duality with $C_c(\bar{X})$ in $\bar{X}$, i.e.* (3.24) *holds.*
3. *There exist $p \in (0,1), K \in \mathbb{R}, N \in (0,\infty)$ such that $(X_j, \mathsf{d}_j, \mathfrak{m}_j, \ll_j, \leq_j, \tau_j)$ satisfies $\mathsf{TCD}_p^e(K,N)$, for each $j \in \mathbb{N}$.*

Then the limit space $(X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, \ll_\infty, \leq_\infty, \tau_\infty)$ satisfies the $\mathsf{wTCD}_p^e(K,N)$ condition.

Proof. Without affecting generality, we will identify X_j with its isomorphic image $\iota_j(X_j) \subset \bar{X}$ and the measure $\mathfrak{m}_j$ with $(\iota_j)_\sharp \mathfrak{m}_j$, for each $j \in \mathbb{N}\cup\{\infty\}$.

Fix $\mu_0^\infty, \mu_1^\infty \in \mathrm{Dom}(\mathrm{Ent}(\cdot|\mathfrak{m}_\infty)) \cap \mathcal{P}_c(X_\infty)$ strongly timelike p-dualisable, i.e. such that there exists $\pi_\infty \in \Pi_{\leq_\infty}^{p\text{-opt}}(\mu_0^\infty, \mu_1^\infty)$ with $\pi_\infty(\{\tau_\infty > 0\}) = 1$ and there exists a measurable ℓ^p-cyclically monotone set

$$\Gamma \subset (X_\infty^2)_{\ll_\infty} \cap (\operatorname{supp} \mu_0^\infty \times \operatorname{supp} \mu_1^\infty)$$

such that a coupling $\pi \in \Pi_{\leq_\infty}(\mu_0^\infty, \mu_1^\infty)$ is ℓ_p-optimal if and only if π is concentrated on Γ. Since $\mu_0^\infty, \mu_1^\infty$ have compact support and $\bar X$ is $\mathcal{K}$-globally hyperbolic, we can restrict all the arguments to a large compact subset of $\bar X$ whose $\mathfrak{m}_\infty$-measure is the limit of its $\mathfrak{m}_j$-measures. For easy of notation, without loss of generality we can thus directly assume that $\bar X$ is compact and that (up to a convergent sequence of normalizations) $\mathfrak{m}_j$ are probability measures converging to $\mathfrak{m}_\infty$ in narrow topology.

Step 1: We prove that, up to a subsequence, for every $j \in \mathbb{N}$ there exists $(\mu_0^j, \mu_1^j) \in \mathcal{P}(X_j)^2$ timelike p-dualisable such that

$$\mu_0^j \to \mu_0^\infty,\ \mu_1^j \to \mu_1^\infty \text{ narrowly in } \bar X \text{ and } \ell_p(\mu_0^j, \mu_1^j) \to \ell_p(\mu_0^\infty, \mu_1^\infty) \text{ as } j \to \infty. \tag{3.31}$$

Since $\bar X$ is compact, the narrow convergence implies $W_q^{(\bar X, \bar{\mathsf{d}})}$ convergence for some (or equivalently every) $q \in [1, \infty)$. In particular $\mathfrak{m}_j \to \mathfrak{m}_\infty$ in $W_2^{(\bar X, \bar{\mathsf{d}})}$. Let

$$\boldsymbol{\gamma}_j \in \Pi(\mathfrak{m}_\infty, \mathfrak{m}_j) \text{ be a } W_2^{(\bar X, \bar{\mathsf{d}})}\text{-optimal coupling.} \tag{3.32}$$

Thanks to the next Lemma 3.15, we can approximate π_∞ by

$$\begin{gathered}\pi_{\infty,n} = \rho_{\infty,n} \mathfrak{m}_\infty \otimes \mathfrak{m}_\infty,\ \rho_{\infty,n} \in L^\infty(\mathfrak{m}_\infty \otimes \mathfrak{m}_\infty),\ \pi_{\infty,n}(\{\bar\tau > 0\}) = 1,\ \pi_{\infty,n} \to \pi_\infty \text{ narrowly} \\ \lim_{n\to\infty} \mathrm{Ent}((P_1)_\sharp \pi_{\infty,n} | \mathfrak{m}_\infty) = \mathrm{Ent}(\mu_0^\infty | \mathfrak{m}_\infty), \quad \lim_{n\to\infty} \mathrm{Ent}((P_2)_\sharp \pi_{\infty,n} | \mathfrak{m}_\infty) = \mathrm{Ent}(\mu_1^\infty | \mathfrak{m}_\infty).\end{gathered} \tag{3.33}$$

Define then

$$\tilde\pi_{j,n}(dx_1 dx_2 dx_3 dx_4) := \rho_{\infty,n}(x_1, x_3)\, \boldsymbol{\gamma}_j(dx_1\, dx_2) \otimes \boldsymbol{\gamma}_j(dx_3\, dx_4), \quad \pi_{j,n} := (P_{24})_\sharp \tilde\pi_{j,n}, \tag{3.34}$$

and observe that $\pi_{j,n} \ll \mathfrak{m}_j \otimes \mathfrak{m}_j$ and that $\pi_{j,n} \to \pi_{\infty,n}$ narrowly as $j \to \infty$. By lower semicontinuity over open subsets, we have that $\liminf_{j\to\infty} \pi_{j,n}(\{\bar\tau > 0\}) \geq \pi_{\infty,n}(\{\bar\tau > 0\}) = 1$. Thus, calling $c_{j,n} := 1/\pi_{j,n}(\{\bar\tau > 0\})$ for j large enough, it holds that

$$\pi'_{j,n} := c_{j,n} \pi_{j,n} \llcorner \{\bar\tau > 0\} \to \pi_{\infty,n} \text{ narrowly} \quad \text{and} \quad \lim_{j\to\infty} \int \bar\tau^p\, \pi'_{j,n} = \int \bar\tau^p\, \pi_{\infty,n} > 0 \tag{3.35}$$

by Lemma 1.17. Let

$$(\mu')_0^{j,n} := (P_1)_\sharp \pi'_{j,n},\ (\mu')_1^{j,n} := (P_2)_\sharp \pi'_{j,n} \tag{3.36}$$

and notice that $\ell_p\big((\mu')_0^{j,n}, (\mu')_1^{j,n}\big) \in (0, \infty)$. Let

$$\pi''_{j,n} \in \Pi_{\leq}^{p\text{-opt}}\big((\mu')_0^{j,n}, (\mu')_1^{j,n}\big) \tag{3.37}$$

be an ℓ_p-optimal coupling (whose existence is ensured by Proposition 2.3).

Combining (3.35) with Lemma 1.15, with Prokhorov Theorem 1.14 and with the causal closeness of $\bar X$ we deduce that there exists $\hat\pi_{\infty,n} \in \Pi_{\leq}((P_1)_\sharp \pi_{\infty,n}, (P_2)_\sharp \pi_{\infty,n})$ such that, up to a subsequence, $\pi''_{j,n} \to \hat\pi_{\infty,n}$ narrowly as $j \to \infty$. Repeating once more the tightness argument, we deduce that there exists $\hat\pi_\infty \in \Pi_{\leq}(\mu_0^\infty, \mu_1^\infty)$ such that, up to a subsequence, $\hat\pi_{n,\infty} \to \hat\pi_\infty$ narrowly as $n \to \infty$. We conclude that there exist sequences $(n_k), (j_k)$ such that

$$\pi''_{j_k,n_k} \to \hat\pi_\infty \text{ narrowly and } \int \bar\tau^p\, \hat\pi_\infty = \lim_{k\to\infty} \int \bar\tau^p\, \pi''_{j_k,n_k} \geq \int \bar\tau^p\, \pi_\infty, \tag{3.38}$$

where the last chain of inequality follows from Lemma 1.17, (3.33), (3.35) and the optimality of π''_{j_k,n_k}. Combining (3.38) with the fact that $\pi_\infty \in \Pi_{\leq_\infty}^{p\text{-opt}}(\mu_0^\infty, \mu_1^\infty)$, we get that $\hat\pi_\infty \in \Pi_{\leq_\infty}^{p\text{-opt}}(\mu_0^\infty, \mu_1^\infty)$ as well. Since by assumption $(\mu_0^\infty, \mu_1^\infty)$ is strongly timelike p-dualisable, we infer that $\hat\pi_\infty\{\bar\tau > 0\} = 1$. Thus $\liminf_{k\to\infty} \pi''_{j_k,n_k}(\{\bar\tau > 0\}) \geq \hat\pi_\infty(\{\bar\tau > 0\}) = 1$. For k large enough, set

$$c''_k := 1/\pi''_{j_k,n_k}(\{\bar\tau > 0\}),\ \pi_k = c''_k \pi''_{j_k,n_k} \llcorner \{\bar\tau > 0\}, \quad \mu_0^k := (P_1)_\sharp \pi_k,\ \mu_1^k := (P_2)_\sharp \pi_k \tag{3.39}$$

and notice that

$$\pi_k \to \hat{\pi}_\infty, \quad \mu_0^k \to \mu_0^\infty, \quad \mu_1^k \to \mu_1^\infty \text{ narrowly.} \tag{3.40}$$

Since the restriction of an optimal coupling is optimal (Lemma 2.10), it follows that $\pi_k \in \Pi_{\leq}^{p\text{-opt}}(\mu_0^k, \mu_1^k)$ and by construction $\pi_k(\{\bar{\tau} > 0\} = 1$. We conclude that (μ_0^k, μ_1^k) is timelike p-dualisable by π_k and $\ell_p(\mu_0^k, \mu_1^k) \to \ell_p(\mu_0^\infty, \mu_1^\infty)$. Up to renaming the indices, the claim (3.31) follows.

Step 2. We prove that the sequences $(\mu_0^j), (\mu_1^j)$ constructed in Step 1 satisfy:

$$\limsup_{j\to\infty} \mathrm{Ent}(\mu_0^j|\mathfrak{m}_j) \leq \mathrm{Ent}(\mu_0^\infty|\mathfrak{m}_\infty), \quad \limsup_{j\to\infty} \mathrm{Ent}(\mu_1^j|\mathfrak{m}_j) \leq \mathrm{Ent}(\mu_1^\infty|\mathfrak{m}_\infty). \tag{3.41}$$

We divide this step into two substeps. Recall the definition (3.34) of $\tilde{\pi}_{j,n}(dx_1dx_2dx_3dx_4)$ and set

$$\mu_0^{j,n} := (P_2)_\sharp \tilde{\pi}_{j,n}, \quad \mu_1^{j,n} := (P_4)_\sharp \tilde{\pi}_{j,n}.$$

Step 2a. We first prove that:

$$\mathrm{Ent}(\mu_0^{j,n}|\mathfrak{m}_j) \leq \mathrm{Ent}((P_1)_\sharp \pi_{\infty,n}|\mathfrak{m}_\infty), \quad \mathrm{Ent}(\mu_1^{j,n}|\mathfrak{m}_j) \leq \mathrm{Ent}((P_2)_\sharp \pi_{\infty,n}|\mathfrak{m}_\infty), \quad \forall j,n \in \mathbb{N}. \tag{3.42}$$

We give the argument for the former in (3.42), the latter being completely analogous. The explicit expression (3.34) of $\tilde{\pi}_{j,n}(dx_1dx_2dx_3dx_4)$ combined with (3.33) and with Fubini's Theorem permits to write

$$\begin{aligned}
&(P_1)_\sharp \pi_{\infty,n} = \rho_0^{\infty,n}\mathfrak{m}_\infty; \quad \rho_0^{\infty,n}(x_1) = \int_X \rho_{\infty,n}(x_1,x_3)\,\mathfrak{m}_\infty(dx_3), \quad (P_1)_\sharp\pi_{\infty,n}\text{-a.e. } x_1 \in X_\infty;\\
&\mu_0^{j,n} = \rho_0^{j,n}\mathfrak{m}_j; \quad \rho_0^{j,n}(x_2) = \int_X \left(\int_{X^2} \rho_{\infty,n}(x_1,x_3)\,\boldsymbol{\gamma}_j(dx_3dx_4)\right)(\boldsymbol{\gamma}_j)_{x_2}(dx_1), \quad \mu_0^{j,n}\text{-a.e. } x_2 \in X_j;
\end{aligned} \tag{3.43}$$

where $\{(\boldsymbol{\gamma}_j)_{x_2}\}$ is the disintegration of $\boldsymbol{\gamma}_j$ with respect to P_2. Since $u(t) = t\log t$ is convex on $[0,\infty)$, Jensen's inequality gives:

$$\begin{aligned}
\mathrm{Ent}(\mu_0^{j,n}|\mathfrak{m}_j) &= \int_X u\big(\rho_0^{j,n}(x_2)\big)\,\mathfrak{m}_j(dx_2)\\
&\leq \int_X\int_X u\left(\int_{X^2} \rho_{\infty,n}(x_1,x_3)\,\boldsymbol{\gamma}_j(dx_3dx_4)\right)(\boldsymbol{\gamma}_j)_{x_2}(dx_1)\,\mathfrak{m}_j(dx_2)\\
&= \int_{X^2} u\big(\rho_0^{\infty,n}(x_1)\big)\,\boldsymbol{\gamma}_j(dx_1dx_2) = \int_X u\big(\rho_0^{\infty,n}(x_1)\,\mathfrak{m}_\infty(dx_1)\\
&= \mathrm{Ent}((P_1)_\sharp\pi_{\infty,n}|\mathfrak{m}_\infty).
\end{aligned}$$

Step 2b. We prove that the sequences (μ_0^j), (μ_1^j) constructed in Step 1 satisfy:

$$\limsup_{j\to\infty} \mathrm{Ent}(\mu_i^j|\mathfrak{m}_j) \leq \limsup_{k\to\infty} \mathrm{Ent}(\mu_i^{j_k,n_k}|\mathfrak{m}_{j_k}), \quad i = 0,1. \tag{3.44}$$

We give the argument for $i = 0$, the case $i = 1$ being completely analogous. From the construction of μ_0^k in Step 1 (see (3.34), (3.35), (3.36), (3.37), (3.39), see also (3.43)) it is not hard to check that $\mu_0^k = \rho_0^k \mathfrak{m}_{j_k}$, where $\rho_0^k \in L^\infty(\mathfrak{m}_{j_k})$ satisfies

$$0 \leq \rho_0^k \leq c_k\,\rho_0^{j_k,n_k} \leq c_k\|\rho_{\infty,n_k}\|_{L^\infty(\mathfrak{m}_\infty\otimes\mathfrak{m}_\infty)} \quad \forall k\in\mathbb{N}, \qquad c_k \to 1 \text{ as } k\to\infty. \tag{3.45}$$

The fact that $u(t) := t\log t$ is convex on $[0,\infty)$ and $u(0) = 0$, easily yields

$$u(t+h) - u(t) \geq u(h), \quad \forall t,h \in [0,\infty).$$

Thus, (3.45) combined with Jensen's inequality gives

$$\begin{aligned}\int u(c_k\rho_0^{j_k,n_k})\mathfrak{m}_{j_k} - \int u(\rho_0^k)\,\mathfrak{m}_{j_k} &\geq \int u(c_k\rho_0^{j_k,n_k} - \rho_0^k)\,\mathfrak{m}_{j_k} \\ &\geq u\left(\int (c_k\rho_0^{j_k,n_k} - \rho_0^k)\mathfrak{m}_{j_k}\right) \\ &= u\,(c_k - 1) \to 0 \text{ as } k \to \infty.\end{aligned} \tag{3.46}$$

The claim (3.44) follows immediately from (3.46). The claim (3.41) is a straightforward consequence of (3.42) combined with (3.33) and (3.44).

Step 3. Passing to the limit in the TCD condition.
For simplicity of presentation we give the argument for the $\mathsf{TCD}_p^e(0,N)$ condition, the one for general $K \in \mathbb{R}$ being analogous just a bit more cumbersome due to the distortion coefficients. Since for each $j \in \mathbb{N}$ the pair $(\mu_0^j, \mu_1^j) \in \big(\mathrm{Dom}(\mathrm{Ent}(\cdot|\mathfrak{m}_j))\big)^2 \subset \mathcal{P}(X_j)^2$ is timelike p-dualisable, the assumption that $(X_j, \mathsf{d}_j, \mathfrak{m}_j, \ll_j, \leq_j, \tau_j)$ satisfies the $\mathsf{TCD}_p^e(0,N)$ condition yields the existence of an ℓ_p-geodesic $(\mu_t^j)_{t\in[0,1]}$ such that

$$U_N(\mu_t^j|\mathfrak{m}_j) \geq (1-t)\,U_N(\mu_0^j|\mathfrak{m}_j) + t\,U_N(\mu_1^j|\mathfrak{m}_j), \quad \forall t \in [0,1],\ \forall j \in \mathbb{N}. \tag{3.47}$$

Since $\bar{X}$ is compact and non-totally imprisoning, from (the proof of) (2.24) we deduce that

$$\sup_{j\in\mathbb{N}} \mathrm{L}_{W_1^{(\bar{X},\bar{\mathsf{d}})}}\left((\mu_t^j)_{t\in[0,1]}\right) \leq \bar{C} < \infty.$$

By the metric Arzelá-Ascoli Theorem we deduce that there exists a limit continuous curve $(\mu_t^\infty)_{t\in[0,1]} \subset \mathcal{P}(X_\infty) \cap \mathcal{P}(\bar{X}, W_1^{(\bar{X},\bar{\mathsf{d}})})$ such that (up to a sub-sequence) $W_1^{(\bar{X},\bar{\mathsf{d}})}\left(\mu_t^j, \mu_t^\infty\right) \to 0$ and thus $\mu_t^j \to \mu_t^\infty$ narrowly in $\bar{X}$, as $j \to \infty$ for every $t \in [0,1]$.
Using that $\bar{\tau}$ is continuous and bounded, (3.31) and that $(\mu_t^j)_{t\in[0,1]}$ is an ℓ_p-geodesic, it follows that

$$\ell_p(\mu_0^\infty, \mu_t^\infty) \geq \lim_{j\to\infty} \ell_p(\mu_0^j, \mu_t^j) = t\,\lim_{j\to\infty} \ell_p(\mu_0^j, \mu_1^j) = t\,\ell_p(\mu_0^\infty, \mu_1^\infty). \tag{3.48}$$

By reverse triangle inequality, we get that the curve $(\mu_t^\infty)_{t\in[0,1]}$ is an ℓ_p-geodesic from μ_0^∞ to μ_1^∞.
The joint upper semicontinuity of U_N under narrow convergence (1.14) yields:

$$U_N(\mu_t^\infty|\mathfrak{m}_\infty) \geq \limsup_{j\in\mathbb{N}} U_N(\mu_t^j|\mathfrak{m}_j), \quad \forall t \in [0,1]. \tag{3.49}$$

The combination of (3.41), (3.47) and (3.49) gives that

$$U_N(\mu_t^\infty|\mathfrak{m}_\infty) \geq (1-t)\,U_N(\mu_0^\infty|\mathfrak{m}_\infty) + t\,U_N(\mu_1^\infty, \mathfrak{m}_\infty), \quad \forall t \in [0,1],$$

as desired. □

In the proof of Theorem 3.14 we made use of the following approximation result.

Lemma 3.15. *Let $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ be a locally causally closed, globally hyperbolic Lorentzian geodesic space.*
Let $\mu, \nu \in \mathcal{P}_c(X)$, $\mu, \nu \ll \mathfrak{m}$ such that there exists $\pi \in \Pi_{\leq}^{p\text{-}opt}(\mu,\nu)$ with $\pi(\{\tau > 0\}) = 1$.
Then there exists a sequence $(\pi_n) \subset \Pi_{\ll}(X^2)$ with the following properties:

1. *$\pi_n = \rho_n \mathfrak{m} \otimes \mathfrak{m} \ll \mathfrak{m} \otimes \mathfrak{m}$ with $\rho_n \in L^\infty(\mathfrak{m} \otimes \mathfrak{m})$;*
2. *$\pi_n \to \pi$ in the narrow convergence;*

3. If $(P_1)_\sharp\pi_n =: \mu_n = \rho_{\mu_n}\mathfrak{m}$ and $(P_2)_\sharp\pi_n =: \nu_n = \rho_{\nu_n}\mathfrak{m}$, it holds that $\rho_{\mu_n} \to \rho_\mu$ and $\rho_{\nu_n} \to \rho_\nu$ in $L^1(\mathfrak{m})$. Moreover,

$$\lim_{n\to\infty} \mathrm{Ent}(\mu_n|\mathfrak{m}) = \mathrm{Ent}(\mu|\mathfrak{m}), \quad \lim_{n\to\infty} \mathrm{Ent}(\nu_n|\mathfrak{m}) = \mathrm{Ent}(\nu|\mathfrak{m}). \tag{3.50}$$

Proof. **Step 1.** Basic approximation by product measures.
First, we cover $\{\tau > 0\}$ with a countable family of products of open subsets $A_i \times B_i \subset \{\tau > 0\}$:

$$\{\tau > 0\} = \bigcup_{i\in\mathbb{N}} A_i \times B_i, \quad \text{with } A_i, B_i \subset X \text{ open subsets.} \tag{3.51}$$

Let $\bar\pi_n := \pi_{\llcorner \cup_{i\le n} A_i\times B_i}$ and define $\pi_n := \bar\pi_n - \bar\pi_{n-1}$, $\pi_0 = 0$. We have the following decomposition:

$$\pi = \sum_{n\in\mathbb{N}} \pi_n, \qquad \pi_n \perp \pi_m, \qquad \pi_n(\{\tau > 0\} \setminus A_n \times B_n) = 0.$$

For $n \ge 1$, consider

$$\mu_n := (P_1)_\sharp\pi_n, \quad \nu_n := (P_2)_\sharp\pi_n, \quad \eta_n := \mu_n \otimes \nu_n/\pi_n(X^2).$$

Observe that $\mu_n(X \setminus A_n) = \nu_n(X \setminus B_n) = \eta_n(X^2 \setminus \{\tau > 0\}) = 0$ and, by linearity of projections, $\mu = \sum_{n\in\mathbb{N}} \mu_n, \quad \nu = \sum_{n\in\mathbb{N}} \nu_n$. Notice moreover that the factor $1/\pi_n(X^2)$ in the definition of η_n is necessary to obtain that $(P_1)_\sharp\eta_n = \mu_n$ and $(P_2)_\sharp\eta_n = \nu_n$. Finally, set $\eta := \sum_{n\in\mathbb{N}} \eta_n$ and note that

$$\eta \in \Pi_\le(\mu,\nu), \quad \eta(X^2_\ll) = 1, \quad \eta \ll \mathfrak{m}\otimes\mathfrak{m}.$$

Notice moreover that, writing $\eta = \rho\, \mathfrak{m}\otimes\mathfrak{m}$, then

$$\rho(x,y) = \rho_{\mu_n}(x)\,\rho_{\nu_n}(y) \le \rho_\mu(x)\,\rho_\nu(y), \quad \eta\text{-a.e. } (x,y) \in A_n \times B_n,$$

where ρ_μ (respectively ρ_ν, ρ_{μ_n}, ρ_{ν_n}) is the density of μ (resp. ν, μ_n, ν_n)) with respect to $\mathfrak{m}$.

