Asymptotics of Radiative Spacetimes

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Overview

- Einstein equations and asymptotic flatness
- Asymmetric structures
- Gravitational waves and memory
- Non-isotropic mass
- New results, including natural "barrier" for the peeling of the curvature components
- Angular momentum well-defined
- Antipodal (non-)symmetry

Photos:

The Seagull Nebula: Courtesy of Carlos Taylor,

NGC 6559: Courtesy of Adam Block, Telescope Live.

Einstein Equations and Spacetimes

Einstein Equations

$$\mathbf{R}_{\mu\nu} - \frac{\mathbf{1}}{\mathbf{2}} \, \mathbf{g}_{\mu\nu} \, \mathbf{R} = \mathbf{8}\pi \, \mathbf{T}_{\mu\nu} \ , \tag{1}$$

with

$\mathbf{R}_{\mu\nu}$ the **Ricci curvature tensor**,

$\mathbf{R}$ the **scalar curvature tensor**,

$\mathbf{g}$ the **metric tensor** and

$\mathbf{T}_{\mu\nu}$ the **energy-momentum tensor**.

Investigate dynamics of spacetimes (M, g), where M a 4-dimensional manifold with Lorentzian metric g solving Einstein's equations (1).

Einstein Vacuum (EV) Equations

$$R_{\mu\nu} \quad = \quad 0 \ . \tag{2}$$

Evolution Equations, Constraints and Lapse

Evolution equations of a maximal foliation:

$$
\begin{aligned}
\frac{\partial \bar{g}_{ij}}{\partial t} &= -2\Phi k_{ij} \\
\frac{\partial k_{ij}}{\partial t} &= -\nabla_i \nabla_j \Phi \; + \; (\bar{R}_{ij} \; - \; 2k_{im} k^m_{\;j})\Phi
\end{aligned}
$$

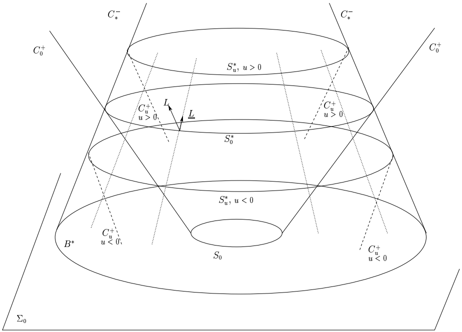
Constraint equations of a maximal foliation:

$$
\begin{aligned}
trk &= 0 \\
\nabla^i \; k_{ij} &= 0 \\
\bar{R} &= \mid k \mid^2
\end{aligned}
$$

Lapse equation of a maximal foliation:

$$\triangle \Phi - \; \mid k \mid^2 \Phi \; = \; 0$$

Foliations of the Spacetime



Foliation by a time function t
$\Rightarrow$ spacelike, complete Riemannian hypersurfaces H_t.

Foliation by a function u
$\Rightarrow$ null hypersurfaces C_u.

$$S_{t,u} = H_t \cap C_u$$

Large Data

In order to investigate gravitational waves, we study the Cauchy problem.

• Stability theorems give precise description of null infinity.

• They are proven for small data. However, important results also hold for large data.

Large data

Main behavior along null hypersurfaces towards future null infinity

$\Rightarrow$ Largely independent from the smallness.

General Spacetimes

From stability proofs, we have a very good understanding of the dynamics and behavior of various classes of spacetimes.

Theorems [D. Christodoulou and S. Klainerman (1991), L. Bieri (2007)]

Every asymptotically flat initial data obeying appropriate smallness assumptions (controlled via weighted Sobolev norms) gives rise to a globally asymptotically flat solution of the Einstein vacuum equations that is causally geodesically complete.

(CK) initial data where at spacelike infinity
$\bar{g}_{ij} = (1 + \frac{2M}{r})\ \delta_{ij} + o_4\ (r^{-\frac{3}{2}})$, and $k_{ij} = o_3\ (r^{-\frac{5}{2}})$.
(B) initial data where at spacelike infinity $\bar{g}_{ij} = \delta_{ij} + o_3\ (r^{-\frac{1}{2}})$ and $k_{ij} = o_2\ (r^{-\frac{3}{2}})$. (ADM energy finite.)

Small data ensures existence.

Large data

Main behavior along null hypersurfaces towards future null infinity

$\Rightarrow$ Largely independent from the smallness.

Non-Isotropic Mass

First, note that in *(CK) spacetimes* with initial metric asymptotically flat of the type

$$\bar{g}_{ij} = (1 + \frac{2M}{r})\ \delta_{ij} + o_4\ (r^{-\frac{3}{2}})$$

the mass M is a constant.

⇒ What about the *more general situation* where we have mass that is non-isotropic, that is

$$M(\theta, \phi)?$$

⇒ Investigate *(A) spacetimes, more general!* (See next slide.)

Asymptotic Flatness

(A) Asymptotically flat initial data set in the sense of (A): an asymptotically flat initial data set $(H_0, \bar{g}, k)$, where $\bar{g}$ and k are sufficiently smooth and for which there exists a coordinate system (x^1, x^2, x^3) in a neighbourhood of infinity such that with $r = (\sum_{i=1}^{3} (x^i)^2)^{\frac{1}{2}} \to \infty$, it is:

$$\bar{g}_{ij} = \delta_{ij} + h_{ij} + o_3 \left(r^{-\frac{3}{2}}\right) \quad (3)$$

$$k_{ij} = o_2(r^{-\frac{5}{2}}) \quad (4)$$

with h_{ij} being homogeneous of degree -1.