Step 2. We iterate the construction taking finer coverings of the form (3.51) to obtain a sequence (η_m) converging in the narrow topology to π.
Fix any $f, g \in C_b(X)$ and observe that

$$\begin{aligned}\int_{X^2} f(x)g(y)\pi(dxdy) &- \int_{X^2} f(x)g(y)\eta(dxdy)\\ &= \sum_{n=1}^{\infty} \int_{X^2} f(x)g(y)\pi_n(dxdy) - \int_X f(x)\mu_n(dx)\int_X g(y)\frac{\nu_n(dy)}{\pi_n(X^2)}.\end{aligned} \tag{3.52}$$

Since μ,ν have compact support, we have that f, g are uniformly continuous on $\operatorname{supp}\mu \cup \operatorname{supp}\nu \Subset X$. Given any $(x_n, y_n) \in A_n \times B_n$, we estimate

$$\left|\int_{X^2} f(x)g(y)\pi_n(dxdy) - f(x_n)g(y_n)\pi_n(X^2)\right| \le \varepsilon\pi_n(X^2)(\|f\|_\infty + \|g\|_\infty), \tag{3.53}$$

where ε is the modulus of continuity of both f and g over A_n and B_n respectively. Analogously,

$$\begin{aligned}&\left|\int_X f(x)\mu_n(dx)\int_X g(y)\frac{\nu_n(dy)}{\pi_n(X^2)} - f(x_n)g(y_n)\pi_n(X^2)\right|\\ &= \left|\int_X f(x)\mu_n(dx)\int_X g(y)\frac{\nu_n(dy)}{\pi_n(X^2)} - f(x_n)\mu_n(X)g(y_n)\right|\\ &\le \left|\int_X (f(x)-f(x_n))\mu_n(dx)\int_X g(y)\frac{\nu_n(dy)}{\pi_n(X^2)}\right| + |f(x_n)\mu_n(X)|\left|\int_X g(y)\frac{\nu_n(dy)}{\pi_n(X^2)} - g(y_n)\right|\\ &\le \varepsilon\mu_n(X)\|g\|_\infty + \varepsilon\mu_n(X)\|f\|_\infty = \varepsilon\pi_n(X)(\|g\|_\infty + \|f\|_\infty).\end{aligned} \tag{3.54}$$

Combining (3.52), (3.53) and (3.54), we obtain

$$\left| \int_{X^2} f(x)g(y)\pi - \int_{X^2} f(x)g(y)\eta \right| \leq 2\varepsilon(\|g\|_\infty + \|f\|_\infty),$$

where ε is the modulus of continuity of both f and g over subsets of $\operatorname{supp}\mu \cup \operatorname{supp}\nu \Subset X$ with diameter at most $\sup_{n\in\mathbb{N}} \max\{\operatorname{diam}(A_n), \operatorname{diam}(B_n)\}$.
Then, considering finer and finer open coverings

$$\{\tau > 0\} = \bigcup_{i\in\mathbb{N}} A_i^m \times B_i^m, \text{ with } A_i^m, B_i^m \subset X \text{ open sets, } \lim_{m\to\infty} \sup_{i\in\mathbb{N}} \max\{\operatorname{diam}(A_i^m), \operatorname{diam}(B_i^m)\} = 0,$$

and the corresponding measures η_m constructed in Step 1, it holds

$$\eta_m \in \Pi_{\leq}(\mu,\nu), \quad \eta_m(X^2_{\ll}) = 1, \quad \eta_m \ll \mathfrak{m}\otimes\mathfrak{m}, \quad \eta_m \to \pi \text{ narrowly.}$$

Step 3. Conclusion by truncation and dominated convergence Theorem.
Let $\eta_m = \rho_m \mathfrak{m}\otimes\mathfrak{m}$ be the sequence constructed in Step 2. For any $C > 0$ define

$$\eta_m^C := \alpha_{C,m} \min\{\rho_m, C\}\, \mathfrak{m}\otimes\mathfrak{m},$$

where $\alpha_{C,m}$ is the normalization constant. It is standard to check that $\eta_m^C \to \eta_m$ narrowly as $C \to \infty$. By a diagonal argument we obtain a sequence $\eta_m^{C_m} \to \pi$ narrowly for some $C_m \to \infty$. Define

$$\mu_m := (P_1)_\sharp \eta_m^{C_m}, \quad \nu_m := (P_2)_\sharp \eta_m^{C_m}.$$

Writing $\mu_m = \rho_{\mu,m}\mathfrak{m}$, it holds

$$\rho_{\mu,m}(x) = \alpha_{C_m,m} \int_X \min\{\rho_m(x,y), C_m\}\mathfrak{m}(dy) \leq \alpha_{C_m,m} \int_X \rho_m(x,y)\mathfrak{m}(dy) = \alpha_{C_m,m}\rho_\mu(x), \tag{3.55}$$

where the last identity follows from $(P_1)_\sharp\eta_m = \mu$ for any $m \in \mathbb{N}$. Hence, by dominated convergence Theorem, $\rho_{\mu,m}(x)/\alpha_{C_m,m}$ is converging to $\rho_\mu(x)$ in the stronger $L^1(\mathfrak{m})$ norm, as $m \to \infty$.

For the last claim (3.50), without loss of generality we can assume $\operatorname{Ent}(\mu|\mathfrak{m}) < \infty$ (otherwise it is trivial). From dominated convergence Theorem and (3.55), we deduce that $\operatorname{Ent}(\mu_m|\mathfrak{m}) \to \operatorname{Ent}(\mu|\mathfrak{m})$. To conclude the proof, it is enough to repeat the last arguments also for ν. □

Remark 3.16. Recalling from Theorem 3.1 that smooth globally hyperbolic spacetimes of dimension $\leq N$ with timelike Ricci curvature bounded below by $K \in \mathbb{R}$ satisfy $\mathsf{TCD}_p^e(K,N)$, Theorem 3.14 yields that their limit spaces (in the sense of Therem 3.14) satisfy $\mathsf{wTCD}_p^e(K,N)$.

3.4 Optimal maps in timelike non-branching $\mathsf{TMCP}_p^e(K,N)$ spaces

In this section we prove some results about existence of optimal transport maps in timelike non-branching $\mathsf{TMCP}_p^e(K,N)$ spaces, from which we will deduce the uniqueness of ℓ_p-geodesics (this section should be compared with [18] where the analogous results were obtained for metric-measure spaces satisfying $\mathsf{MCP}(K,N)$ and essentially non-branching).

Lemma 3.17. *Let $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ be a timelike non-branching, $\mathcal{K}$-globally hyperbolic, Lorentzian geodesic space satisfying $\mathsf{TMCP}_p^e(K,N)$ for some $p \in (0,1), K \in \mathbb{R}, N \in (0,\infty)$. Let $\mu_0 \in \mathcal{P}_c(X)$, with $\mu_0 \in \operatorname{Dom}(\operatorname{Ent}(\cdot|\mathfrak{m}))$ and μ_1 be a finite convex combination of Dirac masses, i.e. $\mu_1 := \sum_{j=1}^n \lambda_j\delta_{x_j}$ for some $\{x_j\}_{j=1,\dots,n} \subset X$ with $x_i \neq x_j$ for $i \neq j$, and $\{\lambda_j\}_{j=1,\dots,n} \subset (0,1]$ with $\sum_{j=1}^n \lambda_j = 1$. Assume that there exists $\pi \in \Pi_{\leq}^{p\text{-}opt}(\mu_0,\mu_1)$ such that $\operatorname{supp}\pi \Subset \{\tau > 0\}$.*

Then π is the unique element in $\Pi_{\leq}^{p\text{-}opt}(\mu_0,\mu_1) = \{\pi\}$ with $\operatorname{supp}\pi \Subset \{\tau > 0\}$. Moreover such a π is induced by a map T, i.e. $\pi = (\mathrm{Id}, T)_\sharp\mu_0$ and

$$\ell_p(\mu_0,\mu_1)^p = \int_X \tau(x, T(x))^p\, \mu_0(dx).$$

Proof. We first show that π is induced by a map, the uniqueness will follow.

Consider the set

$$S := \{x \in X \,:\, \exists x_i \neq x_j \ \text{ with } (x, x_i), (x, x_j) \in \operatorname{supp} \pi\} \subset \operatorname{supp} \mu_0, \tag{3.56}$$

and, since $\operatorname{supp} \mu_0$ is compact, S is easily seen to be a closed set and therefore compact. It will be enough to prove the stronger statement $\mu_0(S) = 0$.

Suppose by contradiction $\mu_0(S) > 0$. Since μ_1 is a finite sum of Dirac masses, up to taking a smaller S and up to relabelling the points x_j, we can assume the existence of

$$T_1, T_2 : S \to X, \qquad \operatorname{graph}(T_1),\ \operatorname{graph}(T_2) \subset \operatorname{supp} \pi,$$

both μ_0-measurable with $T_1(x) = x_1$ and $T_2(x) = x_2$ for all $x \in S$, with $x_1 \neq x_2$.

Possibly restricting to a subset of S, still of positive $\mathfrak{m}$-measure, we also assume that if $1/C \leq \rho_0 \leq C$ over S, where ρ_0 is the density of μ_0 with respect to $\mathfrak{m}$. Thanks to Lemma 2.10 the couplings

$$\frac{\chi_{S \times \{x_1\}}}{\rho_0 \mathfrak{m}(S)} \pi, \quad \frac{\chi_{S \times \{x_2\}}}{\rho_0 \mathfrak{m}(S)} \pi, \tag{3.57}$$

are optimal. Hence, with no loss of generality, we can redefine $\mu_0 := \mathfrak{m}\llcorner_S / \mathfrak{m}(S)$ and consider $\eta^1 \in \operatorname{OptGeo}_{\ell_p}(\mu_0, \delta_{x_1})$ and $\eta^2 \in \operatorname{OptGeo}_{\ell_p}(\mu_0, \delta_{x_2})$ given by Proposition 2.32.

Necessarily $\operatorname{supp} \eta^1 \cap \operatorname{supp} \eta^2 = \emptyset$; indeed for $i = 1, 2$ it holds $\eta^i(\{\gamma \colon \gamma_1 = x_i\}) = 1$ and by construction $x_1 \neq x_2$. Thus, again by Proposition 2.32, it holds

$$(\mathrm{e}_t)_\sharp \eta^1 \perp (\mathrm{e}_t)_\sharp \eta^2, \quad \forall t \in (0, 1]. \tag{3.58}$$

The $\mathsf{TMCP}^e_p(K, N)$ condition (3.13) gives that (see (3.14)), for $i = 1, 2$,

$$\int \rho^i_t \log(\rho^i_t)\, \mathfrak{m} \leq -\log(\mathfrak{m}(S)) - N \log(\sigma^{(1-t)}_{K/N}(\|\tau\|_{L^2((\mathrm{e}_0, \mathrm{e}_1)_\sharp \eta_i)})), \quad \forall t \in [0, 1),\ i = 1, 2, \tag{3.59}$$

where we have written $(\mathrm{e}_t)_\sharp \eta^i = \rho^i_t \mathfrak{m}$. By Jensen's inequality (1.13) we have

$$\int_X \rho^i_t \log(\rho^i_t)\, \mathfrak{m} \geq \, -\log(\mathfrak{m}(\{\rho^i_t > 0\}))$$

which, combined with (3.59), gives

$$\liminf_{t \to 0} \mathfrak{m}\left(\{\rho^i_t > 0\}\right) \geq \mathfrak{m}\,(S) = \mathfrak{m}\left(\{\rho^i_0 > 0\}\right). \tag{3.60}$$

Denote now

$$E := \bigcup_{t \in [0,1], i = 1,2} \operatorname{supp} (\mathrm{e}_t)_\sharp \eta^i$$
$$S^\varepsilon_E := \{y \in E \,:\, \tau(x, y) \leq \varepsilon \text{ for some } x \in S\}$$

and notice that, by $\mathcal{K}$-global hyperbolicity, E (and thus also S^ε_E) is a compact subset of X. Moreover, by Dominated Convergence Theorem, we have $\lim_{\varepsilon \to 0} \mathfrak{m}(S^\varepsilon_E) = \mathfrak{m}(S)$. In particular there exists $\varepsilon_0 > 0$ such that

$$\mathfrak{m}(S^{\varepsilon_0}_E) \leq \frac{3}{2} \mathfrak{m}(S). \tag{3.61}$$

We now claim that there exists a small $t_0 > 0$, such that

$$\mathfrak{m}\left(\{\rho^1_{t_0} > 0\} \cap \{\rho^2_{t_0} > 0\}\right) > 0. \tag{3.62}$$

To this aim notice that, by construction, for $(\mathrm{e}_t)_\sharp \eta^i$-a.e. $x \in X$ there exists a timelike geodesic $\gamma \in \operatorname{TGeo}(X)$ such that $x = \gamma_t$, $\gamma_0 \in S$, $\gamma_1 = x^i$, $i = 1, 2$; in particular, for $t \in [0, \varepsilon_0]$ the measure

$(\mathrm{e}_t)_\sharp\eta^i$ is concentrated on $S_E^{\varepsilon_0}$. But then the combination of (3.60) and (3.61) implies that there exists $t_0 \in (0, \varepsilon_0)$ satisfying the claim (3.62).
Observing that (3.62) contradicts (3.58), we conclude that π is induced by a map.

We now show that there exists a unique element $\pi \in \Pi_{\leq}^{p\text{-opt}}(\mu_0, \mu_1)$ satisfying $\mathrm{supp}\,\pi \Subset \{\tau > 0\}$. Assume by contradiction that there exist $\pi_1, \pi_2 \in \Pi_{\leq}^{p\text{-opt}}(\mu_0, \mu_1)$ satisfying $\mathrm{supp}\,\pi_1, \mathrm{supp}\,\pi_2 \Subset \{\tau > 0\}$ with $\pi_1 \neq \pi_2$. By the first part of the proof, we know that there exist maps $T_1, T_2 : X \to X$ such that $\pi_i = (\mathrm{Id}, T_i)_\sharp\mu_0$; in particular $T_1 \neq T_2$ on a μ_0-nonnegligible subset. It is straightforward to check that $\pi := \frac{1}{2}(\pi_1 + \pi_2)$ satisfies $\pi \in \Pi_{\leq}^{p\text{-opt}}(\mu_0, \mu_1), \mathrm{supp}\,\pi \Subset \{\tau > 0\}$ and that π cannot be induced by a map. This contradicts the first part of the proof. □

Proposition 3.18. *Let $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ be a timelike non-branching, locally causally closed, $\mathcal{K}$-globally hyperbolic, Lorentzian geodesic space satisfying $\mathsf{TMCP}^e_p(K, N)$ for some $p \in (0, 1), K \in \mathbb{R}, N \in (0, \infty)$. Let $\mu_0, \mu_1 \in \mathcal{P}_c(X)$, with $\mu_0 \in \mathrm{Dom}(\mathrm{Ent}(\cdot|\mathfrak{m}))$. Assume that there exists $\pi \in \Pi_{\leq}^{p\text{-opt}}(\mu_0, \mu_1)$ such that* $\mathrm{supp}\,\pi \Subset \{\tau > 0\}$.

Then there exist $\hat{\pi} \in \Pi_{\leq}^{p\text{-opt}}(\mu_0, \mu_1)$ and an ℓ_p-geodesic $(\mu_t)_{t\in[0,1]}$ from μ_0 to μ_1 satisfying

$$U_N(\mu_t|\mathfrak{m}) \geq \sigma_{K/N}^{(1-t)}(\|\tau\|_{L^2(\hat{\pi})})\, U_N(\mu_0|\mathfrak{m}), \quad \forall t \in [0, 1). \tag{3.63}$$

In particular $\mu_t \ll \mathfrak{m}$ for all $t \in [0, 1)$.

Proof. **Step 1.** Additionally assume $\mathrm{supp}\,\mu_0 \times \mathrm{supp}\,\mu_1 \subset \{\tau > 0\}$.
If $\mathrm{supp}\,\mu_1$ is made of finitely many points, an easier variant of the following arguments give the result (more precisely it is enough to take n to be the number of points in $\mathrm{supp}\,\mu_1$ and stop at the end of Step 2). Thus without loss of generality we can assume that $\mathrm{supp}\,\mu_1$ contains infinitely many points.
Let $B_i \subset \mathrm{supp}\,\mu_1$, $i = 1, \dots, n$ be a finite Borel partition of $\mathrm{supp}\,\mu_1$ with $\mu_1(B_i) > 0$ for each i. For every i pick a point $x_1^i \in B_i$ and define

$$\bar{\mu}_1 := \sum_{i=1}^n a_i \delta_{x_1^i},$$

where $a_i := \mu_1(B_i)$. Since $\mathrm{supp}\,\mu_0 \times \mathrm{supp}\,\mu_1 \subset \{\tau > 0\}$, there exists $\bar{\pi} \in \Pi_{\leq}^{p\text{-opt}}(\mu_0, \bar{\mu}_1)$ such that $\mathrm{supp}\,\bar{\pi} \Subset \{\tau > 0\}$. Let $T : X \to X$ be the $\ell_p(\mu_0, \bar{\mu}_1)$-optimal map associated to $\bar{\pi}$ by Lemma 3.17 and define $A_i := T^{-1}(x_1^i)$. Observe that the sets $C_i = A_i \times \{x_1^i\}$ satisfy $C_i \Subset \{\tau > 0\}$ and form a finite Borel partition of $\mathrm{supp}\,\bar{\pi}$. Set $\bar{\pi}^i := \frac{1}{a_i}\bar{\pi}\llcorner_{C_i}$ and

$$\bar{\mu}_0^i := (P_1)_\sharp\bar{\pi}^i, \qquad \bar{\mu}_1^i := (P_2)_\sharp\bar{\pi}^i = \delta_{x_1^i}.$$

Note that, by construction, $\mu_0 = \sum_i a_i \bar{\mu}_0^i$ and

$$\bar{\mu}_0^i \perp \bar{\mu}_0^j, \quad \forall i \neq j. \tag{3.64}$$

Noting that $\mathrm{supp}\,\bar{\pi}^i \Subset \{\tau > 0\}$, the $\mathsf{TMCP}^e_p(K, N)$ condition ensures that there exists an ℓ_p-geodesic $(\bar{\mu}_t^i)_{t\in[0,1]}$ from $\bar{\mu}_0^i$ to $\bar{\mu}_1^i$ satisfying

$$U_N(\bar{\mu}_t^i|\mathfrak{m}) \geq \sigma_{K/N}^{(1-t)}(\|\tau(\cdot, x_1^i)\|_{L^2(\bar{\mu}_0^i)})\, U_N(\bar{\mu}_0^i|\mathfrak{m}), \quad \forall i = 1, \dots, n,\ t \in [0, 1). \tag{3.65}$$

Step 2. Taking the logarithm of (3.65) and summing over i (recalling that $\sum_i a_i = 1$), we obtain

$$-\frac{1}{N}\sum_i a_i \mathrm{Ent}(\bar{\mu}_t^i|\mathfrak{m}) \geq \sum_i a_i \log\left(\sigma_{K/N}^{(1-t)}(\|\tau(\cdot, x_1^i)\|_{L^2(\bar{\mu}_0^i)})\right) - \frac{1}{N}\sum_i a_i \mathrm{Ent}(\bar{\mu}_0^i|\mathfrak{m}). \tag{3.66}$$

Call $\bar{\eta}^i \in \mathrm{OptGeo}_{\ell_p}(\bar{\mu}_0^i, \bar{\mu}_1^i)$ the ℓ_p-optimal dynamical plan representing the ℓ_p-geodesic $(\bar{\mu}_t^i)_{t\in[0,1]}$ given by Proposition 2.32 point 2. Since by construction $\bar{\mu}_1^i = \delta_{x_1^i}$ and $x_1^i \neq x_1^j$ for $i \neq j$, it follows that $\mathrm{supp}\,\bar{\eta}^i \cap \mathrm{supp}\,\bar{\eta}^j = \emptyset$ for $i \neq j$. Proposition 2.32 point 6 implies that

$$\bar{\mu}_t^i \perp \bar{\mu}_t^j, \quad \forall t \in (0, 1), \quad \forall i \neq j. \tag{3.67}$$

Calling $\bar\mu_t := \sum_i a_i \bar\mu_t^i$ and using (3.64), (3.67) it follows that

$$\mathrm{Ent}(\bar\mu_t|\mathfrak{m}) = \mathrm{Ent}\left(\sum_i a_i\bar\mu_t^i|\mathfrak{m}\right) = \sum_i a_i \mathrm{Ent}(\bar\mu_t^i|\mathfrak{m}) + \sum_i a_i\log(a_i), \quad \forall t\in[0,1). \tag{3.68}$$

Hence adding $-\frac{1}{N}\sum_i a_i\log(a_i)$ to both sides of (3.66), using (3.68), and the convexity of the function $(-\infty,\pi^2)\ni k\to\log\sigma_k^{(t)}(1)$ (recall that $\sigma_k^{(t)}(\vartheta)=\sigma_{k\vartheta^2}^{(t)}(1)$) we obtain

$$U_N(\bar\mu_t|\mathfrak{m}) \ge \sigma_{K/N}^{(1-t)}(\|\tau\|_{L^2(\bar\pi)})\, U_N(\mu_0|\mathfrak{m}), \quad \forall t\in[0,1).$$

Step 3. Taking finer partitions of $\operatorname{supp}\mu_1$ we can construct a sequence $\{\bar\mu_1^k\}_{k\in\mathbb{N}}\subset\mathcal{P}_c(X)$ such that each $\bar\mu_1^k$ is a finite convex combination of Dirac masses, $\operatorname{supp}\bar\mu_1^k\subset\operatorname{supp}\mu_1$ for each k, and $\bar\mu_1^k\to\mu_1$ narrowly. We then invoke Theorem 2.16 to obtain another sequence, that we still denote by $\bar\mu_1^k$, that is converging to μ_1 narrowly and which is absolutely continuous with the previous $\bar\mu_1^k$, hence still obtained as a finite convex combination of Dirac deltas.

For each k let $(\bar\mu_t^k = (\mathrm{e}_t)_\sharp\bar\eta^k)_{t\in[0,1]}$ be the ℓ_p-geodesic from μ_0 to $\bar\mu_1^k$ and $\bar\pi_k\in\Pi_\le^{p\text{-opt}}(\mu_0,\bar\mu_1^k)$ the optimal coupling constructed in Step 2 satisfying

$$U_N(\bar\mu_t^k|\mathfrak{m}) \ge \sigma_{K/N}^{(1-t)}(\|\tau\|_{L^2((\mathrm{e}_0,\mathrm{e}_1)_\sharp\bar\eta^k)})\, U_N(\mu_0|\mathfrak{m}), \quad \forall t\in[0,1). \tag{3.69}$$

Notice indeed that by construction, $(\mathrm{e}_0,\mathrm{e}_1)_\sharp\bar\eta^k=\bar\pi_k$.

We aim to construct a limit ℓ_p-geodesic $(\mu_t)_{t\in[0,1]}$ from μ_0 to μ_1 satisfying (3.63). First of all notice that by $\mathcal{K}$-global hyperbolicity,

$$\bar K := \bigcup_{t\in[0,1]} \mathfrak{I}(\operatorname{supp}\mu_0,\operatorname{supp}\mu_1,t)\Subset X$$

is a compact subset, see (1.6),(1.7). It is easily seen that

$$\operatorname{supp}\bar\mu_t^k\subset\mathfrak{I}(\operatorname{supp}\mu_0,\operatorname{supp}\mu_1,t)\subset\bar K, \quad \forall t\in[0,1],\ k\in\mathbb{N}. \tag{3.70}$$

From (2.24) we deduce that

$$\sup_{k\in\mathbb{N}} \mathrm{L}_{W_1}\left((\bar\mu_t^k)_{t\in[0,1]}\right)\le\bar C<\infty.$$

By the metric Arzelá-Ascoli Theorem we deduce that there exists a limit continuous curve $(\mu_t)_{t\in[0,1]}\subset(\mathcal{P}(\bar K),W_1)$ such that (up to a sub-sequence) $W_1\left(\bar\mu_t^k,\mu_t\right)\to0$ and thus $\bar\mu_t^k\to\mu_t$ narrowly, as $n\to\infty$. Recalling the assumption of $\pi\in\Pi_\le^{p\text{-opt}}(\mu_0,\mu_1)$ with $\operatorname{supp}\pi\Subset\{\tau>0\}$, Theorem 2.16 and Lemma 2.12 yield that

$$t\ell_p(\mu_0,\mu_1) = t\lim_{k\to\infty}\ell_p(\mu_0,\bar\mu_1^k) = \lim_{k\to\infty}\ell_p(\mu_0,\bar\mu_t^k)\le\ell_p(\mu_0,\mu_t). \tag{3.71}$$

Thus, by reverse triangle inequality, the curve $(\mu_t)_{t\in[0,1]}$ is an ℓ_p-geodesic from μ_0 to μ_1 and any narrow limit $\hat\pi$ of $(\bar\pi_k)$ is ℓ_p-optimal. The upper-semicontinuity (3.4) of $U_N(\cdot|\mathfrak{m})$ in narrow topology yields

$$\limsup_{j\to\infty} U_N(\bar\mu_t^{k_j}|\mathfrak{m})\le U_N(\mu_t|\mathfrak{m}), \quad \forall t\in[0,1). \tag{3.72}$$

Combining (3.69) and (3.72) gives the desired (3.63).

Step 4. Removing the assumption $\operatorname{supp}\mu_0\times\operatorname{supp}\mu_1\subset\{\tau>0\}$.
By assumption there exists $\pi\in\Pi_\le^{p\text{-opt}}(\mu_0,\mu_1)$ such that $\operatorname{supp}\pi\Subset\{\tau>0\}$. Since $\operatorname{supp}\pi$ is compact, we can find finitely many products of open subsets $A_i\times B_i\Subset\{\tau>0\}$, $i=1,\dots,n$, such that $\operatorname{supp}\pi\subset\bigcup_{i=1}^n A_i\times B_i$. Argueing by induction over $n\in\mathbb{N}$ noticing that

$$\bigcup_{i=1}^n A_i\times B_i = \left(\left(A_n\setminus\bigcup_{i=1}^{n-1}A_i\right)\times B_n\right)\cup\left(\bigcup_{i=1}^{n-1}(A_i\cap A_n)\times(B_i\cup B_n)\right)\cup\left(\bigcup_{i=1}^{n-1}(A_i\setminus A_n)\times B_i\right),$$

it is easy to see that we can assume with no loss in generality that $A_i \cap A_j = \emptyset$, provided we admit A_i to be Borel. In this way we obtain that $\operatorname{supp}\pi \subset \bigcup_{i=1}^n A_i \times B_i$ and $\operatorname{supp}\mu_0 \subset \bigcup_{i=1}^n A_i$ are both finite Borel pairwise disjoint unions with $A_i \times B_i \Subset \{\tau > 0\}$ for every $i = 1, \dots, n$. Up to taking a subset of indices, we can assume that $\pi(A_i \times B_i) > 0$, for all $i = 1, \dots, n$.

Setting $\bar{\pi}_i := \pi\llcorner_{A_i \times B_i}$, we obtain the following decomposition:

$$\pi = \sum_{i \le n} \bar{\pi}_i, \qquad \bar{\pi}_i \perp \bar{\pi}_j \text{ for } i \neq j, \qquad \bar{\pi}_i(\{\tau > 0\} \setminus A_i \times B_i) = 0.$$

Finally, set $\pi_i := \bar{\pi}_i / \bar{\pi}_i(X \times X)$ and $\mu_{0,i} := (P_1)_\sharp \pi_i$, $\mu_{1,i} := (P_2)_\sharp \pi_i$. Clearly, it holds $\mu_{0,i} \perp \mu_{0,j}$ if $i \neq j$. By restriction property, $\pi_i \in \Pi_{\ll}^{p\text{-opt}}(\mu_{0,i}, \mu_{1,i})$ and we can apply the previous part of the proof to the marginals $\mu_{0,i}, \mu_{1,i}$: there exists $\hat{\pi}_i \in \Pi_{\le}^{p\text{-opt}}(\mu_{0,i}, \mu_{1,i})$ and an ℓ_p-geodesic $(\mu_{t,i})_{t \in [0,1]}$ from $\mu_{0,i}$ to $\mu_{1,i}$ satisfying

$$U_N(\mu_{t,i}|\mathfrak{m}) \ge \sigma_{K/N}^{(1-t)}(\|\tau\|_{L^2(\hat{\pi}_i)})\, U_N(\mu_{0,i}|\mathfrak{m}), \quad \forall t \in [0,1).$$

In particular $\mu_{t,i} \ll \mathfrak{m}$ for all $t \in [0,1)$. We can then sum over i the previous inequality and, reasoning like in Step 2 by using mutual orthogonality of $\mu_{0,i}$, we have the claim. □

Theorem 3.19. *Let $(X, \mathsf{d}, \mathfrak{m}, \ll, \le, \tau)$ be a timelike non-branching, locally causally closed, $\mathcal{K}$-globally hyperbolic, Lorentzian geodesic space satisfying $\mathsf{TMCP}_p^e(K,N)$ for some $p \in (0,1), K \in \mathbb{R}, N \in (0,\infty)$. Let $\mu_0, \mu_1 \in \mathcal{P}_c(X)$, with $\mu_0 \in \mathrm{Dom}(\mathrm{Ent}(\cdot|\mathfrak{m}))$. Assume that there exists $\pi \in \Pi_{\le}^{p\text{-}opt}(\mu_0, \mu_1)$ such that $\pi(\{\tau > 0\}) = 1$.*

Then there exists a unique optimal coupling $\pi \in \Pi_{\le}^{p\text{-}opt}(\mu_0, \mu_1)$ such that $\pi(\{\tau > 0\}) = 1$ and it is induced by a map T, i.e. $\pi = (\mathrm{Id}, T)_\sharp \mu_0$ and

$$\ell_p(\mu_0, \mu_1)^p = \int_X \tau(x, T(x))^p \, \mu_0(dx).$$

Proof. The arguments are along the same lines of the proof of Lemma 3.17 but with some (non-completely trivial) modifications that we briefly discuss.

Step 1. Let $\Gamma \subset X^2_{\ll}$ be an ℓ_p-monotone subset such that $\pi(\Gamma) = 1$, given by Proposition 2.8. Define

$$\Gamma(x) := P_2\Big(\Gamma \cap (\{x\} \times X)\Big), \tag{3.73}$$

and S the set of those $x \in X$ such that $\Gamma(x)$ is not a singleton. Note that the set S is Suslin. It will be enough to prove the stronger statement $\mu_0(S) = 0$.

So suppose by contradiction $\mu_0(S) > 0$. By Von Neumann Selection Theorem, there exists

$$T_1, T_2 : S \to X, \qquad \mathrm{graph}(T_1),\ \mathrm{graph}(T_2) \subset \Gamma,$$

both μ_0-measurable and $\mathsf{d}(T_1(x), T_2(x)) > 0$, for all $x \in S$. By Lusin Theorem, there exists a compact set $S_1 \subset S$ such that the maps T_1 and T_2 are both continuous when restricted to S_1 and $\mu_0(S_1) > 0$. In particular

$$\inf_{x \in S_1} \mathsf{d}(T_1(x), T_2(x)) = \min_{x \in S_1} \mathsf{d}(T_1(x), T_2(x)) = 2r > 0.$$

Then one can deduce the existence of a couple of points $x_1, x_2 \in X$, of a positive $r > 0$ and of a compact set $S_2 \subset S_1$, again with $\mu_0(S_2) > 0$, such that

$$\{T_1(x) \colon x \in S_2\} \subset B_r(x_1), \qquad \{T_2(x) \colon x \in S_2\} \subset B_r(x_2),$$

with $\mathsf{d}(x_1, x_2) > 2r$, where $B_r(x_i)$ is the open ball centred in x_i and radius r, for $i = 1, 2$ with respect to d. By the continuity of τ, up to further reducing $r > 0$, we can also suppose that $S_2 \times (B_r(x_1) \cup B_r(x_2)) \Subset \{\tau > 0\}$.

Step 2. Following the arguments of the proof of Lemma 3.17 (see in particular (3.57)), we can invoke Lemma 2.10 and assume with no loss of generality μ_0 to be restricted and renormalised to S_2. In particular we redefine $\mu_0 := \mathfrak{m}\llcorner_{S_2} / \mathfrak{m}(S_2)$; the following measures are well defined as well

$$\mu_1^1 := (T_1)_\sharp \mu_0, \qquad \mu_1^2 := (T_2)_\sharp \mu_0;$$

in particular μ_1^1, μ_1^2 are Borel probability measures with $\operatorname{supp}\mu_1^1 \cap \operatorname{supp}\mu_1^2 = \emptyset$.
By Proposition 3.18 we know there exist ℓ_p-geodesics $(\mu_t^i)_{t\in[0,1]}$ from μ_0 to μ_1^i, $i = 1, 2$, satisfying

$$U_N(\mu_t^i|\mathfrak{m}) \geq \sigma_{K/N}^{(1-t)}(\|\tau\|_{L^2(\hat{\pi}^i)})\, U_N(\mu_0|\mathfrak{m}), \quad \forall t \in [0,1),\ i = 1, 2. \tag{3.74}$$

Using (3.74), one can now follow verbatim the proof of Lemma 3.17 and conclude. □

Theorem 3.20. *Let $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ be a timelike non-branching, locally causally closed, $\mathcal{K}$-globally hyperbolic, Lorentzian geodesic space satisfying $\mathsf{TMCP}_p^e(K,N)$ for some $p \in (0,1), K \in \mathbb{R}, N \in (0,\infty)$. Let $\mu_0, \mu_1 \in \mathcal{P}_c(X)$, with $\mu_0 \in \mathrm{Dom}(\mathrm{Ent}(\cdot|\mathfrak{m}))$. Assume that there exists $\pi \in \Pi_{\leq}^{p\text{-}opt}(\mu_0, \mu_1)$ such that $\pi(\{\tau > 0\}) = 1$.*

Then there exists a unique $\eta \in \mathrm{OptGeo}_{\ell_p}(\mu_0, \mu_1)$ with $(\mathrm{e}_0, \mathrm{e}_1)_\sharp\eta(\{\tau > 0\}) = 1$ and such η is induced by a map, i.e. there exists $\mathfrak{T} : X \to \mathrm{TGeo}(X)$ such that $\eta = \mathfrak{T}_\sharp\mu_0$.

Proof. As usual, it is sufficient to show that every $\eta \in \mathrm{OptGeo}_{\ell_p}(\mu_0, \mu_1)$ with $(\mathrm{e}_0, \mathrm{e}_1)_\sharp\eta(\{\tau > 0\}) = 1$ is induced by a map; indeed if there exist $\eta_1 \neq \eta_2 \in \mathrm{OptGeo}_{\ell_p}(\mu_0, \mu_1)$ with $(\mathrm{e}_0, \mathrm{e}_1)_\sharp\eta_i(\{\tau > 0\}) = 1$ then also $\bar\eta := \frac{1}{2}(\eta_1 + \eta_2)$ would be an element of $\mathrm{OptGeo}_{\ell_p}(\mu_0, \mu_1)$ with $(\mathrm{e}_0, \mathrm{e}_1)_\sharp\bar\eta(\{\tau > 0\}) = 1$ but $\bar\eta$ cannot be given by a map.

Assume by contradiction there exists $\eta \in \mathrm{OptGeo}_{\ell_p}(\mu_0, \mu_1)$ not induced by a map. In particular, given the disintegration of η with respect to $\mathrm{e}_0 : \mathrm{TGeo}(X) \to X$

$$\eta = \int_X \eta_x\, \mu_0(dx),$$

there exists a compact subset $D \subset \mathrm{supp}(\mu_0)$ with $\mu_0(D) > 0$ such that for μ_0-a.e. $x \in D$ the probability measure η_x is not a Dirac mass. Via a selection argument, for μ_0-a.e. $x \in D$ we can also assume that η_x is the sum of two Dirac masses. Then for μ_0-a.e. $x \in D$ there exist $t = t(x) \in (0,1)$ such that $(\mathrm{e}_t)_\sharp\eta_x$ is not a Dirac mass over X. Then by continuity there exists an open interval $I = I(x) \subset (0,1)$ containing $t(x)$ above such that $(\mathrm{e}_s)_\sharp\eta_x$ is still not a Dirac mass over X, for every $s \in I(x)$.
It follows that we can find a subset $\bar D \subset D \subset X$ still satisfying $\mu_0(\bar D) > 0$ with the following property: there exists $\bar q \in \mathbb{Q} \cap (0,1)$ such that $(\mathrm{e}_{\bar q})_\sharp\eta_x$ is not a Dirac mass, for every $x \in \bar D$.

Indeed, since $D = \bigcup_{q\in\mathbb{Q}\cap(0,1)} D_q$ where

$$D_q := \{x \in D\,:\, (\mathrm{e}_q)_\sharp\eta_x \text{ is not a Dirac mass}\}$$

and since $\mu_0(D) > 0$, there must exist $\bar q \in \mathbb{Q} \cap (0,1)$ with $\mu_0(D_{\bar q}) > 0$; we then set $\bar D := D_{\bar q}$. Set now

$$\bar\eta = \frac{1}{\mu_0(\bar D)} \int_{\bar D} \eta_x\, \mu_0(dx).$$

Note that $\bar\eta$ is an ℓ_p-optimal dynamical plan satisfying $(\mathrm{e}_0, \mathrm{e}_{\bar q})_\sharp\bar\eta(\{\tau > 0\}) = 1$. But $(\mathrm{e}_0, \mathrm{e}_{\bar q})_\sharp\bar\eta$ is an ℓ_p-optimal coupling which is not given by a map, contradicting Theorem 3.19. □

Corollary 3.21. *Let $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ be a timelike non-branching, locally causally closed, $\mathcal{K}$-globally hyperbolic, Lorentzian geodesic space satisfying $\mathsf{TMCP}_p^e(K,N)$ for some $p \in (0,1), K \in \mathbb{R}, N \in [1,\infty)$. Let $\mu_0, \mu_1 \in \mathcal{P}_c(X)$ be two probability measures with $\mu_0 = \mathfrak{m}\llcorner_{A_0}/\mathfrak{m}(A_0)$, $A_0 \subset M$ compact subset. Assume that there exists $\pi \in \Pi_{\leq}^{p\text{-}opt}(\mu_0, \mu_1)$ such that $\pi(\{\tau > 0\}) = 1$.*

Then there exists a unique ℓ_p-geodesic $(\mu_t)_{t\in[0,1]}$ from μ_0 to μ_1. Moreover, it satisfies $\mu_t = \rho_t\mathfrak{m} \ll \mathfrak{m}$ and

$$\mathfrak{m}(\{\rho_t > 0\}) \geq \sigma_{K/N}^{(1-t)}(\|\tau\|_{L^2(\pi)})^N \mathfrak{m}(A_0), \tag{3.75}$$

where π is the unique element of $\Pi_{\leq}^{p\text{-}opt}(\mu_0, \mu_1)$ concentrated on $\{\tau > 0\}$.
In particular, calling $A_1 = \operatorname{supp}\mu_1$ and using the notation of Proposition 3.4, the following timelike half-Brunn-Minkowski inequality holds:

$$\mathfrak{m}(A_t)^{1/N} \geq \sigma_{K/N}^{(1-t)}(\Theta)\, \mathfrak{m}(A_0)^{1/N}.$$

Proof. From Theorem 3.19 there exists a unique ℓ_p-geodesic $(\mu_t)_{t\in[0,1]}$ from μ_0 to μ_1 and from Proposition 3.18 we deduce that $U_N(\mu_t|\mathfrak{m}) \geq \sigma^{(1-t)}_{K/N}(\|\tau\|_{L^2(\pi)})\mathfrak{m}(A)^{1/N}$, where π is the unique element of $\Pi^{p\text{-opt}}_{\leq}(\mu_0,\mu_1)$ concentrated on $\{\tau > 0\}$. We conclude applying twice Jensen's inequality as in (3.10). □

4 Localization of Timelike Measure Contraction Property

4.1 Transport relation and disintegration associated to a time separation function

From now on we make the standing assumptions that $(X, \mathsf{d}, \ll, \leq, \tau)$ is a globally hyperbolic Lorentzian geodesic space and $V \subset X$ is an achronal FTC Borel subset (see Definition 1.7). Recall that, associated to V, we have the signed time-separation function $\tau_V : X \to [-\infty, +\infty]$ defined in (1.8).

Lemma 4.1. *For each $x \in I^+(V)$ there exists a point $y_x \in V$ with $\tau_V(y_x) = \tau(y_x, x) > 0$. Moreover:*

$$\tau_V(z) - \tau_V(x) \geq \tau(y_x, z) - \tau(y_x, x) \geq \tau(x, z), \quad \forall x, z \in I^+(V) \cup V,\ x \leq z. \tag{4.1}$$

Proof. The first claim follows directly from Lemma 1.8. If $x \leq z$ and $\tau(y_x, x) > 0$, then also $y_x \leq z$ by transitivity. By reverse triangle inequality (1.1), we deduce (4.1). □

Notice that (4.1) can be extended to the whole X^2 simply by replacing τ with ℓ (recall (2.2)):

$$\tau_V(z) - \tau_V(x) \geq \ell(x, z), \qquad \forall x, z \in (I^+(V) \cup V)^2. \tag{4.2}$$

We can therefore naturally associate to V the following transport relation:

$$\Gamma_V := \{(x, z) \in (I^+(V) \cup V)^2 \cap X^2_{\leq} :\ \tau_V(z) - \tau_V(x) = \tau(x, z) > 0\} \cup \{(x, x) : x \in I^+(V) \cup V\}. \tag{4.3}$$

Recalling the Definition 2.6, inequality (4.2) readily yields:

Lemma 4.2. *The set Γ_V is ℓ-cyclically monotone.*

Proof. Take $(x_1, z_1), \dots, (x_n, z_n) \in \Gamma_V$ and sum

$$\sum_{i=1}^n \ell(x_i, z_i) = \sum_{i=1}^n \tau(x_i, z_i) = \sum_{i=1}^n \tau_V(z_i) - \tau_V(x_i) \geq \sum_{i=1}^n \ell(x_{i+1}, z_i).$$

□

A consequence of ℓ-cyclical monotonicity is the alignment along geodesics of the couples:

Lemma 4.3. *Consider $(x, z) \in \Gamma_V$ with $x \neq z$, $x \notin V$. Then there exist $y \in V, \gamma \in \mathrm{TGeo}(y, z)$ and $t \in (0, 1)$ such that*

$$x = \gamma_t, \qquad \tau(y, \gamma_s) = \tau_V(\gamma_s) \quad \forall s \in [0, 1], \qquad (\gamma_s, \gamma_t) \in \Gamma_V \quad \forall s \in [0, t].$$

Proof. From Lemma 4.1, we have the existence of $y \in V$ such that $\tau_V(x) = \tau(y, x) > 0$. Moreover from $(x, z) \in \Gamma_V$ we get $(y, z) \in X^2_{\leq}$ and

$$\tau(y, z) \leq \tau_V(z) = \tau_V(x) + \tau(x, z) = \tau(y, x) + \tau(x, z) \leq \tau(y, z),$$

yielding $0 < \tau(y, x) + \tau(x, z) = \tau(y, z)$ and $(y, z) \in \Gamma_V$. Hence we can concatenate a timelike geodesic from y to x with a timelike geodesic from x to z (whose existence is guaranteed by fact that X is a Lorentzian geodesic space) in order to obtain $\gamma \in \mathrm{TGeo}(y, z)$ and $t \in (0, 1)$ such that $\gamma_t = x$, proving first claim. In order to show the second claim, observe that for any $s \in [0, 1]$ it holds:

$$\tau_V(\gamma_s) = \tau_V(\gamma_1) - \tau_V(\gamma_1) + \tau_V(\gamma_s) = \tau(y, z) - \tau_V(\gamma_1) + \tau_V(\gamma_s).$$

From (4.1) we know that $\tau_V(\gamma_1) - \tau_V(\gamma_s) \geq \tau(\gamma_s, \gamma_1)$ hence it follows that

$$\tau(y, \gamma_s) \leq \tau_V(\gamma_s) \leq \tau(y, z) - \tau(\gamma_s, z) = \tau(y, \gamma_s),$$

proving the second point. For the last point, simply observe that

$$\tau_V(\gamma_t) - \tau_V(\gamma_s) = \tau(y, \gamma_t) - \tau(y, \gamma_s) = \tau(\gamma_s, \gamma_t).$$

□

Next, we set $\Gamma_V^{-1} := \{(x, y) : (y, x) \in \Gamma_V\}$ and we consider the *transport relation* R_V and the *transport set with endpoints* $\mathcal{T}_V^e$:

$$R_V := \Gamma_V \cup \Gamma_V^{-1}, \qquad \mathcal{T}_V^e := P_1(R_V \setminus \{x = y\}). \tag{4.4}$$

The transport relation will be an equivalence relation on a specific subset of $\mathcal{T}_V^e$ that we will now construct. Firstly we consider the following subsets of $\mathcal{T}_V^e$:

$$\begin{aligned} \mathfrak{a}(\mathcal{T}_V^e) &:= \{x \in \mathcal{T}_V^e : \nexists y \in \mathcal{T}_V^e \ s.t.\ (y, x) \in \Gamma_V, y \neq x\} \\ \mathfrak{b}(\mathcal{T}_V^e) &:= \{x \in \mathcal{T}_V^e : \nexists y \in \mathcal{T}_V^e \ s.t.\ (x, y) \in \Gamma_V, y \neq x\}, \end{aligned} \tag{4.5}$$

called the set of *initial* and *final points*, respectively. Define the *transport set without endpoints*

$$\mathcal{T}_V := \mathcal{T}_V^e \setminus (\mathfrak{a}(\mathcal{T}_V^e) \cup \mathfrak{b}(\mathcal{T}_V^e)). \tag{4.6}$$

Lemma 4.4. *It holds* $I^+(V) = (\mathcal{T}_V \cup \mathfrak{b}(\mathcal{T}_V^e)) \setminus V$ *and* $V \supset \mathfrak{a}(\mathcal{T}_V^e)$.

Proof. By definition $R_V \subset (I^+(V) \cup V)^2$ and since V is achronal $I^+(V) \cap V = \emptyset$; hence the inclusion $I^+(V) \supset (\mathcal{T}_V \cup \mathfrak{b}(\mathcal{T}_V^e)) \setminus V$ is trivial. To show the converse inclusion, for every $x \in I^+(V)$ Lemma 4.1 ensures the existence of $y \in V$ such that $\tau_V(x) = \tau(y, x) > 0$. Thus $(x, y) \in \Gamma_V^{-1} \subset R_V$, giving that $x \in \mathcal{T}_V^e \setminus \mathfrak{a}(\mathcal{T}_V^e) = \mathcal{T}_V \cup \mathfrak{b}(\mathcal{T}_V^e)$. The argument for the second inclusion is trivial. □

Proposition 4.5. *Assume in addition to the previous assumptions that X is timelike (backward and forward) non-branching. Then the transport relation R_V is an equivalence relation over $\mathcal{T}_V$.*

Proof. The reflexive property $(x, x) \in R_V$ for all $x \in \mathcal{T}_V$, as well as symmetry, hold by the very definitions of Γ_V and R_V. We are then left to show transitivity: for every $(x, y), (y, z) \in R_V$ we next prove that $(x, z) \in R_V$. Clearly we can assume $x \neq y \neq z$, otherwise the claim is trivial.
Case 1: $(x, y), (y, z) \in \Gamma_V$. Using (4.1) and reverse triangle inequality we have

$$\tau_V(z) - \tau_V(x) \geq \tau(x, z) \geq \tau(x, y) + \tau(y, z) = \tau_V(y) - \tau_V(x) + \tau_V(z) - \tau_V(y) = \tau_V(z) - \tau_V(x).$$

Hence $\tau(x, z) = \tau_V(z) - \tau_V(x)$ and therefore $(x, z) \in \Gamma_V \subset R_V$.
Case 2: $(x, y), (y, z) \in \Gamma_V^{-1}$. Hence $(z, y), (y, x) \in \Gamma_V$ and therefore $(z, x) \in \Gamma_V$ from case 1.
Case 3: $(x, y) \in \Gamma_V$ and $(y, z) \in \Gamma_V^{-1}$. Hence $(x, y), (z, y) \in \Gamma_V$. Since $y \notin \mathfrak{b}(\mathcal{T}_V^e)$, there exists $w \in \mathcal{T}_V$ such that $(y, w) \in \Gamma_V$ and $y \neq w$. Hence from $(x, y), (z, y), (y, w) \in \Gamma_V$ we deduce like in case 1 that

$$\tau(x, y) + \tau(y, w) = \tau(x, w) > 0, \qquad \tau(z, y) + \tau(y, w) = \tau(z, w) > 0.$$

Since by assumption X is a Lorentzian geodesic space, there exist $\gamma^1 \in \mathrm{TGeo}(x, w)$, $\gamma^2 \in \mathrm{TGeo}(z, w)$ with common intermediate point y. Then from the backward non-branching assumption, necessarily $\gamma^1_{[0,1]} \subset \gamma^2_{[0,1]}$ (or the other inclusion) holds true. The last claim of Lemma 4.3 finally gives $(x, z) \in R_V$.
Case 4: $(x, y) \in \Gamma_V^{-1}$ and $(y, z) \in \Gamma_V$. The argument is analogous to case 3: since $y \notin \mathfrak{a}(\mathcal{T}_v^e)$, there exists $w \in \mathcal{T}_V$ such that $(w, y) \in \Gamma_V$ and $w \neq y$. Then from the Lorentzian geodesic and (now forward) non-branching assumption, necessarily all the points w, y, x, z lie on the same strictly timelike geodesics, giving that $(x, z) \in R_V$. □

Lemma 4.6. *For each equivalence class $[x]$ of $(\mathcal{T}_V, R_V)$ there exists a convex set $I \subset \mathbb{R}$ of the Real line and a bijective map $F : I \to [x]$ satisfying:*

$$\tau(F(t_1), F(t_2)) = t_2 - t_1, \quad \forall\, t_1 \le t_2 \in I. \tag{4.7}$$

Moreover, if in addition X is locally causally closed, calling $\overline{\{z \in [x]\}}$ the topological closure of $\{z \in [x]\} \subset X$, it holds

$$\overline{\{z \in [x]\}} \setminus \{z \in [x]\} = \overline{\{z \in [x]\}} \setminus \mathcal{T}_V \subset \mathfrak{a}(\mathcal{T}_V^e) \cup \mathfrak{b}(\mathcal{T}_V^e). \tag{4.8}$$

Proof. For any $x \in \mathcal{T}_V$, denote with $[x]$ the associated equivalence class. Consider the maps

$$F : (0, \infty) \cap \mathrm{Dom}(F) \ni t \mapsto \{y : (x, y) \in \Gamma_V,\ \tau(x, y) = t\}$$

and

$$F : (-\infty, 0) \cap \mathrm{Dom}(F) \ni t \mapsto \{y : (y, x) \in \Gamma_V,\ \tau(y, x) = -t\}$$

and $F(0) = x$. First observe that F is surjective: for each $y \in [x], y \neq x$ with $(x, y) \in \Gamma_V$ (resp. $(x, y) \in \Gamma_V^{-1}$) it holds $\tau(x, y) \in (0, \infty)$, hence $\tau(x, y) \in \mathrm{Dom}(F)$ and $y \in F(\tau(x, y))$ (resp. $\tau(y, x) \in \mathrm{Dom}(F)$ and $y \in F(-\tau(y, x))$).
The fact that F is injective follows readily from its definition.
We next show that F is a single valued map. Assume by contradiction $y \neq z \in F(t)$ for some $t > 0$ (resp. $t < 0$); since x is not an initial (resp. final) point, using the geodesic assumption like in the proof of Proposition 4.5 would produce a forward (resp. backward) branching time-like geodesic giving a contradiction with the non-branching assumption.
Given $t \in \mathrm{Dom}(F)$, with a slight abuse of notation, we identify $F(t)$ with $\{F(t)\}$.
For $t_1 < t_2 \in \mathrm{Dom}(F)$, Lemma 4.3 implies that the interval $[t_1, t_2] \subset F$ (i.e. $\mathrm{Dom}(F) \subset \mathbb{R}$ is a convex subset) and that (4.7) holds.

We now show (4.8). Let $(z_n) \subset [x]$ be with $\inf_n \tau_V(z_n) > 0$ and $z_n \to \bar{z}$. It is easily seen that there exists $\bar{x} \in [x]$ such that $(\bar{x}, z_n) \in \Gamma_V$ and $\bar{x} \neq \bar{z}$. Using the continuity of τ (by global hyperbolicity), the lower semicontinuity of τ_V, (4.1) and the local causal closeness, it is easy to check that $(\bar{x}, \bar{z}) \in \Gamma_V \subset R_V$ and thus $\bar{z} \in \mathcal{T}_V^e$. Since by Proposition (4.5) the equivalence classes of R_V form a partition of $\mathcal{T}_V$, it follows that if $\bar{z} \notin [x]$ then $\bar{z} \notin \mathcal{T}_V$; more precisely it is easily seen that $\bar{z} \in \mathfrak{b}(\mathcal{T}_V^e)$.
Let now $(z_n) \subset [x]$ be with $\tau_V(z_n) \to 0$ and $z_n \to \bar{z}$. By lower semicontinuity of τ_V, it follows that $\tau_V(\bar{z}) = 0$. Using the continuity of τ it is easy to check that $\bar{z} \in \mathcal{T}_V^e$ and $(\bar{z}, x) \in R_V$. Arguing as above, it follows that if $\bar{z} \notin [x]$ then $\bar{z} \notin \mathcal{T}_V$; more precisely it is easily seen that $\bar{z} \in \mathfrak{a}(\mathcal{T}_V^e)$. □

4.2 Disintegration of $\mathfrak{m}$ associated to τ_V

We start with some measurability properties of the sets we have considered so far. We recall that for any $x \in X$ the set $I^+(x) = \{y \in M : \tau(x, y) > 0\}$ is open by continuity of τ (ensured by global hyperbolicity). Accordingly $I^+(V) = \bigcup_{x \in V} I^+(x)$ is an open subset of X.

By the very definition (1.8), τ_V is sup of continuous functions thus it is lower semi-continuous. It follows that the set Γ_V is Borel measurable (see (4.3)).

It follows that also R_V is Borel measurable, yielding that $\mathcal{T}_V^e$ defined in (4.4) is a Suslin set. To conclude, we obtain measurability of the transport set $\mathcal{T}_V$ defined in (4.6).

Lemma 4.7. *The set $\mathcal{T}_V$ is Suslin.*

Proof. Just notice that $\mathcal{T}_V$ coincides with the following set

$$P_2\{(x, y, z) \in I^+(V) \times I^+(V) \times I^+(V) : (x, y) \in \Gamma_V, (y, z) \in \Gamma_V,\ \mathsf{d}(z, y) \neq 0,\ \mathsf{d}(x, y) \neq 0\}.$$

Being the projection of a Borel set, the claim follows. □

It is not hard to show that $\mathfrak{a}(\mathcal{T}_V^e)$ and $\mathfrak{b}(\mathcal{T}_V^e)$ are co-Suslin sets, meaning their complement is Suslin. We next build an $\mathfrak{m}$-measurable quotient map $\mathfrak{Q}$ of the equivalence relation R_V over $\mathcal{T}_V$.

Lemma 4.8. *There exists an $\mathfrak{m}$-measurable quotient map $\mathfrak{Q} : \mathcal{T}_V \to X$ of the equivalence relation R_V over $\mathcal{T}_V$, i.e.*

$$\mathfrak{Q} : \mathcal{T}_V \to \mathcal{T}_V, \qquad (x, \mathfrak{Q}(x)) \in R_V, \qquad (x, y) \in R_V \Rightarrow \mathfrak{Q}(x) = \mathfrak{Q}(y). \tag{4.9}$$

Proof. First consider the following (saturated) family of subsets of $\mathcal{T}_V$:

$$E_n := \{y \in \mathcal{T}_V \,:\, (x, y) \in R_V \text{ for some } x \in \mathcal{T}_V \text{ with } \tau_V(x) > 1/n\}, \quad \forall n \in \mathbb{N}, n \geq 1.$$

By definition E_n is Suslin, $E_n \subset E_{n+1}$ and $\mathcal{T}_V = \cup_n E_n$. Set $F_n := E_n \setminus E_{n-1}$, with $F_1 = E_1$. Define the map $\mathfrak{Q}$ via its graph:

$$\text{graph}(\mathfrak{Q}) = \bigcup_n \{(x, y) \in F_n \times F_n \colon (x, y) \in R_V,\ \tau_V(y) = (1/n + 1/(n-1))/2\}.$$

Notice that the map $\mathfrak{Q}$ is well defined and its graph is $\mathcal{A}$-measurable where $\mathcal{A}$ denotes the σ-algebra generated by Suslin sets. Since $\mathcal{A}$-measurable sets are universally measurable (meaning they belong to every completion of the Borel σ-algebra with respect to any probability measure), in particular this ensures that $\mathfrak{Q}$ is $\mathfrak{m}$-measurable. □

Notation. From now on we will denote $Q := \mathfrak{Q}(\mathcal{T}_V) \subset X$ the quotient set (which is $\mathcal{A}$-measurable). The equivalence classes of R_V inside $\mathcal{T}_V$ will be called *rays* and denoted with X_α, with $\alpha \in Q$.

Applying the same trick used in [19, Section 3.1], Lemma 4.8 allows to apply Disintegration Theorem [32, Section 452] (see also [15, Section 6.3]), provided the measure $\mathfrak{m}$ is suitably modified into a finite measure. To this aim, it will be useful the next elementary lemma (for its proof see [19, Lemma 3.3]).

Lemma 4.9. *There exists a Borel function $f : X \to (0, \infty)$ satisfying*

$$\inf_{\mathcal{K}} f > 0, \text{ for any bounded subset } \mathcal{K} \subset X, \qquad \int_{\mathcal{T}_V} f\, \mathfrak{m} = 1. \tag{4.10}$$

Then, given $f : X \to (0, \infty)$ satisfying (4.10), set $\mu := f\, \mathfrak{m}\llcorner_{\mathcal{T}_V}$, and define the normalized quotient measure $\mathfrak{q} := \mathfrak{Q}_\sharp\, \mu \in \mathcal{P}(X)$. It is straightforward to check that

$$\mathfrak{Q}_\sharp(\mathfrak{m}\llcorner_{\mathcal{T}_V}) \ll \mathfrak{q}.$$

Take indeed $E \subset Q$ with $\mathfrak{q}(E) = 0$; then by definition $\int_{\mathfrak{Q}^{-1}(E)} f(x)\, \mathfrak{m}(dx) = 0$, implying $\mathfrak{m}(\mathfrak{Q}^{-1}(E)) = 0$, since $f > 0$. From the Disintegration Theorem [32, Section 452], we deduce the existence of a map

$$Q \ni \alpha \longmapsto \mu_\alpha \in \mathcal{P}(X)$$

verifying the following properties:

(1) for any μ-measurable set $B \subset X$, the map $\alpha \mapsto \mu_\alpha(B)$ is $\mathfrak{q}$-measurable;

(2) for $\mathfrak{q}$-a.e. $\alpha \in Q$, μ_α is concentrated on $\mathfrak{Q}^{-1}(\alpha)$;

(3) for any μ-measurable set $B \subset X$ and $\mathfrak{q}$-measurable set $C \subset Q$, the following disintegration formula holds:
$$\mu(B \cap \mathfrak{Q}^{-1}(C)) = \int_C \mu_\alpha(B)\, \mathfrak{q}(d\alpha).$$

Finally the disintegration is $\mathfrak{q}$-essentially unique, i.e. if any other map $Q \ni \alpha \longmapsto \bar{\mu}_\alpha \in \mathcal{P}(X)$ satisfies the previous three points, then

$$\bar{\mu}_\alpha = \mu_\alpha, \qquad \mathfrak{q}\text{-a.e. } \alpha \in Q. \tag{4.11}$$

Hence once $\mathfrak{q}$ is given (recall that $\mathfrak{q}$ depends on f from Lemma 4.9), the disintegration is unique up to a set of $\mathfrak{q}$-measure zero. In the case $\mathfrak{m}(X) < \infty$, the natural choice, that we tacitly assume, is to take as f the characteristic function of $\mathcal{T}_V$ normalised by $\mathfrak{m}(\mathcal{T}_V)$ so that $\mathfrak{q} := \mathfrak{Q}_\sharp(\mathfrak{m}\llcorner_{\mathcal{T}_V}/\mathfrak{m}(\mathcal{T}_V))$.

All the previous properties will be summarized saying that $Q \ni \alpha \mapsto \mu_\alpha$ is a *disintegration of* μ *strongly consistent with respect to* $\mathfrak{Q}$. It follows from [32, Proposition 452F] that

$$\int_X g(x)\mu(dx) = \int_Q \int g(x)\mu_\alpha(dx)\,\mathfrak{q}(d\alpha),$$

for every $g : X \to \mathbb{R} \cup \{\pm\infty\}$ such that $\int g\mu$ is well-defined in $\mathbb{R} \cup \{\pm\infty\}$. Hence picking $g = 1/f$ (where f is the one used to define μ), we get that

$$\mathfrak{m}\llcorner_{\mathcal{T}_V} = \int_Q \frac{\mu_\alpha}{f}\,\mathfrak{q}(d\alpha),$$

where the identity has to be understood in duality with test functions as the previous formula.

Defining $\mathfrak{m}_\alpha := \mu_\alpha/f$, we obtain that $\mathfrak{m}_\alpha$ (called *conditional measure*) is a Radon non-negative measure over X, verifying all the measurability properties (with respect to $\alpha \in Q$) of μ_α and giving a disintegration of $\mathfrak{m}\llcorner_{\mathcal{T}_V}$ strongly consistent with respect to $\mathfrak{Q}$. Moreover, for every bounded subset $\mathcal{K} \subset X$, it holds

$$\frac{1}{\sup_{\mathcal{K}} f}\mu_\alpha(K) \leq \mathfrak{m}_\alpha(\mathcal{K}) = \frac{\mu_\alpha}{f}(\mathcal{K}) \leq \frac{1}{\inf_{\mathcal{K}} f}, \quad \text{for } \mathfrak{q}\text{-a.e. } \alpha \in Q.$$

In the next statement, we summarize what obtained so far (cf. [19]). We denote by $\mathcal{M}_+(X)$ the space of non-negative Radon measures over (X, d).

Theorem 4.10. *Let $(X, \mathsf{d}, \ll, \leq, \tau)$ be a globally hyperbolic Lorentzian geodesic space, and $V \subset X$ a Borel achronal FTC subset.*

Then the measure $\mathfrak{m}$ restricted to the transport set without endpoints $\mathcal{T}_V$ admits the following disintegration formula:

$$\mathfrak{m}\llcorner_{\mathcal{T}_V} = \int_Q \mathfrak{m}_\alpha\,\mathfrak{q}(d\alpha),$$

where $\mathfrak{q}$ is a Borel probability measure over $Q \subset X$ such that $\mathfrak{Q}_\sharp(\mathfrak{m}\llcorner_{\mathcal{T}_V}) \ll \mathfrak{q}$ and the map $Q \ni \alpha \mapsto \mathfrak{m}_\alpha \in \mathcal{M}_+(X)$ satisfies the following properties:

(1) for any $\mathfrak{m}$-measurable set B, the map $\alpha \mapsto \mathfrak{m}_\alpha(B)$ is $\mathfrak{q}$-measurable;

(2) for $\mathfrak{q}$-a.e. $\alpha \in Q$, $\mathfrak{m}_\alpha$ is concentrated on $\mathfrak{Q}^{-1}(\alpha) = X_\alpha$ (strong consistency);

(3) for any $\mathfrak{m}$-measurable set B and $\mathfrak{q}$-measurable set C, the following disintegration formula holds:

$$\mathfrak{m}(B \cap \mathfrak{Q}^{-1}(C)) = \int_C \mathfrak{m}_\alpha(B)\,\mathfrak{q}(d\alpha);$$

(4) For every bounded subset $\mathcal{K} \subset X$ there exists a constant $C_{\mathcal{K}} \in (0, \infty)$ such that

$$\mathfrak{m}_\alpha(\mathcal{K}) \leq C_{\mathcal{K}}, \quad \text{for } \mathfrak{q}\text{-a.e. } \alpha \in Q.$$

Moreover, fixed any $\mathfrak{q}$ as above such that $\mathfrak{Q}_\sharp(\mathfrak{m}\llcorner_{\mathcal{T}_V}) \ll \mathfrak{q}$, the disintegration is $\mathfrak{q}$-essentially unique (in the sense of (4.11)*).*

4.3 ℓ^p-cyclically monotone subsets contained in the transport set $\mathcal{T}_V$

We will now obtain two results permitting to include ℓ^p-cyclically monotone sets inside ℓ-cyclically monotone sets. This technique has been introduced in [14] and pushed further in [16, 17] for the metric setting, to generalize localization paradigm to metric measure spaces using the equivalence between optimality and cyclical monotonicity.

In the present setting, since the cost ℓ^p may take the value $-\infty$, ℓ^p-cyclical monotonicity does not directly imply optimality. Nontheless using [9] and its consequences included in Proposition 2.8, we will use cyclically monotone sets to construct *locally optimal* couplings and to deduce local estimates on the disintegration that will be then globalized.

There is a simple and natural way to construct Wasserstein geodesics with $0 < p < 1$: translate along transport rays by a constant "distance". Notice that $0 < p < 1$ plays a crucial role, as an analogous statement in the Riemannian setting does not hold true for W_2.

Proposition 4.11. *Consider $\Lambda \subset \Gamma_V$ with the following property: there exists $t > 0$ such that for each $(x, y) \in \Lambda$, $\tau(x, y) = t$. Then for each $0 < p < 1$ the set Λ is ℓ^p-cyclically monotone.*

Proof. Given $(x_1, y_1), \dots, (x_n, y_n) \in \Lambda$, we need to prove

$$\sum_{i=1}^{n} \ell(x_i, y_i)^p \geq \sum_{i=1}^{n} \ell(x_{i+1}, y_i)^p,$$

that can be rewritten as

$$t \geq \left(\frac{1}{n} \sum_{i=1}^{n} \ell(x_{i+1}, y_i)^p \right)^{1/p}. \tag{4.12}$$

From Lemma 4.2 the corresponding inequality for $p = 1$ is valid:

$$nt = \sum_{i=1}^{n} \ell(x_i, y_i) \geq \sum_{i=1}^{n} \ell(x_{i+1}, y_i);$$

we rewrite it as

$$t \geq \frac{1}{n} \sum_{i=1}^{n} \ell(x_{i+1}, y_i). \tag{4.13}$$

Since by assumption $0 < p < 1$, the concavity of the function $\mathbb{R} \ni s \mapsto s^p$ implies

$$\left(\frac{1}{n} \sum_{i=1}^{n} \ell(x_{i+1}, y_i) \right)^p \geq \frac{1}{n} \sum_{i=1}^{n} \ell(x_{i+1}, y_i)^p,$$

which, combined with (4.13), gives (4.12). □

In the next proposition we give a second way to construct ℓ^p-cyclically monotone sets (cf. [14]).

Proposition 4.12. *Let $\Delta \subset \Gamma_V$ be such that*

$$(\tau_V(x_0) - \tau_V(x_1))(\tau_V(y_0) - \tau_V(y_1)) \geq 0, \quad \text{for all } (x_0, y_0), (x_1, y_1) \in \Delta. \tag{4.14}$$

Then Δ is ℓ^p-cyclically monotone for each $p \in (0, 1)$.

Proof. Let $\{(x_1, y_1), \dots, (x_N, y_N)\} \subset \Delta$ be an arbitrary finite subset of Δ. Define $s_i := \tau_V(x_i)$, $t_i := \tau_V(y_i)$ and consider the auxiliary measures

$$\eta_0 := \frac{1}{N} \sum_{i=1}^{N} \delta_{s_i}, \qquad \eta_1 := \frac{1}{N} \sum_{i=1}^{N} \delta_{t_i}.$$

Notice that the support η_0 and η_1 are confined inside a compact real interval, say I. Consider finally the map $F : I \to I$ defined by

$$F(s) = \begin{cases} t_i & \text{if } s = s_i, \\ 0 & \text{elsewhere .} \end{cases} \tag{4.15}$$

Trivially $F_\sharp \eta_0 = \eta_1$; moreover, by (4.14), F is monotone on $\operatorname{supp} \eta_0$. This implies that graph(F) is also $|\cdot|^2$-cyclically monotone on $\operatorname{supp} \eta_0$ and in particular

$$\int |x - F(x)|^2 \eta_0(dx) = W_2(\eta_0, \eta_1),$$

where W_2 is intended to be defined over $\mathcal{P}_2(\mathbb{R})$. By [77, Remark 2.19 (ii)], F is optimal for any cost $c(x, y) = h(|x - y|)$, with h strictly convex and non-negative. For $p \in (0, 1)$, consider the function $h(r) := -r^p + a$, where a can be taken to be

$$a > 2 \sup_{s \in I} |s|^p.$$

Thus $\bar{c}(s, t) := -|t - s|^p + a$ is non-negative and falls into the hypothesis of [77, Remark 2.19 (ii)]. Hence graph(F) restricted to $\operatorname{supp} \eta_0$ is also $\bar{c}$-cyclically monotone. We can now conclude as follows:

$$\begin{aligned} \sum_{i=1}^N \ell(x_i, y_i)^p &= \sum_{i=1}^N (\tau_V(y_i) - \tau_V(x_i))^p = -\sum_{i=1}^N \bar{c}(s_i, t_i) + Na \\ &\geq -\sum_{i=1}^N \bar{c}(s_i, t_{i+1}) + Na = \sum_{i=1}^N (|\tau_V(y_{i+1}) - \tau_V(x_i)|)^p \\ &\geq \sum_{i=1}^N \ell(x_i, y_{i+1})^p, \end{aligned}$$

where in the last inequality we used (4.2). □

4.4 Regularity of the conditional measures

Recall that by the Disintegration Theorem 4.10 we can write $\mathfrak{m} = \int_Q \mathfrak{m}_\alpha \mathfrak{q}(d\alpha)$, where $\mathfrak{m}_\alpha$ is a non-negative Radon measure on X concentrated on the ray X_α, for $\mathfrak{q}$-a.e. $\alpha \in Q$. The goal of this section is to prove that the conditional measures $\mathfrak{m}_\alpha$'s are absolutely continuous with respect to the Hausdorff measure $\mathcal{H}^1$ restricted to the ray X_α, for $\mathfrak{q}$-a.e. $\alpha \in Q$. Such a regularity of $\mathfrak{m}_\alpha$ can be inferred from the behavior of $\mathfrak{m}$ with respect to translation along the transport set $\mathcal{T}_V$ (cf. [10]).
Let us set some notation. First recall the definition (4.4) of transport set with endpoints $\mathcal{T}_V^e$. For any Borel set $A \subset \mathcal{T}_V^e$ and $t \in [0, +\infty)$ we can associate its "forward" translation

$$A_t := P_2\{(x, y) \in (A \times \mathcal{T}_V^e) \cap \Gamma_V : \tau(x, y) = t\}.$$

If A is a Suslin set, A_t is Suslin as well. In particular, for $A \subset \mathcal{T}_V^e$ having $\mathfrak{m}(A) > 0$ it makes sense to consider the set

$$\{t \in [0, +\infty) \colon \mathfrak{m}(A_t) > 0\}.$$

and to evaluate its Lebesgue measure.

Proposition 4.13. *Let $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ be a timelike non-branching, locally causally closed, $\mathcal{K}$-globally hyperbolic, Lorentzian geodesic space satisfying $\mathsf{TMCP}_p^e(K, N)$ for some $p \in (0, 1), K \in \mathbb{R}, N \in [1, \infty)$. For any Suslin set $A \subset \mathcal{T}_V^e \setminus \mathfrak{b}(\mathcal{T}_V^e)$ having $\mathfrak{m}(A) > 0$ there exists $s > 0$ and a compact subset $B \subset A$ such that*

$$\bigcup_{t \in [0,s]} B_t \Subset X, \qquad B_t \subset \mathcal{T}_V^e \setminus \mathfrak{b}(\mathcal{T}_V^e) \quad \text{and} \quad \mathfrak{m}(B_t) > 0 \quad \forall t \in [0, s). \tag{4.16}$$

In particular, $|\{t \in [0, +\infty) \colon \mathfrak{m}(A_t) > 0\}| > 0$.

Proof. Consider $A \subset \mathcal{T}_V^e \setminus \mathfrak{b}(\mathcal{T}_v^e)$ with $\mathfrak{m}(A) > 0$. Take $s \in [0, +\infty)$ and consider the following subset of Γ_V:

$$\Lambda_s := \{(x, y) \in (A \times \mathcal{T}_V^e) \cap \Gamma_V : \tau(x, y) = s\}.$$

From Proposition 4.11 we deduce that Λ_s is ℓ^p-cyclically monotone, for each $s \in [0, +\infty)$. We also observe that

$$0 \le s_1 \le s_2 \quad \Longrightarrow \quad P_1(\Lambda_{s_1}) \subset P_1(\Lambda_{s_2}) \subset A.$$

Moreover, since $A \subset \mathcal{T}_V^e \setminus \mathfrak{b}(\mathcal{T}_V^e)$, it follows that for each $x \in A$ there exist $s \in (0, +\infty)$ and $z \in \mathcal{T}_V$ such that $(x, z) \in \Lambda_s$, showing that

$$\bigcup_{s>0} P_1(\Lambda_s) = A.$$

In particular, by monotone convergence, we have $\lim_{s\downarrow 0} \mathfrak{m}(P_1(\Lambda_s)) = \mathfrak{m}(A) > 0$. Define then $B := P_1(\Lambda_s)$ for $s > 0$ small enough so that $\mathfrak{m}(B) > 0$. We can also find a compact subset of B of positive $\mathfrak{m}$-measure, that we still denote by B, and a measurable map $T : B \to \mathcal{T}_V$ such that $(x, T(x)) \in \Lambda_s$ for all $x \in B$. We then consider the following measures

$$\mu_0 := \mathfrak{m}\llcorner_B/\mathfrak{m}(B), \qquad \mu_1 := T_\sharp \mu_0.$$

By construction, the coupling associated to T, i.e. $\pi_T = (\mathrm{Id}, T)_\sharp \mu_0$ verifies the following two conditions:

$$\int \tau(x, y)^p \pi_T(dxdy) = s^p \in (0, +\infty).$$

Since π_T is ℓ^p-cyclically monotone and $\pi_T(\{\tau > 0\}) = 1$, Proposition 2.8 ensures it is an ℓ_p-optimal coupling. Up to further restricting π, we can assume that $\operatorname{supp} \pi \Subset \{\tau > 0\}$. Then by Theorem 3.20 and Corollary 3.21, there is a unique ℓ_p-geodesic $(\mu_t)_{t\in[0,1]}$ between μ_0 and μ_1, and $\mu_t \ll \mathfrak{m}$ for all $t \in [0, 1)$. $\mathcal{K}$-global hyperbolicity implies that $\bigcup_{t\in[0,1]} \operatorname{supp} \mu_t \Subset X$.

Since T is a translation of length s, it follows that μ_t is concentrated inside $B_{ts} \subset A_{ts}$; being absolutely continuous, it implies that

$$\mathfrak{m}(A_{ts}) > \mathfrak{m}(B_{ts}) > 0, \quad \forall t \in [0, 1),$$

proving the claim. □

Corollary 4.14. *Under the same assumptions of Proposition 4.13, it holds* $\mathfrak{m}(\mathfrak{a}(\mathcal{T}_V^e)) = 0$.

Proof. Assume by contradiction $\mathfrak{m}(\mathfrak{a}(\mathcal{T}_V^e)) > 0$. Setting $A = \mathfrak{a}(\mathcal{T}_V^e)$ in Proposition 4.13, we obtain $B \subset A$ compact subset satisfying (4.16).

Step 1. With the same notation of Proposition 4.13, we first claim that

$$B_{t_0} \cap B_{t_1} = \emptyset, \qquad \text{for any } 0 < t_0 < t_1 < s. \tag{4.17}$$

Indeed, if by contradiction there exists $y \in B_{t_0} \cap B_{t_1}$ then there exist $x, z \in \mathfrak{a}(\mathcal{T}_V^e)$ such that $\tau(x, y) = t_0$, $\tau(z, y) = t_1$, $(x, y) \in \Gamma_V$ and $(z, y) \in \Gamma_V$. Since $y \notin \mathfrak{b}(\mathcal{T}_V^e)$, we can repeat the argument in Case 3 of the proof of Proposition 4.5 and get that $(z, x) \in \Gamma_V$ contradicting that $x \in \mathfrak{a}(\mathcal{T}_V^e)$.

Step 2. From Proposition 4.13 we have that there are uncountably many $t \in [0, s)$ satisfying $\mathfrak{m}(B_t) > 0$ and (4.17). Hence, on the one hand,

$$\mathfrak{m}\left(\bigcup_{t\in(0,s)} B_t\right) = +\infty. \tag{4.18}$$

On the other hand, since by (4.16) $\bigcup_{t\in[0,s]} B_t$ is relatively compact and $\mathfrak{m}$ is by assumption a Radon measure, we have $\mathfrak{m}\left(\bigcup_{t\in[0,s]} B_t\right) < \infty$ contradicting (4.18). □

Of course, if we assume that X endowed with the reversed causal structure satisfies the assumptions of Proposition 4.13, then also $\mathfrak{m}(\mathfrak{b}(\mathcal{T}_V^e)) = 0$.

Proposition 4.15. *Under the same assumptions of Proposition 4.13, the conditional measure $\mathfrak{m}_\alpha$ (given in the Disintegration Theorem 4.10) is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^1\llcorner_{X_\alpha}$ along the ray X_α, for $\mathfrak{q}$-a.e. $\alpha \in Q$.*

Proof. Assume by contradiction there is a Borel subset $\hat{Q} \subset Q$ with $\mathfrak{q}(\hat{Q}) > 0$ such that $\mathfrak{m}_\alpha \not\ll \mathcal{L}^1\llcorner_{X_\alpha}$ for each $\alpha \in \hat{Q}$.
Let $\mathfrak{m}_\alpha = h_\alpha \mathcal{L}^1\llcorner_{X_\alpha} + \mathfrak{m}_\alpha^\perp$ be the Lebesgue decomposition of $\mathfrak{m}_\alpha$ with respect to $\mathcal{L}^1\llcorner_{X_\alpha}$, with $\mathfrak{m}_\alpha^\perp \perp \mathcal{L}^1\llcorner_{X_\alpha}$. Then, for every $\alpha \in \hat{Q}$ there exists a Borel subset $A^\alpha \subset X_\alpha$ such that

$$\mathcal{L}^1(A^\alpha) = 0 \quad \text{and} \quad \mathfrak{m}_\alpha^\perp = \mathfrak{m}_\alpha^\perp\llcorner_{A^\alpha}. \tag{4.19}$$

Define $A := \bigcup_{\alpha \in \hat{Q}} A^\alpha \subset \mathcal{T}_V$ and observe that the Disintegration Theorem 4.10 gives

$$\mathfrak{m}(A) = \int_Q \mathfrak{m}_\alpha(A)\, \mathfrak{q}(d\alpha) = \int_{\hat{Q}} \mathfrak{m}_\alpha^+(A^\alpha)\, \mathfrak{q}(d\alpha) > 0.$$

Proposition 4.13 implies

$$0 < \int_{\mathbb{R}^+} \mathfrak{m}(A_t)\, dt = \int_{\mathbb{R}^+} \left(\int_Q \mathfrak{m}_\alpha(A_t)\, \mathfrak{q}(d\alpha) \right) dt = \int_Q \left(\int_{\mathbb{R}^+} \mathfrak{m}_\alpha(A_t)\, dt \right) \mathfrak{q}(d\alpha), \tag{4.20}$$

where in the second equality we used the Disintegration Theorem 4.10, and the third equality follows by Fubini-Tonelli's Theorem. In order to simplify the notation, for the rest of the proof we identify X_α with an interval in the Real line (see Lemma 4.6). Observe that

$$\begin{aligned} \int_{\mathbb{R}^+} \mathfrak{m}_\alpha(A_t)\, dt &= \mathcal{L}^1 \otimes \mathfrak{m}_\alpha \left\{ (t,x) \,:\, t > 0,\, x \in X_\alpha,\, x - t \in A^\alpha \right\} \\ &= \int_{X_\alpha} \mathcal{L}^1(\{ t > 0 \,:\, x - t \in A^\alpha \})\, \mathfrak{m}_\alpha(dx) = 0, \end{aligned} \tag{4.21}$$

where in the last equality we used that

$$\mathcal{L}^1(\{ t > 0 \,:\, x - t \in A^\alpha \}) = \mathcal{L}^1(A_\alpha) = 0,$$

by the invariance properties of the Lebesgue measure and (4.19).
Plugging (4.21) into (4.20) gives the contradiction $0 < 0$. □

We summarise the content of this subsection, combined with Lemma 4.4 and the Disintegration Theorem 4.10, in the next statement.

Theorem 4.16. *Let $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ be a timelike non-branching, locally causally closed, $\mathcal{K}$-globally hyperbolic, Lorentzian geodesic space satisfying $\mathsf{TMCP}_p^e(K,N)$ for some $p \in (0,1), K \in \mathbb{R}, N \in [1,\infty)$, and assume that the causally-reversed structure satisfies the same conditions. Let $V \subset X$ be a Borel achronal FTC subset, $\mathcal{T}_V^e, \mathfrak{a}(\mathcal{T}_V^e), \mathfrak{b}(\mathcal{T}_V^e)$ and $\mathcal{T}_V$ be defined in (4.4), (4.5), (4.6).*

Then $\mathfrak{m}(\mathfrak{a}(\mathcal{T}_V^e)) = \mathfrak{m}(\mathfrak{b}(\mathcal{T}_V^e) = 0$ and the following disintegration formula holds true:

$$\mathfrak{m}\llcorner_{I^+(V)} = \mathfrak{m}\llcorner_{\mathcal{T}_V^e} = \mathfrak{m}\llcorner_{\mathcal{T}_V} = \int_Q \mathfrak{m}_\alpha\, \mathfrak{q}(d\alpha) = \int_Q h(\alpha,\cdot)\, \mathcal{L}^1\llcorner_{X_\alpha}\, \mathfrak{q}(d\alpha), \tag{4.22}$$

where

- *$\mathfrak{q}$ is a probability measure over the Borel quotient set $Q \subset \mathcal{T}_V$;*
- *$h(\alpha,\cdot) \in L^1_{loc}(X_\alpha, \mathcal{L}^1\llcorner_{X_\alpha})$ for $\mathfrak{q}$-a.e. $\alpha \in Q$;*
- *the map $\alpha \mapsto \mathfrak{m}_\alpha(A) = h(\alpha,\cdot)\mathcal{L}^1\llcorner_{X_\alpha}(A)$ is $\mathfrak{q}$-measurable for every Borel set $A \subset \mathcal{T}_V$.*

4.5 Localization of $\mathsf{TMCP}^e_p(K,N)$

In this section we localize the curvature condition $\mathsf{TMCP}^e_p(K,N)$ to the one dimensional metric measures spaces $(X_\alpha, |\cdot|, \mathfrak{m}_\alpha)$ decomposing $\mathcal{T}_V$, in the sense of the Disintegration Theorem 4.16 (cf. [10, 19]).

Theorem 4.17. *Let $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ and $V \subset X$ be as in Theorem 4.16 with $N \in (1,\infty)$, and recall the Disintegration formula (4.22).*

Then, for $\mathfrak{q}$-a.e. $\alpha \in Q$, the density $h(\alpha,\cdot)$ has an almost everywhere representative that is locally Lipschitz and strictly positive in the interior of X_α, continuous on its closure, and satisfying

$$\left(\frac{\mathfrak{s}_{K/(N-1)}(b-\tau_V(x_1))}{\mathfrak{s}_{K/(N-1)}(b-\tau_V(x_0))}\right)^{N-1} \leq \frac{h(\alpha,x_1)}{h(\alpha,x_0)} \leq \left(\frac{\mathfrak{s}_{K/(N-1)}(\tau_V(x_1)-a)}{\mathfrak{s}_{K/(N-1)}(\tau_V(x_0)-a)}\right)^{N-1}, \tag{4.23}$$

for all $x_0, x_1 \in X_\alpha$, with $0 \leq a < \tau_V(x_0) < \tau_V(x_1) < b < \pi\sqrt{(N-1)/(K \vee 0)}$.

In other words, for $\mathfrak{q}$-a.e. $\alpha \in Q$, the one-dimensional metric measure space $(X_\alpha, |\cdot|, \mathfrak{m}_\alpha)$ satisfies $\mathsf{MCP}(K,N)$.

Proof. For $x \in \mathcal{T}_V$ we will write $R(x)$ to denote its equivalence class in $(\mathcal{T}_V, R_V)$, i.e. the "ray passing through x" (recall Proposition 4.5). For a subset $B \subset \mathcal{T}_V$, we denote $R(B) := \bigcup_{x\in B} R(x)$.
Let $\bar{Q} \subset Q$ be an arbitrary compact subset of positive $\mathfrak{q}$-measure for which there exist $\varepsilon > 0$ and $0 < a_0 < a_1$ such that

$$\sup_{x,y\in X_\alpha} \tau(x,y) > \varepsilon, \qquad X_\alpha \cap \{\tau_V = a_0\} \neq \emptyset, \qquad X_\alpha \cap \{\tau_V = a_1\} \neq \emptyset \qquad \forall \alpha \in \bar{Q},$$
$$R(\bar{Q}) \cap \tau_V^{-1}([a_0,a_1]) \Subset X, \quad \{(x,y) \in \Gamma_V \,:\, x,y \in R(\bar{Q}),\, \tau_V(x) = a_0,\, \tau_V(y) = a_1\} \Subset \{\tau > 0\}.$$

For any $A_0 \in (a_0,a_1)$ and $L_0 > 0$ satisfying $A_0 + L_0 < a_1$, consider the probability measure

$$\mu_0 := c_{\bar{Q},A_0,L_0} \cdot \mathfrak{m}\llcorner_{\tau_V^{-1}(A_0,A_0+L_0)\cap R(\bar{Q})},$$

where $c_{\bar{Q},A_0,L_0}$ is the normalization constant so that $\mu_0 \in \mathcal{P}_c(X)$.
Let $T_{a_1} : R(\bar{Q}) \to R(\bar{Q}) \cap \tau_V^{-1}(a_1)$ be the "ray-projection map" defined by $T_{a_1}(x) = \tau_V^{-1}(a_1) \cap R(x)$ and set $\mu_1 := (T_{a_1})_\sharp \mu_0$. Notice that $\{(x, T_{a_1}(x)) \,:\, x \in \operatorname{supp}\mu_0\} \Subset \{\tau > 0\}$. Moreover, Proposition 4.12 implies that the associated coupling $\pi_{T_{a_1}} = (\mathrm{Id}, T_{a_1})_\sharp \mu_0$ is ℓ^p-cyclically monotone and thus, by Proposition 2.8, ℓ_p-optimal. Analogously, setting $T^t(x) := \tau_V^{-1}((1-t)\tau_V(x) + ta_1) \cap R(x)$, it follows that the curve of probability measures $\bar{\mu}_t = T^t_\sharp \mu_0$ is an ℓ_p-geodesic. Notice that

$$\bar{\mu}_t\left(\tau_V^{-1}(A_t, A_t + L_t) \cap R(\bar{Q})\right) = 1, \tag{4.24}$$

where $A_t := (1-t)A_0 + ta_1$ and $L_t := (1-t)L_0$.
Since by Corollary 3.21 there is a unique ℓ_p-geodesic $(\mu_t)_{t\in[0,1]}$ between μ_0 and μ_1, it must be $(\mu_t)_{t\in[0,1]} = (\bar{\mu}_t)_{t\in[0,1]}$. Thus, combining (4.24) with (3.75), we get

$$\mathfrak{m}(\tau_V^{-1}(A_t, A_t+L_t) \cap R(\bar{Q})) \geq \sigma^{(1-t)}_{K/N}(\|\tau\|_{L^2(\pi_{T_{a_1}})})^N \mathfrak{m}(\tau_V^{-1}(A_0, A_0+L_0) \cap R(\bar{Q})),$$

that can be rewritten using the Disintegration formula (4.22) as

$$\int_{\bar{Q}} \mathfrak{m}_\alpha(\tau_V^{-1}(A_t, A_t+L_t)\, \mathfrak{q}(d\alpha) \geq \sigma^{(1-t)}_{K/N}(\|\tau\|_{L^2(\pi_{T_{a_1}})})^N \int_{\bar{Q}} \mathfrak{m}_\alpha(\tau_V^{-1}(A_0, A_0+L_0))\, \mathfrak{q}(d\alpha).$$

Recalling that $\mathfrak{m}_\alpha = h(\alpha,\cdot)\mathcal{L}^1$, the arbitrariness of $\bar{Q}, a_0, a_1, A_0, L_0$ (letting $L_0 \downarrow 0$) implies that

$$(1-t)h_\alpha((1-t)A_0 + ta_1) \geq \sigma^{(1-t)}_{K/N}(a_1 - A_0)^N h_\alpha(A_0)$$

for $\mathfrak{q}$-a.e. $\alpha \in Q$, $\mathcal{L}^1$-a.e. $t \in (0,1)$, that can be rewritten as

$$\frac{b-s}{b-a} h_\alpha(s) \geq \sigma^{(\frac{b-s}{b-a})}_{K/N}(b-a)^N h_\alpha(a), \quad \text{for } \mathfrak{q}\text{-a.e. } \alpha \in Q,\ \mathcal{L}^1\text{-a.e. } s \in (a,b) \subset X_\alpha. \tag{4.25}$$

It is a standard trick to obtain the first inequality in (4.23) out of (4.25). We anyway include few details for the case $K > 0$, the other one being completely analogous. Using the notation of [73] and of [4] we consider $\tau_{K,N}^{(t)}(\vartheta) := t^{1/N}\sigma_{K/(N-1)}^{(t)}(\vartheta)^{\frac{N-1}{N}}$. While $\tau_{K,N}^{(t)}(\vartheta)$ is always larger than $\sigma_{K/N}^{(t)}(\vartheta)$, for $\vartheta \ll 1$ the two coefficients are almost identical: to be precise if $0 < K' < \tilde{K} < K$ we can choose $\vartheta^* > 0$ so that for all $0 \leq \vartheta \leq \vartheta^*$ and all $t \in [0,1]$ the reverse inequality $\tau_{K',N}^{(t)}(\vartheta) \leq \sigma_{\tilde{K}/N}^{(t)}(\vartheta)$ is valid. Hence (4.25) becomes:

$$\frac{b-s}{b-a}h_\alpha(s) \geq \tau_{K',N}^{(\frac{b-s}{b-a})}(b-a)^N h_\alpha(a),$$

provided $0 < b-a < \vartheta^*$, that can be rewritten in the following form:

$$h_\alpha(s) \geq \sigma_{K'/(N-1)}^{(\frac{b-s}{b-a})}(b-a)^{N-1}h_\alpha(a), \qquad \text{for } \mathfrak{q}\text{-a.e. } \alpha \in Q,\ \mathcal{L}^1\text{-a.e. } s \in (a,b) \subset X_\alpha,\ b-a < \vartheta^*. \tag{4.26}$$

We have therefore proved that for each $K' < K$ the following is true: for any point a there exists a neighborhood of a where (4.26) is valid. As shown for instance in [4, 21] this implies that the same inequality is valid on the whole domain of h_α (local-to-global property). Taking then the limit as $K' \to K$ from below we obtain the first inequality of (4.23).
Applying the analogous procedure to the causal-reversed structure we obtain the second inequality of (4.23). □

Remark 4.18 (The case $N = 1$)**.** In case $N = 1$, under the same assumptions of Theorem 4.17 one can follow the proof up to (4.25) and obtain that

$$\frac{b-s}{b-a}h_\alpha(s) \geq \sigma_K^{(\frac{b-s}{b-a})}(b-a)h_\alpha(a), \quad \text{for } \mathfrak{q}\text{-a.e. } \alpha \in Q,\ \mathcal{L}^1\text{-a.e. } s \in (a,b) \subset X_\alpha.$$

If $K \geq 0$, then $\sigma_K^{(\frac{b-s}{b-a})}(b-a) \geq \frac{b-s}{b-a}$ implying $h_\alpha(s) \geq h_\alpha(a)$; reversing the causal structure, it follows that h_α has to be constant. For $K < 0$ we compute the Taylor expansion

$$\sigma_K^{(t)}(\theta) = t\left[\frac{1 + t^2\frac{\theta^2}{6}(-K) + o(\theta^4)}{1 + \frac{\theta^2}{6}(-K) + o(\theta^4)}\right] = t\left[1 - \frac{\theta^2}{6}(-K)(1-t) + o(\theta^4)\right].$$

Hence we can conclude that $\liminf_{b\to a}(h_\alpha(b) - h_\alpha(a))/(b-a) \geq 0$. Again reversing the causal structure we obtain that h_α is locally Lipschitz and the reverse inequality holds, yielding h_α constant as well.

5 Applications

5.1 Synthetic mean curvature bounds for achronal FTC subsets

In this section we will work under the standing assumptions of Theorem 4.16.

Recall that, thanks to Lemma 1.8 and Lemma 4.4, $\mathcal{T}_V^e \subset \{\tau_V > 0\} \cup V$. For each $t \geq 0$, we consider the map $f_t : \mathrm{Dom}(f_t) \subset Q \to \mathcal{T}_V^e$, where

$$\mathrm{Dom}(f_t) := \{\alpha \in Q \colon \bar{X}_\alpha \cap \{\tau_V = t\} \setminus \mathfrak{b}(\mathcal{T}_V^e) \neq \emptyset\}, \qquad f_t(\alpha) := \bar{X}_\alpha \cap \{\tau_V = t\} \setminus \mathfrak{b}(\mathcal{T}_V^e), \tag{5.1}$$

where $\bar{X}_\alpha$ denotes the closure of the ray $X_\alpha \subset X$.
Proposition 4.5 ensures that f_t is single valued for every $t \geq 0$ and injective for $t > 0$. Moreover $f_0(\alpha) \in V$ for all $\alpha \in \mathrm{Dom}\,(f_0)$ (see Proposition 4.8).
Thus, for each $\mathfrak{m}$-measurable subset $A \subset \mathcal{T}_V^e$ having $\mathfrak{m}(A) < \infty$ the next identities hold true:

$$\mathfrak{m}(A) = \int_Q \int_{A\cap X_\alpha} h(\alpha,t)dt\,\mathfrak{q}(d\alpha) = \int_{[0,+\infty)} (f_t)_\sharp(h(\cdot,t)\,\mathfrak{q}(d\alpha))(A)\,dt, \tag{5.2}$$

where the first identity is the Disintegration formula (4.22) and the second identity follows from Fubini-Tonelli's Theorem. Define then

$$\mathcal{H}_t := (f_t)_\sharp\, h(\cdot, t)\mathfrak{q}, \quad \text{for all } t \geq 0. \tag{5.3}$$

By definition, $\mathcal{H}_t$ is concentrated on the level set $\{\tau_V = t\}$. In particular $\mathcal{H}_0$ is concentrated on V. An expert reader will recognise that $\mathcal{H}_t(\{\tau_V = t\})$ is a kind of τ-Minkowski content of the set $\{\tau_V = t\}$, with respect to $\mathfrak{m}$. We summarize this construction in the following

Proposition 5.1. *The following coarea-type formula holds true:*

$$\mathfrak{m}\llcorner_{\mathcal{T}_V^e} = \int_0^\infty \mathcal{H}_t \, dt,$$

meaning that for each measurable set $A \subset \mathcal{T}_V^e$ with $\mathfrak{m}(A) < \infty$, the map $[0, \infty) \ni t \mapsto \mathcal{H}_t(A)$ is measurable and

$$\mathfrak{m}(A) = \int_0^\infty \mathcal{H}_t(A) dt = \int_0^\infty \mathcal{H}_t(A \cap \{\tau_V = t\}) dt.$$

We use the previous codimension-one measures to propose the following weak notion of upper bound on the mean curvature of V. Notice that, even if $\mathcal{H}_0$ (as well as $\mathcal{H}_t$ for every $t \geq 0$) is a well defined measure, in general it not finite (even locally). Since from a geometric point of view the mean curvature is the first variation of the area, in order to speak of the former it is natural to assume that the latter is locally finite. In what follows, we will thus assume that $\mathcal{H}_0$ is a non-negative Radon measure.

In the next definition we use the "initial-point projection map" $\mathfrak{a} : \mathcal{T}_V \to V$, $\mathfrak{a} := f_0 \circ \mathfrak{Q}$. It is not hard to check it is $\mathfrak{m}$-measurable: notice indeed that

$$\operatorname{graph}(\mathfrak{a}) = \{(x, y) \in \mathcal{T}_V \times V : \tau_V(x) = \tau(y, x)\},$$

showing that $\operatorname{graph}(\mathfrak{a})$ is Borel.

Definition 5.2. The Borel achronal FTC subset $V \subset X$ has *forward mean curvature bounded below by $H_0 \in \mathbb{R}$* if $\mathcal{H}_0$ is a non-negative Radon measure and for any normal variation

$$V_{t,\phi} := \{x \in \mathcal{T}_V : 0 \leq \tau_V(x) \leq t\phi(\mathfrak{a}(x))\},$$

the following inequality holds true:

$$\limsup_{t \to 0} \frac{\mathfrak{m}(V_{t,\phi}) - t \int_V \phi \mathcal{H}_0}{t^2/2} \geq H_0 \int_V \phi^2 \mathcal{H}_0,$$

for any bounded Borel function $\phi : V \to [0, \infty)$ with compact support. Analogously V has *forward mean curvature bounded above by $H_0 \in \mathbb{R}$* if $\mathcal{H}_0$ is a non-negative Radon measure and for any normal variation $V_{t,\phi}$ as above the following inequality holds true:

$$\liminf_{t \to 0} \frac{\mathfrak{m}(V_{t,\phi}) - t \int_V \phi \mathcal{H}_0}{t^2/2} \leq H_0 \int_V \phi^2 \mathcal{H}_0, \tag{5.4}$$

for any bounded Borel function $\phi : V \to [0, \infty)$ with compact support.

Remark 5.3 (The disintegration formula, the measures $\mathcal{H}_t$ and the mean curvature bounds in the smooth setting)**.** Let (M^n, g) be a $2 \leq n$-dimensional smooth globally hyperbolic space-time and $V \subset M$ be a smooth compact achronal spacelike hypersurface without boundary. Then, the signed time-separation function τ_V from V is smooth on a neighbourhood U of V and $\nabla \tau_V$ is the smooth timelike past-pointing unit normal vector field along V. More precisely,

$$\nabla \tau_V(x) \perp T_x V, \quad g(\nabla \tau_V(x), \nabla \tau_V(x)) = -1, \quad \forall x \in V.$$

Denote with Vol_g the volume measure of (M^n, g) and with Vol_V the induced $(n-1)$-dimensional volume measure on V. By compactness of V, there exists $\delta > 0$ such that the g-geodesic $[0,\delta] \ni t \mapsto \exp_x(-t\nabla\tau_V(x))$ is a future pointing maximal geodesic, for every $x \in V$. Define

$$\mathcal{U} := V \times [0,\delta] \subset V \times \mathbb{R}, \quad \Phi : \mathcal{U} \to M, \; \Phi(x,t) := \exp_x(-t\nabla\tau_V(x)).$$

For $\delta > 0$ small enough it is a standard fact (tubular neighbourhood theorem) that Φ is a diffeomorphism onto its image and that the following integration formula holds true:

$$\int_M \varphi \, d\mathrm{Vol}_g = \int_V \int_0^\delta \varphi \circ \Phi(x,t) \, \det D\Phi_{(x,t)}|_{T_x V} \, dt \, \mathrm{Vol}_V(dx), \quad \forall \varphi \in C_c(\Phi(\mathcal{U})). \tag{5.5}$$

Consider also the map $\mathfrak{Q} : \Phi(\mathcal{U}) \to V$ given by $\mathfrak{Q} := P_1 \circ \Phi^{-1}$. Notice that, for every $x \in V$, it holds

$$\mathfrak{Q}^{-1}(V) = \mathcal{T}_V^e \cap \Phi(\mathcal{U}) = \mathcal{T}_V \cap \Phi(\mathcal{U}), \quad \mathfrak{Q}^{-1}(x) = R(x) \cap \Phi(\mathcal{U}) = [x]_{(\mathcal{T}_V, R_V)} \cap \Phi(\mathcal{U}),$$

i.e. $\mathfrak{Q}^{-1}(x)$ is the transport ray associated to τ_V intersected with $\Phi(\mathcal{U})$. Moreover,

$$\mathfrak{q} := \mathfrak{Q}_\sharp(\mathrm{Vol}_{g \llcorner \Phi(\mathcal{U})}) = \psi \, \mathrm{Vol}_V \ll \mathrm{Vol}_V, \quad \text{where } \psi(x) := \left(\int_0^\delta \det D\Phi_{(x,t)}|_{T_x V} \, dt \right), \, \forall x \in V.$$

Hence, we can identify Q with V, and the quotient measure $\mathfrak{q}$ with $\psi \, \mathrm{Vol}_V$. The integration formula (5.5) can be thus rewritten as

$$\int_M \varphi \, d\mathrm{Vol}_g = \int_V \frac{1}{\psi(x)} \int_0^\delta \varphi \circ \Phi(x,t) \, \det D\Phi_{(x,t)}|_{T_x V} \, dt \, \mathfrak{q}(dx), \quad \forall \varphi \in C_c(\Phi(\mathcal{U})). \tag{5.6}$$

The uniqueness statement (4.11) in the disintegration formula combined with (4.22) and (5.6) gives:

$$h_\alpha(t) = \frac{1}{\psi(\alpha)} \det D\Phi_{(\alpha,t)}|_{T_\alpha V}, \quad h_\alpha(0) = \frac{1}{\psi(\alpha)}, \quad \forall \alpha \in V, \; \forall t \in [0,\delta].$$

Moreover, observing that $\Phi(\alpha, t) = f_t(\alpha)$ where the latter was defined in (5.1), it follows that the measure $\mathcal{H}_t$ defined in (5.3) can written as

$$\mathcal{H}_t := (f_t)_\sharp \, h(\cdot, t)\mathfrak{q} = \Phi(\cdot, t)_\sharp \left(\det D\Phi_{(\alpha,t)}|_{T_\alpha V} \, \mathrm{Vol}_V(d\alpha) \right), \quad \text{for all } t \geq 0,$$

in particular, $\mathcal{H}_0 = \mathrm{Vol}_V$, $\mathcal{H}_t$ is the $(n-1)$-volume measure on the hypersurface $\{\Phi(x,t) : x \in V\}$ and Proposition 5.1 reduces to the standard co-area formula. The definition 5.2 of mean curvature bounds also reduces to the classical notions. Indeed, for $\phi \in C^\infty(V; \mathbb{R}_{\geq 0})$, the region $V_{t,\phi}$ is the domain trapped between V and the normal graph of ϕ. The first variation of the volume is thus $\frac{d}{dt}\mathrm{Vol}_g(V_{t,\phi}) = \mathcal{H}^{n-1}(\{\Phi(x, t\phi(x)) : x \in V\})$, where $\mathcal{H}^{n-1}$ is the standard $(n-1)$-volume of the hypersurface $\{\Phi(x, t\phi(x)) : x \in V\}$; in particular, $\frac{d}{dt}\big|_{t=0} \mathrm{Vol}_g(V_{t,\phi}) = \mathrm{Vol}_V(V) = \mathcal{H}_0(V)$. The left hand side in (5.4), corresponding to the second variation of volume, is thus the first variation of the area which gives the mean curvature $\vec{H}_V$ of V:

$$\begin{aligned} \lim_{t \downarrow 0} \frac{\mathfrak{m}(V_{t,\phi}) - t \int_V \phi \mathcal{H}_0}{t^2/2} &= \frac{d}{dt^2}\bigg|_{t=0} \mathrm{Vol}_g(V_{t,\phi}) = \frac{d}{dt}\bigg|_{t=0} \mathcal{H}^{n-1}(\{\Phi(x, t\phi(x)) : x \in V\}) \\ &= \int_V \phi \, g(\vec{H}_V, \nabla\tau_V) \, \mathrm{Vol}_V. \end{aligned}$$

Remark 5.4 (Example of a surface with a conical singularity)**.** The notion of forward mean curvature bound should be compared with the recent related definition proposed by Ketterer [48]. In the notation of [48], in order to have finite bound H_0 one needs that the rays X_α are extendable passing through V, which corresponds to have an interior & exterior ball condition (equivalent, in the smooth setting, to a local L^∞ bound on the full second fundamental form), see [48, Remark 5.9]. The notion proposed above in Definition 5.2 instead works well even if the set V has corners or conical singularities. Indeed, for instance, it is not hard to see that the set

$$V = \{(x,t) \subset \mathbb{R}^{n,1} : t = \alpha|x|\}, \quad \alpha \in (0,1),$$

in the $(n+1)$-dimensional Minkowski space-time $\mathbb{R}^{n,1}$ is an achronal topological hypersurface, smooth outside the origin (where it is Lipschitz) and having forward mean curvature bounded above by $H_0 = 0$ in the sense of Definition 5.2. Notice that for any compact subset, one could choose the upper bound on the mean curvature to be strictly negative, but such an upper bound approaches zero as $|x| \to \infty$.

5.2 Hawking Singularity theorem in a synthetic framework

Let us define $D_{H_0,K,N} > 0$ as follows:

$$D_{H_0,K,N} := \begin{cases} \frac{\pi}{2}\sqrt{\frac{N-1}{K}} & \text{if } K>0,\ N>1,\ H_0=0 \\ \sqrt{\frac{N-1}{K}}\cot^{-1}\left(\frac{-H_0}{\sqrt{K(N-1)}}\right) & \text{if } K>0,\ N>1,\ H_0\in\mathbb{R}\setminus\{0\} \\ -\frac{N-1}{H_0} & \text{if } K=0,\ N>1,\ H_0<0 \\ \sqrt{-\frac{N-1}{K}}\coth^{-1}\left(\frac{-H_0}{\sqrt{-K(N-1)}}\right) & \text{if } K<0,\ N>1,\ H_0<-\sqrt{-K(N-1)}. \end{cases} \tag{5.7}$$

Theorem 5.5 (Hawking Singularity Theorem for $\mathsf{TMCP}^e_p(K,N)$ spaces)**.** *Let $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ be a timelike non-branching, locally causally closed, $\mathcal{K}$-globally hyperbolic, Lorentzian geodesic space satisfying $\mathsf{TMCP}^e_p(K,N)$ for some $p \in (0,1)$, $K \in \mathbb{R}$, $N \in [1,\infty)$ and assume that the causally-reversed structure satisfies the same conditions.*
Let $V \subset X$ be a Borel achronal FTC subset having forward mean curvature bounded above by H_0 in the sense of Definition 5.2. If

1. *$K > 0$, $N > 1$ and $H_0 \in \mathbb{R}$, or*
2. *$K = 0$, $N > 1$ and $H_0 < 0$, or*
3. *$K < 0$, $N > 1$ and $H_0 < \sqrt{-K(N-1)} < 0$,*

then for every $x \in I^+(V)$ it holds $\tau_V(x) \leq D_{H_0,K,N}$. In particular, for every timelike geodesic $\gamma \in$ TGeo(X) with $\gamma_0 \in V$, the maximal (on the right) domain of definition is contained in $\big[0, D_{H_0,K,N}\big]$. In case $N = 1$, $H_0 < 0$, it holds that $I^+(V) = \emptyset$.

Proof. **Step 1**: we show that $\sup_{x\in I^+(V)} \tau_V(x) \leq D_{H_0,K,N}$, case $N \in (1,\infty)$.
Recall that, from Lemma 4.4, it holds $I^+(V) = \mathcal{T}^e_V \setminus V$. Moreover, from Theorem 4.16 we have the disintegration formula

$$\mathfrak{m}\llcorner_{I^+(V)} = \mathfrak{m}\llcorner_{\mathcal{T}^e_V} = \int_Q h(\alpha,\cdot)\,\mathcal{L}^1\llcorner_{X_\alpha}\,\mathfrak{q}(d\alpha), \tag{5.8}$$

where the closure $\bar{X}_\alpha$ of each X_α is a timelike geodesic starting at a point $\mathfrak{a}_\alpha \in V$ and parametrized by arclength on a (apriori possibly unbounded) closed Real interval $I_\alpha := [0, d_\alpha] \subset [0,\infty)$ in terms of $\tau_V(\cdot) = \tau(\mathfrak{a}_\alpha, \cdot)$, see Lemma 4.6 . For simplicity of notation, in the rest of the proof we will identify $\bar{X}_\alpha$ with the closed Real interval $I_\alpha \subset [0,\infty)$.
From Theorem 4.17, for $\mathfrak{q}$-a.e. $\alpha \in Q$, the density $h(\alpha,\cdot)$ in (5.8), has an almost everywhere representative that is locally Lipschitz and strictly positive in the interior of I_α and continuous on I_α satisfying

$$h(\alpha,t) \geq h(\alpha,0)\left(\frac{\mathfrak{s}_{K/(N-1)}(b_\alpha - t)}{\mathfrak{s}_{K/(N-1)}(b_\alpha)}\right)^{N-1} \text{ for all } t \in [0,b_\alpha],\ b_\alpha \in I_\alpha. \tag{5.9}$$

Recalling the notation of Definition 5.2 and using (5.8)-(5.9), for every bounded Borel function $\phi : V \to [0,\infty)$ with compact support satisfying $\phi(f_0(\alpha)) \in I_\alpha$ for every $\alpha \in Q$, and for any $\mathfrak{q}$-measurable assignment $Q \ni \alpha \mapsto b_\alpha \in I_\alpha$ with $b_\alpha \geq \phi(f_0(\alpha))$, $b_\alpha > 0$ for every $\alpha \in Q$ it holds:

$$\begin{aligned}
\mathfrak{m}(V_{t,\phi}) - t\int_V \phi\mathcal{H}_0 &= \int_Q \left(\int_{[0,t\phi(f_0(\alpha))]} h(\alpha,x)dx\right)\mathfrak{q}(d\alpha) - t\int_Q \phi(f_0(\alpha))h(\alpha,0)\,\mathfrak{q}(d\alpha)\\
&= \int_Q \left(\int_{[0,t]} h(\alpha, s\phi(f_0(\alpha)))\phi(f_0(\alpha))ds\right)\mathfrak{q}(d\alpha) - t\int_Q \phi(f_0(\alpha))h(\alpha,0)\,\mathfrak{q}(d\alpha)\\
&= \int_Q \left(\int_{[0,t]} (h(\alpha,s\phi(f_0(\alpha))) - h(\alpha,0))ds\right)\phi(f_0(\alpha))\,\mathfrak{q}(d\alpha)\\
&\geq \int_Q\int_{[0,t]} \left(\left(\frac{\mathfrak{s}_{K/(N-1)}(b_\alpha - s\phi(f_0(\alpha)))}{\mathfrak{s}_{K/(N-1)}(b_\alpha)}\right)^{N-1} - 1\right) ds\,\phi(f_0(\alpha))\,h(\alpha,0)\,\mathfrak{q}(d\alpha)\\
&= \int_Q\int_{[0,t]} \left(-\sqrt{|K|(N-1)}\frac{\mathfrak{c}_{K/(N-1)}(b_\alpha)}{\mathfrak{s}_{K/(N-1)}(b_\alpha)} s\phi(f_0(\alpha)) + o(s)\right) ds\,\phi(f_0(\alpha))\,h(\alpha,0)\,\mathfrak{q}(d\alpha)\\
&= \int_Q \left(-\sqrt{|K|(N-1)}\frac{\mathfrak{c}_{K/(N-1)}(b_\alpha)}{\mathfrak{s}_{K/(N-1)}(b_\alpha)}\frac{t^2}{2} + o(t^2)\right)\phi(f_0(\alpha))^2\,h(\alpha,0)\mathfrak{q}(d\alpha), \quad \forall t \in (0,1).
\end{aligned}$$

Taking lim inf of both sides of the last inequality, using Fatou's Lemma and the assumption that the forward mean curvature of V is bounded above by H_0 we deduce that

$$H_0\int_V \phi^2\mathcal{H}_0 \geq \liminf_{t\to 0}\frac{\mathfrak{m}(V_{t,\phi}) - t\int_V \phi\mathcal{H}_0}{t^2/2} \geq \int_V -\sqrt{|K|(N-1)}\frac{\mathfrak{c}_{K/(N-1)}(b_\alpha)}{\mathfrak{s}_{K/(N-1)}(b_\alpha)}\phi^2\mathcal{H}_0,$$

implying

$$b_\alpha \leq D_{H_0,K,N} \quad \mathfrak{q}\text{-a.e. } \alpha \in Q.$$

By the arbitrariness of the assignment $Q \ni \alpha \mapsto b_\alpha \in I_\alpha = [0, d_\alpha] \subset [0,\infty)$, it follows that

$$d_\alpha \leq D_{H_0,K,N} \quad \mathfrak{q}\text{-a.e. } \alpha \in Q. \tag{5.10}$$

Since by construction $d_\alpha = \sup_{x\in X_\alpha} \tau_V(x)$, the combination of (5.10) and the disintegration formula (5.8) yields

$$\tau_V(x) \leq D_{H_0,K,N}, \quad \mathfrak{m}\text{-a.e. } x \in I^+(V). \tag{5.11}$$

The lower semi-continuity of τ_V permits to promote (5.11) to every $x \in I^+(V)$.

Step 2. Consider any timelike geodesic γ parametrized by arclength and defined on a maximal (on the right) interval $[0,a) \subset [0,\infty)$ such that $\gamma_0 \in V$. We claim that $a \leq D_{H_0,K,N}$. Indeed, if by contradiction for some $s_0 \in [0,a)$

$$\tau(\gamma_0, \gamma_{s_0}) = s_0 > D_{H_0,K,N},$$

the very definition (1.8) of τ_V would imply $\tau_V(\gamma_{s_0}) > D_{H_0,K,N}$ contradicting Step 1.

Step 3: The case $N = 1, H_0 < 0$.
Recalling Remark 4.18, in case $N = 1$ the density $h_\alpha(\cdot)$ is constant on I_α. Thus, argueing along the lines of Step 1, we get that $H_0\int_V \phi^2\mathcal{H}_0 \geq 0$ which gives a contradiction unless $I^+(V) = \emptyset$.

□

5.3 Timelike Bishop-Gromov, Bonnet-Myers and Poincaré inequalities for $\mathsf{TMCP}^e_p(K,N)$

In order to state the next result we need to introduce a bit of notation. Given a Borel achronal FTC subset $V \subset X$, we say that a subset $E \subset I^+(V) \cup V$ is (τ_V, R_0)-*conically shaped* if

$$E = \{x \in I^+(V) \cup V : \tau_V(x) \leq R_0,\ y_x \in E \text{ for all } y_x \in V \text{ with } \tau(y_x, x) = \tau_V(x)\}.$$

Note that, for a closed subset E, the condition is equivalent to ask that for every $x \in E \cap \mathcal{T}_V$ the intersection $E \cap [x]_{(\mathcal{T}_V, R_V)}$ corresponds to the interval $[0, R_0]$ via the map F of Lemma 4.6.

Proposition 5.6 (A Bishop-Gromov type inequality for achronal FTC sets in $\mathsf{TMCP}^e_p(K,N)$ spaces)**.** *Let $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ be a timelike non-branching, locally causally closed, $\mathcal{K}$-globally hyperbolic, Lorentzian geodesic space satisfying $\mathsf{TMCP}^e_p(N,N)$ for some $p \in (0,1), K \in \mathbb{R}, N \in (1,\infty)$ and assume that the causally-reversed structure satisfies the same conditions.*
Let $V \subset X$ be a Borel achronal FTC subset. Then, for every compact (τ_V, R_0)-conically shaped subset $E \subset I^+(V) \cup V$ it holds (recall the definition (5.3) of $\mathcal{H}_t$):

$$\frac{\mathcal{H}_r(\{\tau_V = r\} \cap E)}{\mathcal{H}_R(\{\tau_V = R\} \cap E)} \geq \left(\frac{\mathfrak{s}_{K/(N-1)}(r)}{\mathfrak{s}_{K/(N-1)}(R)} \right)^{N-1}, \quad \text{for all } 0 \leq r \leq R \leq R_0 \tag{5.12}$$

$$\frac{\mathfrak{m}(\{\tau_V \leq r\} \cap E)}{\mathfrak{m}(\{\tau_V \leq R\} \cap E)} \geq \frac{\int_0^r \left(\mathfrak{s}_{K/(N-1)}(t)\right)^{N-1} dt}{\int_0^R \left(\mathfrak{s}_{K/(N-1)}(t)\right)^{N-1} dt}, \quad \text{for all } 0 \leq r \leq R \leq R_0. \tag{5.13}$$

Proof. In order to show (5.12) observe that the combination of (5.2), (5.3) and Theorem 4.17 gives

$$\begin{aligned} \mathcal{H}_r(\{\tau_V = r\} \cap E) &= \int_{V \cap E} h(\alpha, r)\, \mathfrak{q}(d\alpha) \\ &\geq \left(\frac{\mathfrak{s}_{K/(N-1)}(r)}{\mathfrak{s}_{K/(N-1)}(R)} \right)^{N-1} \int_{V \cap E} h(\alpha, R)\, \mathfrak{q}(d\alpha) \\ &= \left(\frac{\mathfrak{s}_{K/(N-1)}(r)}{\mathfrak{s}_{K/(N-1)}(R)} \right)^{N-1} \mathcal{H}_r(\{\tau_V = R\} \cap E). \end{aligned}$$

The claim (5.13) follows from (5.12) by recalling (5.2), (5.3) and the classical Gromov's Lemma (see for instance [22, Lemma III.4.1]). □

Notice that, in particular, if $\{\tau_V \leq R_0\} \subset X$ is a compact subset then (5.12) and (5.13) remain valid without capping with the cutoff set E in the left hand side.

Let us introduce some notation for the next result. For $u : X \to \mathbb{R}$ we will use the short-hand notation $u(\alpha, t)$ to denote $u(\bar{X}_\alpha \cap \{\tau_V = t\})$. Notice that if u is Lipschitz then, for every $\alpha \in Q$, the function $t \mapsto u(\alpha, t)$ is locally Lipschitz and thus $\mathcal{L}^1$-a.e. differentiable with derivative denoted as $\frac{\partial}{\partial t} u(\alpha, t)$. For u with compact support, we will also use the notation

$$u_\alpha := \frac{1}{\mathfrak{m}_\alpha(\operatorname{supp} u)} \int_{X_\alpha} u\, \mathfrak{m}_\alpha \quad \text{if } \mathfrak{m}_\alpha(\operatorname{supp} u) \neq 0, \text{ and } u_\alpha := 0 \text{ otherwise},$$

to denote the average of u on $(X_\alpha, \mathfrak{m}_\alpha)$.

Proposition 5.7 (A timelike Poincaré inequality for $\mathsf{TMCP}^e_p(K,N)$)**.** *For every $(K, N, D) \in \mathbb{R} \times (1, \infty) \times (0, \infty)$, there exists a constant $\lambda_{\mathsf{MCP}_{K,N,D}} > 0$ with the following property.*
Let $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ and $V \subset X$ be as in Proposition 5.6. Then, for every $u : X \to \mathbb{R}$ Lipschitz with compact support contained in $I^+(V)$ it holds

$$\int_X |u - u_\alpha|^2\, \mathfrak{m} \leq \lambda_{\mathsf{MCP}_{K,N,D}} \int_X \left| \frac{\partial}{\partial t} u(\alpha, t) \right|^2 \mathfrak{m}, \tag{5.14}$$

where $D := \sup_{\alpha \in Q} \sup_{x,y \in X_\alpha \cap \operatorname{supp} u} \tau(x, y) \leq \sup_{x,y \in \operatorname{supp} u} \tau(x,y) < \infty$.

Proof. Since from Theorem 4.17 each ray $(X_\alpha, \mathfrak{m}_\alpha)$ is an $\mathsf{MCP}(K,N)$ space, from [41] we know that

$$\int_{X_\alpha} |u(\alpha, t) - u_\alpha|^2\, \mathfrak{m}_\alpha(dt) \leq \lambda_{\mathsf{MCP}_{K,N,D}} \int_{X_\alpha} \left| \frac{\partial}{\partial t} u(\alpha, t) \right|^2 \mathfrak{m}_\alpha(dt).$$

The claimed (5.14) follows then from the disintegration formula (4.22). □

It is possible to give quite precise estimates on the constan $\lambda_{\mathsf{MCP}_{K,N,D}}$, the interested reader is referred to [41].

Finally we take advantage of the techniques developed in the second part of the paper to sharpen, for timelike non-branching spaces, the timelike Bishop-Gromov inequality obtained in Proposition 3.5 and the timelike Bonnet-Myers inequality obtained in Proposition 3.6.

Proposition 5.8 (A timelike Bishop-Gromov inequality for timelike non-branching $\mathsf{TMCP}^e_p(K,N)$)**.** *Let $(X,\mathsf{d},\mathfrak{m},\ll,\leq,\tau)$ be as in Proposition 5.6. Then, for each $x_0 \in X$, each compact subset $E \subset I^+(x_0)\cup\{x_0\}$ τ-star-shaped with respect to x_0, and each $0<r<R\leq \pi\sqrt{(N-1)/(K\vee 0)}$, it holds:*

$$\frac{s(E,r)}{s(E,R)} \geq \left(\frac{\mathfrak{s}_{K/(N-1)}(r)}{\mathfrak{s}_{K/(N-1)}(R)}\right)^{N-1}, \quad \frac{v(E,r)}{v(E,R)} \geq \frac{\int_0^r \mathfrak{s}_{K/(N-1)}(t)^{N-1}dt}{\int_0^R \mathfrak{s}_{K/(N-1)}(t)^{N-1}dt}. \tag{5.15}$$

Proof. Consider $\tau_{x_0}(\cdot) := \tau(x_0,\cdot) : I^+(x_0)\to\mathbb{R}$. One can repeat verbatim (actually here it would be slightly easier) the constructions of Section 4 replacing τ_V by τ_{x_0} and obtain a partition (up to a set of $\mathfrak{m}$-measure zero) of $I^+(x_0)$ into transport rays $\{X_\alpha\}_{\alpha\in Q}$ associated to τ_{x_0}, i.e. each X_α is a future pointing radial τ-geodesic emanating from x_0. One can disintegrate $\mathfrak{m}\llcorner_{I^+(x_0)}$ accordingly as $\mathfrak{m}\llcorner_{I^+(x_0)}=\int_Q \mathfrak{m}_\alpha\, \mathfrak{q}(d\alpha)$ where each $\mathfrak{m}_\alpha$ is concentrated on X_α, and $(X_\alpha,|\cdot|,\mathfrak{m}_\alpha)$ is a 1-dim. $\mathsf{MCP}(K,N)$ m.m.s.. One can now prove (5.15) along the same lines of the proof of Proposition 5.6. □

Proposition 5.9 (A timelike Bonnet-Myers inequality for timelike non-branching $\mathsf{TMCP}^e_p(K,N)$)**.** *Let $(X,\mathsf{d},\mathfrak{m},\ll,\leq,\tau)$ be as in Proposition 5.6, with $K>0$. Then*

$$\sup_{x,y\in X} \tau(x,y) \leq \pi\sqrt{\frac{N-1}{K}}. \tag{5.16}$$

Proof. Assume by contradiction that there exist $x_0,x_1\in X$ with $\tau(x_0,x_1)\geq \pi\sqrt{(N-1)/K}+2\varepsilon$, for some $\varepsilon>0$. Let $\delta>0$ be such that

$$\inf\{\tau(x_0,y) : y\in B^{\mathsf{d}}(x_1,\delta)\} \geq \pi\sqrt{(N-1)/K}+\varepsilon.$$

Consider the disintegration $\mathfrak{m}\llcorner_{I^+(x_0)}=\int_Q \mathfrak{m}_\alpha\, \mathfrak{q}(d\alpha)$ associated to τ_{x_0}, as outlined in the proof of Proposition 5.8. Since $\mathfrak{m}(B^{\mathsf{d}}(x_1,\delta))>0$, it follows that $L_\tau(X_\alpha)\geq \pi\sqrt{(N-1)/K}+\varepsilon$ for a $\mathfrak{q}$-non negligible subset of rays. But since every $(X_\alpha,|\cdot|,\mathfrak{m}_\alpha)$ is a 1-dim. $\mathsf{MCP}(K,N)$ m.m.s. with full support, its diameter is at most $\pi\sqrt{(N-1)/K}$ (as it's easily seen from (4.23)). Contradiction. □

Remark 5.10 (Sharpness)**.** The Lorentzian model spaces are: for $K<0$ (scaled) de Sitter space, $K=0$ Minkowski space, $K>0$ (scaled) anti-de Sitter space. Recall that the standard de Sitter space (M^n,g_{dS}) has constant sectional curvature equal to 1, thus $\mathrm{Ric}_{g_{dS}}(v,v)=-(n-1)g_{dS}(v,v)$ for v timelike, and hence it is the model space for $K=-(n-1)$. The Minkowski space has null sectional (and thus Ricci) curvatures, thus is the model space for $K=0$. The anti de-Sitter space (M^n,g_{adS}) has constant sectional curvature equal to -1, thus $\mathrm{Ric}_{g_{adS}}(v,v)=(n-1)g_{dS}(v,v)$ for v timelike, and hence it is the model space for $K=n-1$. It is well known that any globally hyperbolic subset of (M^n,g_{adS}) has timelike diameter at most π, with sharp bound; this shows the sharpeness of Proposition 5.9. Using that in the model spaces the sectional curvature is constant, direct volume computations via Jacobi fields show that equality is achieved in (5.15); thus also Proposition 5.8 is sharp. Choosing V to be a level set of the natural time-function in the model spaces, it is possible to check that equality is achieved in (5.12) and (5.13) as well.

5.4 The case of a spacetime with continuous metric

Next we specialise Theorem 5.5 to the case of a spacetime with a continuous metric. As observed in [24], spacetimes with continuous metrics may present pathological causal behaviour. For instance [24] (see also [50, Section 5.1]) gives examples of spacetimes with Hölder-continous metrics where the null curves emanating from a point cover a set with non-empty interior, a phenomenon called "bubbling". In order to prevent such a pathological behavior, [24] proposed the notion of "causally plain" metric. Let us briefly recall it together with the needed notation.

The notion of causally plain Lorentzian metric. Let $\check{g}, g$ be two Lorentzian metrics. We write $\check{g} \prec g$ if $\{v \neq 0 : \check{g}(v,v) \leq 0\} \subset \{v : g(v,v) < 0\}$. For a neighbourhood U of $x \in M$, set

$$\check{I}^+_g(x,U) := \{y \in U : \exists \text{ a smooth Lorentzian metric } \check{g} \prec g \text{ and a future pointing } \check{g}\text{-timelike curve} \\ \gamma : [0,1] \to U, \text{ with } \gamma_0 = x,\ \gamma_1 = y \text{ and } \check{g}(\dot\gamma,\dot\gamma) < 0\}.$$

The set $\check{I}^-_g(x,U)$ is defined analogously. It is clear that $\check{I}^\pm_g \subset I^\pm_g$ and equality holds for smooth metrics. Let us also recall that a *cylindrical neighbourhood* of a point $x \in X$ with respect to g, is a relatively compact chart domain containing x such that, in this chart, g equals the Minkowski metric at x and the slopes of the light cones of g stay close to 1 (for the precise definition see [24, Def. 1.8]).

A spacetime (M,g) is said to be *causally plain* if every $x \in M$ admits a cylindrical neighbourhood U such that $\partial \check{I}^\pm_g(x,U) = \partial J^\pm(x,U)$; otherwise (M,g) is said to be *bubbling* [24, Def. 1.16].
The rough idea is that (M,g) is causally plain provided, for every $x \in M$, the span of all null curves emanating from x has empty interior.
It was proved in [24, Corollary 1.17] that a spacetime with locally Lipschitz continuous Lorentzian metric is causally plain. In the same paper [24, Section 1.1] (see also [50, Section 5.1]) examples of Hölder-continuos bubbling Lorentzian metrics are discussed.

Let (M,g) be a spacetime with a C^0-Lorentzian metric. Recall from Remark 1.12 that if (M,g) is globally hyperbolic and causally plain then the associated Lorentzian synthetic space is causally closed, $\mathcal{K}$-globally hyperbolic and geodesic. Recall also that any Cauchy hypersurface is causally complete.
It is then clear that Proposition 3.4, Proposition 3.5, Proposition 3.6 and Remark 3.8 give the following:

Corollary 5.11 (Geometric properties of a globally hyperbolic causally plain spacetime with C^0 metric, with synthetic timelike Ricci bounded below)**.** *Let (M,g) be a $2 \leq n$-dimensional globally hyperbolic, causally plain spacetime with a C^0-Lorentzian metric.*

- Timelike Bishop-Gromov: *If (M,g) satisfies $\mathsf{TMCP}^e_p(K,N)$ for some $p \in (0,1), K \in \mathbb{R}, N \in [1,\infty)$, then for each $x_0 \in M$, each compact subset $E \subset I^+(x_0) \cup \{x_0\}$ τ-star-shaped with respect to x_0, and each $0 < r < R \leq \pi\sqrt{N/(K \vee 0)}$, inequality* (3.11) *holds.*
- Timelike Bonnet-Myers: *If (M,g) satisfies $\mathsf{TMCP}^e_p(K,N)$ for some $p \in (0,1), K > 0, N \in [1,\infty)$, then* (3.12) *holds. In particular (M,g) is not timelike geodesically complete.*
- Timelike Brunn-Minkowski: *If (M,g) satisfies $\mathsf{wTCD}^e_p(K,N)$ for some $p \in (0,1), K \in \mathbb{R}, N \in [1,\infty)$, then* (3.9) *holds.*

In Corollary 5.14 below, taking advantage of the techniques developed in Section 3.4 and Section 4, the results of Corollary 5.11 will be improved to sharp forms in case of timelike non-branching structures.

It is clear that Theorem 5.5 implies the following result for a spacetime with C^0-Lorentzian metric.

Corollary 5.12 (Hawking Singularity Theorem for a spacetime with a C^0-Lorentzian metric)**.** *Let (M,g) be a $2 \leq n$-dimensional timelike non-branching, globally hyperbolic, causally plain spacetime with a C^0-Lorentzian metric satisfying $\mathsf{TMCP}^e_p(K,N)$ for some $p \in (0,1), K \in \mathbb{R}, N \in (1,\infty)$ and assume that the causally-reversed structure satisfies the same conditions.*
Let $V \subset M$ be a Borel achronal FTC subset (or, more strongly, let V be a Cauchy hypersurface) having forward mean curvature bounded above by $H_0 < 0$ in the sense of Definition 5.2.

Then for every $x \in I^+(V)$ it holds $\tau_V(x) \leq D_{H_0,K,N}$, provided H_0, K, N fall in the range specified in Theorem 5.5 . In particular, for every timelike geodesic $\gamma \in \mathrm{TGeo}(M)$ with $\gamma_0 \in V$, the maximal (on the right) domain of definition is contained in $\left[0, D_{H_0,K,N}\right]$; in particular (M,g) is not timelike geodesically complete.

Remark 5.13. [Literature about Hawking singularity Theorem] Hawking singularity Theorem was proved in [43, Theorem 4, p. 272] for smooth space-times (the proof works for C^2 metrics) assuming that V is a *compact* spacelike slice. The result was extended to $C^{1,1}$ metrics in [51] and to C^1 metrics in [37], by approximating the metric of low regularity with smoother metrics. The extension to non-compact future causally complete V was established in [33, Theorem 3.1] (see also [38]) in the smooth setting, and extended to $C^{1,1}$ metrics in [36]. Theorem 5.5 and Corollary 5.12, already in the smooth setting, relax the future causal completeness with the weaker future timelike completeness (in addition to extend the results to a synthetic framework, including C^0 metrics). Hawking (as well as Penrose and Hawking-Penrose) singularity Theorem was also extended to (smooth) closed cone structures [59] and smooth weighted Lorentz-Finsler manifolds [55]. Let us mention that a first synthetic singularity theorem was recently shown in [1] under the stronger assumptions that the space is a synthetic warped product with lower bound on sectional curvature in the sense of comparision triangles (á la Alexandrov).

Specialising Proposition 5.6, Proposition 5.7, Proposition 5.8, Proposition 5.9 to the case of a spacetime with a C^0-Lorentzian metric give:

Corollary 5.14 (Timelike Bishop-Gromov, Bonnet-Myers and Poincare' inequalities)**.** *Let (M, g) be a $2 \leq n$-dimensional timelike non-branching, globally hyperbolic, causally plain spacetime with a C^0-Lorentzian metric satisfying $\mathsf{TMCP}^e_p(K,N)$ for some $p \in (0,1), K \in \mathbb{R}, N \in (1,\infty)$ and assume that the causally-reversed structure satisfies the same conditions.*
Let $V \subset M$ be a Borel achronal FTC subset (or, more strongly, let V be a Cauchy hypersurface). Then:

- Timelike Bishop-Gromov I: *For every compact (τ_V, R_0)-conically shaped subset $E \subset I^+(V) \cup V$ the inequalities* (5.12) *and* (5.13) *hold. In particular, if $\{\tau_V \leq R_0\} \subset X$ is a compact subset then* (5.12) *and* (5.13) *remain valid without capping with the cutoff set E in the left hand side.*

- Timelike Bishop-Gromov II: *For each $x_0 \in M$, each compact subset $E \subset I^+(x_0) \cup \{x_0\}$ τ-star-shaped with respect to x_0, and each $0 < r < R \leq \pi\sqrt{(N-1)/(K \vee 0)}$, the inequalities* (5.15) *hold.*

- Timelike Poincaré: *For every $u : M \to \mathbb{R}$ Lipschitz with compact support contained in $I^+(V)$, the inequality* (5.14) *holds.*

- Timelike Bonnet-Myers: *If $K > 0$, then inequality* (5.16) *holds.*

A Appendix - $\mathsf{TMCP}^e_p(K,N)$ on smooth Lorentzian manifolds

Theorem A.1. *Let (M^n, g) be a globally hyperbolic smooth spacetime of dimension $n \geq 2$ without boundary. Then the associated Lorentzian geodesic space satisfies $\mathsf{TMCP}^e_p(K,n)$ if only if $\mathrm{Ric}_g(v,v) \geq -Kg(v,v)$ for every timelike vector $v \in TM$.*

Proof. **Step 1**: “If” implication.
From Theorem 3.1, the Lorentzian geodesic space associated to (M^n, g) satisfies the $\mathsf{TCD}^e_p(K,n)$ condition, which in turn implies $\mathsf{TMCP}^e_p(K,n)$ by Proposition 3.11 (see also Remark 1.12).

Step 2 : “Only if” implication.
Fix $x \in M$ and $v \in T_pM$ future pointing with $g(v,v) = -1$. Let U be a compact subset of $\{w \in T_xM :$ w is future pointing with $g(w,w) < 0\}$, star-shaped with respect to 0, such that $rv \in U$ for $r > 0$ small enough and such that the exponential map $\exp^g_x : U \to M$ of g based at x is a diffeomorphism onto its image when restricted to U. Calling $d\mathrm{Vol}_g$ the volume density on M associated to g, recall that it can be represented as

$$d\mathrm{Vol}_g(y) = (\exp^g_x)_\sharp\left(\mathrm{A}_x(r,\xi)drd\xi\right), \quad \text{for all } y = \exp^g_x(r\xi) \in \exp^g_x(U), \tag{A.1}$$

where $\mathrm{A}_x(r,\xi)$ denotes the volume density on $\{r\xi \in U : g(\xi,\xi) = -1\}$ induced by g.

Fix a g-orthonormal basis $e_1, e_2, \dots, e_n$ of T_pM with $e_1 = v$ and denote by κ_i the sectional curvature of the plane spanned by e_1 and e_i, for $i = 2, \dots, n$. Recalling the definitions (3.6), (3.7) of $\mathfrak{s}_\kappa(\vartheta)$ and $\sigma_\kappa^{(t)}(\vartheta)$ respectively, it is easy to check that for small $r > 0$ it holds:

$$\sigma_\kappa^{(1/2)}(2r) = \frac{\mathfrak{s}_\kappa(r)}{\mathfrak{s}_\kappa(2r)} = \frac{1}{2}\left(1 + \frac{\kappa}{2}r^2 + O(r^4)\right). \tag{A.2}$$

Standard Jacobi-fields computations (see for instance [27] for the Lorentzian setting) give that

$$\frac{\mathrm{A}_x(r,v)}{\mathrm{A}_x(2r,v)} = \prod_{i=2}^{n} \frac{\mathfrak{s}_{\kappa_i}(r)}{\mathfrak{s}_{\kappa_i}(2r)} + O(r^3). \tag{A.3}$$

Plugging (A.2) into (A.3) yields

$$\begin{aligned}\frac{\mathrm{A}_x(r,v)}{\mathrm{A}_x(2r,v)} &= \frac{1}{2^{n-1}} \prod_{i=2}^{n}\left(1 + \frac{\kappa_i}{2}r^2\right) + O(r^3) = \frac{1}{2^{n-1}}\left(1 + \sum_{i=2}^{n} \kappa_i r^2\right) + O(r^3) \\ &= \frac{1}{2^{n-1}}\left(1 + \mathrm{Ric}_g(v,v) r^2\right) + O(r^3).\end{aligned} \tag{A.4}$$

We next relate $\mathrm{A}_x(r,v)/\mathrm{A}_x(2r,v)$ with the $\mathsf{TMCP}^e_p(K,n)$ condition via localisation.

Consider $\tau_x(\cdot) := \tau(x,\cdot) : I^+(x) \supset \exp^g_x(U) \to \mathbb{R}$. By the very definitions, we have $\tau_x\left(\exp^g_x(r\xi)\right) = r$ for every $r\xi \in U$, $g(\xi,\xi) = -1$. In other terms, the partition of $\exp^g_x(U) \setminus \{x\}$ by future pointing g-geodesics emanating from x coincides with the partition by transport rays induced by τ_x.

Under this identification, the disintegration of $d\mathrm{Vol}_g$ induced by τ_x is nothing but (A.1). Theorem 4.17 then gives that $r \mapsto \mathrm{A}_x(r,v)$ is an $\mathsf{MCP}(K,n)$ density on an interval $(0,\varepsilon_v)$ (see for instance the proof of [62, Theorem 3.2]): it thus satisfies

$$\frac{\mathrm{A}_x(r,v)}{\mathrm{A}_x(2r,v)} \geq \left(\frac{\mathfrak{s}_{K/(n-1)}(r)}{\mathfrak{s}_{K/(n-1)}(2r)}\right)^{n-1}. \tag{A.5}$$

Combining (A.4) with (A.5), we obtain

$$\begin{aligned}\mathrm{Ric}_g(v,v)r^2 &\geq \left(\frac{2\mathfrak{s}_{K/(n-1)}(r)}{\mathfrak{s}_{K/(n-1)}(2r)}\right)^{n-1} - 1 + O(r^3) = \left(1 + \frac{K}{n-1}r^2\right)^{n-1} - 1 + O(r^3) \\ &= Kr^2 + O(r^3).\end{aligned}$$

Dividing both sides by r^2 and sending $r \downarrow 0$, we thus obtain $\mathrm{Ric}_g(v,v) \geq K = -Kg(v,v)$.
By the arbitrariness of x and v, the proof is complete. □

Corollary A.2. *Let (M^n, g) be a globally hyperbolic smooth spacetime of dimension $n \geq 2$ without boundary.*

1. *If $\mathrm{Ric}_g(v,v) \geq -Kg(v,v)$ for every timelike vector $v \in TM$, then the associated Lorentzian geodesic space satisfies $\mathsf{TMCP}^e_p(K',N')$ for every $K' \leq K$ and $N' \geq N$.*
2. *If the Lorentzian geodesic space associated to (M^n,g) satisfies $\mathsf{TMCP}^e_p(K,N)$, then $n \leq N$.*

Proof. The first statement follows from Theorem A.1 and Lemma 3.10 (or from Theorem 3.1 and Proposition 3.11).
We now prove the second statement. We will build on the proof of Theorem A.1. Fix $x \in M$ and let

$$U \subset \{w \in T_xM : w \text{ is future pointing with } g(w,w) < 0\}$$

be compact star-shaped with respect to 0, with non-empty interior, such that the exponential map $\exp^g_x : U \to M$ of g based at x is a diffeomorphism onto its image when restricted to U. Calling

$$B^g_r(x,U) := \exp^g_x(\{w \in U : |g(w,w)| \leq r\}), \quad r > 0,$$

it is easy to see that there exists $c = c_U > 0$ such that

$$\mathrm{Vol}_g(B^g_r(x,U)) = c\, r^n + O(r^{n+1}), \quad \text{for small } r > 0. \tag{A.6}$$

On the other hand, using that $r \mapsto \mathrm{A}_x(r,v)$ is an $\mathsf{MCP}(K,N)$ density (see the discussion before (A.5)) and recalling (A.1), we obtain via the classical Gromov's Lemma (see for instance [22, Lemma III.4.1]) that

$$(0,\varepsilon) \ni r \mapsto \frac{\mathrm{Vol}_g(B^g_r(x,U))}{\int_0^r \left[\mathfrak{s}_{K/(N-1)}(t)\right]^{N-1} dt} \quad \text{is monotone non-increasing.} \tag{A.7}$$

Since $\mathfrak{s}_{K/(N-1)}(t) = O(t)$ for small $t > 0$, it is easy to see that the combination of (A.6) and (A.7) yields $n \leq N$. □

Remark A.3. In general, $\mathsf{TMCP}^e_p(K,N)$ on a globally hyperbolic smooth spacetime *does not* imply that $\mathrm{Ric}_g(v,v) \geq -Kg(v,v)$ for every timelike vector $v \in TM$. It follows that $\mathsf{TMCP}^e_p(K,N)$ is a *strictly weaker* condition than $\mathsf{wTCD}^e_p(K,N)$. More precisely, the following holds: For each $N > 1$ there exists a constant $c_N > 0$ such that each globally hyperbolic smooth spacetime with timelike Ricci curvature ≥ 0, dimension $\leq N-1$ and τ-diameter $\leq L$ satisfies $\mathsf{TMCP}^e_p(K,N)$ for each positive $K \leq c_N/L^2$ (compare with [73, Remark 5.6] for the Riemannian setting).

Proof. From Theorem A.1 we know that $\mathsf{TMCP}^e_p(0,N-1)$ holds. Recalling that the $\mathsf{TMCP}^e_p(K,N)$ condition is equivalent to (3.14), it is sufficient to show that

$$t^{N-1} \geq \left(\sigma^{(t)}_{K/N}(\vartheta)\right)^N, \quad \forall t \in [0,1],\ \vartheta \in [0,L].$$

Now, for sufficiently small $c_N \in (0,1)$ and all $K\vartheta^2 \leq c_N$, the right-hand side can be estimated from above by $t^N(1+(1-t^2)K\vartheta^2)$. But clearly $t^{N-1} \geq t^N\left(1+(1-t^2)K\vartheta^2\right)$, for all $K\vartheta^2 \leq c_N$. □

References

[1] S.B. Alexander, M. Graf, M. Kunzinger and C. Sämann, Generalized cones as Lorentzian length spaces: Causality, curvature, and singularity theorems, *Preprint* arXiv:1909.09575.

[2] L. Ambrosio, N. Gigli and G. Savaré, Gradient flows in metric spaces and in the space of probability measures, *Lectures in Math. ETH Zürich*, Birkhäuser Verlag, (2005).

[3] L. Ambrosio and A. Pratelli, Existence and stability results in the L^1 theory of optimal transportation, In *Optimal transportation and applications (Martina Franca, 2001)*, Vol. 1813 of *Lecture Notes in Math.*, pages 123–160, Springer, Berlin, (2003).

[4] K. Bacher and K.-T. Sturm, Localization and tensorization properties of the curvature-dimension condition for metric measure spaces, *J. Funct. Anal.,* **259**, (2010), 28–56.

[5] M. Beiglböck and W. Schachermayer, Duality for Borel measurable cost functions, *Trans. Amer. Math. Soc.*, **363**, 8, 4203–4224.

[6] P. Bernard and S. Suhr, Lyapounov functions of closed cone fields: from Conley theory to time functions, *Comm. Math. Phys.*, **359**, (2), 467–498, (2018).

[7] J. Bertrand and M. Puel, The optimal transport problem for relativistic costs, *Calc. Var. Partial Differential Equations*, **46**, 1–2, (2013), 353–374.

[8] J. Bertrand, A. Pratelli and M. Puel, Kantorovich potentials and continuity of total cost for relativistic cost functions, *J. Math. Pures Appl.*, **110**, 9, (2018), 93–122.

[9] S. Bianchini and L. Caravenna, On the extremality, uniqueness and optimality of transference plans, *Bull. Inst. Math. Acad. Sin.(N.S.)*, **4**, (2009), 353–454.

[10] S. Bianchini and F. Cavalletti, The Monge problem for distance cost in geodesic spaces, *Commun. Math. Phys.*, **318**, (2013), 615 – 673.

[11] L. Bombelli, J. Lee, D. Meyer and R.D. Sorkin, Space-time as a causal set, *Phys. Rev. Lett.*, **59**, (5), 521–524, (1987).

[12] A. Burtscher and P.G. LeFloch, The formation of trapped surfaces in sphericallysymmetric Einstein-Euler spacetimes with bounded variation, *J. Math. Pures Appl.*, **102** (6), 1164–1217, (2014).

[13] S. Carroll, Spacetime and Geometry: An Introduction to General Relativity, *Cambridge Univ. Press*, Cambridge, (2019).

[14] F. Cavalletti, Decomposition of geodesics in the Wasserstein space and the globalization property, *Geom. Funct. Anal.*, **24**, (2014) 493 – 551.

[15] F. Cavalletti and E. Milman, The Globalization Theorem for the Curvature Dimension Condition, *preprint* arXiv:1612.07623.

[16] F. Cavalletti and A. Mondino, Sharp and rigid isoperimetric inequalities in metric-measure spaces with lower Ricci curvature bounds, *Invent. math.*, **208**, (2017), no. 3, 803–849.

[17] ———, Sharp and rigid geometric and functional inequalities in metric-measure spaces with lower Ricci curvature bounds, *Geom. Topol.*, **21**, (2017) 603–645.

[18] ———, Optimal maps in essentially non-branching spaces, *Commun. Contemp. Math.*, **19**, (6), 1750007 (2017)

[19] ———, New formulas for the Laplacian of distance functions and applications, *Analysis & PDE*, to appear, preprint arXiv:1803.09687.

[20] ———, in preparation.

[21] F. Cavalletti and K.-T. Sturm, Local curvature-dimension condition implies measure-contraction property, *J. Funct. Anal.*, **262**, 5110 – 5127, 2012.

[22] I. Chavel, Riemannian geometry. A modern introduction. Second edition. Cambridge Studies in Advanced Math., **98**, Cambridge Univ. Press, Cambridge, (2006).

[23] D. Christodoulou, The formation of black holes in general relativity, *EMS Monographs in Math.*, Europ. Math. Soc., (2009).

[24] P.T. Chrúsciel and J.D.E. Grant, On Lorentzian causality with continuous metrics, *Class. Quantum Grav.*, **29**, (14), 145001, 32 (2012).

[25] M. Dafermos, The mathematical analysis of black holes in general relativity, *Proceedings of the ICM*, Seoul 2014, Vol III, pp. 747–772.

[26] M. Eckstein and T. Miller, Causality for Nonlocal Phenomena, *Ann. Henri Poincaré*, **18**, (2017), 3049–3096.

[27] P. E. Ehrlich and M. Sanchéz, Some semi-Riemannian volume comparison Theorems, *Tohoku Math. J.*, **52**, (2000), 331–348.

[28] M. Erbar, K. Kuwada and K.T. Sturm, On the Equivalence of the Entropic Curvature-Dimension Condition and Bochner's Inequality on Metric Measure Space, *Invent. Math.*, **201**, (2015), no. 3, 993–1071.

[29] A. Fathi and A. Siconolfi, On smooth time functions, *Math. Proc. Cambridge Philos. Soc.*, **152**, no. 2, (2012), 303–339.

[30] F. Finster, Causal Fermion Systems: A Primer for Lorentzian Geometers, *J. Phys.: Conf. Ser.* **968** 012004, (2018).

[31] F. Finster, The Continuum Limit of Causal Fermion Systems. Fundamental Theories of Physics, *Fundamental Theories of Physics*, **186**, Springer, (2016).

[32] D. H. Fremlin, *Measure Theory*, vol. 4, Part I, Torres Fremlin (2003).

[33] G. Galloway, Curvature, causality and completeness in space-times with causally complete spacelike slices, *Math. Proc. Cambridge Philos. Soc.*, **99**, (1986), 367–375.

[34] R. Geroch and J. Traschen, Strings and other distributional sources in general relativity, *Phys. Rev. D* **36** (4), 1017–1031, (1987).

[35] N. Gigli, A. Mondino and G. Savaré, Convergence of pointed non-compact metric measure spaces and stability of Ricci curvature bounds and heat flows, *Proc. London Math. Soc.*, **111**, (5), (2015), 1071–1129.

[36] M. Graf, Volume comparison for $C^{1,1}$-metrics, *Ann. Glob. Anal. Geom.* **50** (2016), 209–235.

[37] ———, Singularity theorems for C^1-Lorentzian metrics, *Preprint* arxiv:1910.13915v2.

[38] J.D.E. Grant and J.H. Treude, Volume comparison for hypersurfaces in Lorentzian manifolds and singularity theorems, *Ann. Glob. Anal. Geom.*, **43**, (2013), 233–251.

[39] J.D.E. Grant, M. Kunzinger and C. Sämann, Inextendibility of spacetimes and Lorentzian length spaces, *Ann. Glob. Anal. Geom.*, **55**, (2018), 133–147.

[40] J.P. Griffiths and J. Podolský, Exact Space-Times in Einstein's General Relativity *Cambridge Monographs on Math. Phys.*, Cambridge Univ. Press, Cambridge, (2009)

[41] B. Han and E. Milman, Sharp Poincaré inequality under Measure Contraction Property, *Preprint* Arxiv: 1905.05465.

[42] S.W. Hawking, The occurrence of singularities in cosmology I, *Proc. Roy. Soc. Ser. A*, **294**, (1966), 511–521.

[43] S.W. Hawking and G.F.R. Ellis, The large scale structure of space-time, Cambridge Univ. Press, London, (1973)

[44] S.W. Hawking and R. Penrose, The singularities of gravitational collapse and cosmology, *Proc. Roy. Soc. London Ser. A*, **314**, (1970), 529–548.

[45] M. Kell, On interpolation and curvature via Wasserstein geodesics, *Adv. Calc. Var.*, **10**, (2017), 125–167.

[46] M. Kell and S. Suhr, On the existence of dual solutions for Lorentzian cost functions, *Annales IHP, Anal. non lin.*, in press, DOI:10.1016/j.anihpc.2019.09.005.

[47] S. Krantz and H. Parks, Geometric integration theory, *Cornerstones*, Birkhäuser Boston Inc., (2008).

[48] C. Ketterer, The Heintze-Karcher inequality for metric measure spaces, *Proc. Amer. Math. Soc.*, to appear, preprint arXiv:1908.06146.

[49] S. Klainerman, I. Rodnianski and J. Szeftel, The bounded L^2 curvature conjecture, *Invent. Math.*, **202**, (2015), no. 1, 91–216.

[50] M. Kunzinger and C. Sämann, Lorentzian length spaces, *Ann. Global Anal. Geom.*, **54**, (3), (2018), 399–447.

[51] M. Kunzinger, R. Steinbauer, M. Stojković, J.A. Vickers, Hawking's singularity theorem for $C^{1,1}$-metrics, *Class. Quantum Grav.*, **32**, 075012, (2015)

[52] A. Lichnerowicz, Théories relativistes de la gravitation et de l'électromagnétisme, Relativité générale et théories unitaires, Masson et Cie, Paris (1955).

[53] S. Lisini, Characterization of absolutely continuous curves in Wasserstein spaces, *Calc. Var. PDE*, **28**, (2006), 85–120.

[54] J. Lott and C. Villani, Ricci curvature for metric-measure spaces via optimal transport, *Ann. of Math.* (2) **169**, (2009), 903–991.

[55] Y. Lu, E. Minguzzi, S.I. Ohta, Geometry of weighted Lorentz–Finsler manifolds I: Singularity theorems, *Preprint* arXiv: 1908.03832.

[56] M. Mars and J.M.M. Senovilla, Geometry of general hypersurfaces in spacetime: junction conditions, *Class. Quantum Grav.*, **10**, 1865–1897, (1993).

[57] K. Martin and P. Panangaden, A domain of spacetime intervals in general relativity, *Comm. Math. Phys.*, **267** (3), 563–586, (2006).

[58] R.J. McCann, Displacement convexity of Boltzmann's entropy characterizes the strong energy condition from general relativity, *Preprint* arXiv1808.1536v2.

[59] E. Minguzzi, Causality theory for closed cone structures with applications, *Reviews in Math. Phys.*, **31**, 5, 1930001, (2019).

[60] E. Minguzzi and S. Suhr, Some regularity results for Lorentz-Finsler spaces, *Ann. Glob. Anal. Geom.,* https://doi.org/10.1007/s10455-019-09681-w, 2019.

[61] A. Mondino and S. Suhr, An optimal transport formulation of the Einstein equations of general relativity, *Preprint* arXiv:1810.13309v2.

[62] S.I.. Ohta, On the measure contraction property of metric measure spaces, *Comment. Math. Helv.*, **82**, (2007), 805–828.

[63] B. O'Neill, Semi-Riemannian geometry, *Pure and Applied Math.*, **103**, Academic Press Inc., New York, (1983).

[64] R. Penrose, Gravitational collapse and space-time singularities, *Phys. Rev. Lett.*, **14**, (1965), 57–59.

[65] ———, The geometry of impulsive gravitational waves, In: General Relativity (papers in honour of J. L. Synge), pp. 101–115. Clarendon Press, Oxford (1972).

[66] A.D. Rendall, Theorems on existence and global dynamics for the Einstein equations, *Living Rev. Relativity*, **5**, (2002), 2002-6, 62 pp.

[67] C. Sämann, Global hyperbolicity for spacetimes with continuous metrics, *Ann. Henri Poincaré*, **17**, (2016), 1429–1455.

[68] J. Sbierski, The C^0-inextendibility of the Schwarzschild spacetime and the spacelike diameter in Lorentzian Geometry, *J. Differential Geom.*, **108** (2018), no. 2, 319–378.

[69] C. Sormani and C. Vega, Null distance on a spacetime, *Class. Quantum Grav.*, **33**, (8), 085001, (2016).

[70] S.M. Srivastava, A course on Borel sets, *Graduate Texts in Math.*, Springer, (1998).

[71] R. Steinbauer and J.A. Vickers, On the Geroch-Traschen class of metrics, *Class. Quantum Grav.*, **26**, 065001, (2009).

[72] K.T. Sturm, On the geometry of metric measure spaces. I, *Acta Math.*, **196** (2006), 65–131.

[73] ———, On the geometry of metric measure spaces. II, *Acta Math.*, **196** (2006), 133–177.

[74] S. Suhr, Theory of optimal transport for Lorentzian cost functions, *Münster J. of Math.*, **11** (2018), 13–47.

[75] J.A. Vickers, Quasi-regular singularities and cosmic strings, *Class. Quantum Grav.*, **7**, (5), 731–741, (1990).

[76] J.A. Vickers and J.P. Wilson, Generalized hyperbolicity in conical spacetimes, *Class. Quantum Grav.*, **17**, (6), 1333–1360, (2000).

[77] C. Villani, Topics in optimal transportation, *Graduate Studies in Math.*, **58**, Am. Math. Soc., Providence, RI, (2003).

[78] ———, Optimal transport. Old and new, *Grund. Math. Wiss.*, **338**, Springer-Verlag, Berlin, (2009).

[79] R. Wald, General relativity, University of Chicago Press, Chicago, IL, (1984).

[80] E. Woolgar and W. Wylie, Cosmological singularity theorems and splitting theorems for N-Bakry-Émery spacetimes, *J. Math. Phys.*, **57**, 022504, 12, (2016).