In particular, h may include a non-isotropic mass term $M(\theta, \phi)$ depending on the angles. The spacetime metric will include a resulting term, being homogeneous of degree -1 with corresponding limit $M(u, \theta, \phi)$ at future null infinity depending on the retarded time u.

Asymptotic Flatness

(A*) Asymptotically flat initial data set in the sense of (A*): an asymptotically flat initial data set $(H_0, \bar{g}, k)$, where $\bar{g}$ and k are sufficiently smooth and for which there exists a coordinate system (x^1, x^2, x^3) in a neighbourhood of infinity such that with $r = (\sum_{i=1}^{3}(x^i)^2)^{\frac{1}{2}} \to \infty$, it is:

$$\begin{aligned} \bar{g}_{ij} &= \delta_{ij} + h_{ij} + o_3\,(r^{-\frac{3}{2}}) && (5) \\ k_{ij} &= O_2(r^{-2}) && (6) \end{aligned}$$

with h_{ij} being homogeneous of degree -1.

Summary of main results:

For (A) spacetimes the following hold:

- Peeling of the Weyl curvature components at future null infinity stops.
- Dynamical behavior with different properties: Different behavior and fall-off properties at various levels, in particular, at future null infinity and spacelike infinity. We also *derive different behavior of crucial curvature components and their derivatives*.
- Energy and momenta at future null infinity are well-defined. In particular, angular momentum can be defined and is finite despite the slow decay for β and its derivatives.
- Address antipodal (non-)symmetries.

Asymptotic Flatness

(B) (L. Bieri 2007) (General asymptotically-flat spacetimes with finite energy.) Asymptotically flat initial data set in the sense of (B): an asymptotically flat initial data set $(H_0, \bar{g}, k)$, where $\bar{g}$ and k are sufficiently smooth and for which there exists a coordinate system (x^1, x^2, x^3) in a neighbourhood of infinity such that with $r = (\sum_{i=1}^{3}(x^i)^2)^{\frac{1}{2}} \to \infty$, it is:

$$\bar{g}_{ij} = \delta_{ij} + o_3\,(r^{-\frac{1}{2}}) \tag{7}$$

$$k_{ij} = o_2\,(r^{-\frac{3}{2}})\,. \tag{8}$$

$$\tag{9}$$

We can also take the following:

$$\bar{g}_{ij} = \delta_{ij} + o_3\,(r^{-\alpha}) \tag{10}$$

$$k_{ij} = o_2\,(r^{-1-\alpha}) \tag{11}$$

for $\alpha > 0$ or $0 < \alpha < 1$.

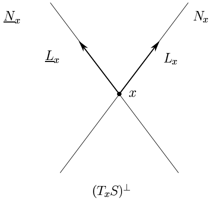
See also recent stability proof by D. Shen for $0 < \alpha < \frac{1}{2}$.

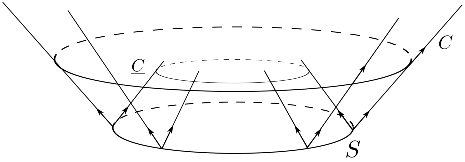
Vectorfields

Start with an outgoing null vectorfield L, define a conjugate (incoming) null vectorfield $\underline{L}$ by requiring that

$$g(L, \underline{L}) = -2 \ .$$

L and $\underline{L}$ are orthogonal to $S_{t,u}$.





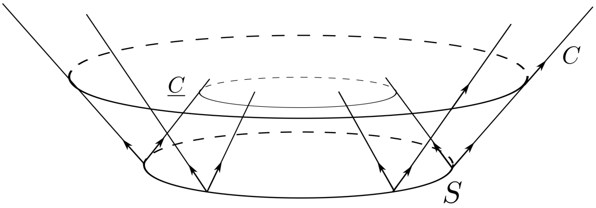
Notation: Denote L by e_4 and $\underline{L}$ by e_3.
Complement e_4 and e_3 with an orthonormal frame e_1, e_2 on $S_{t,u}$
$\Rightarrow$ We obtain a **null frame**.

The **null decomposition** of a tensor relative to a null frame e_4, e_3, e_2, e_1 is obtained by taking **contractions** with the vectorfields e_4, e_3.

Shears and Expansion Scalars

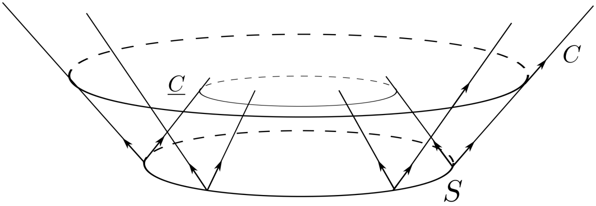
Viewing S as a hypersurface in C, respectively $\underline{C}$:

- Denote the second fundamental form of S in C by χ, and the second fundamental form of S in $\underline{C}$ by $\underline{\chi}$.
- Their traceless parts are the shears and denoted by $\hat{\chi}$, $\hat{\underline{\chi}}$ respectively.
- The traces $tr\chi$ and $tr\underline{\chi}$ are the expansion scalars.
- Null Limits of the Shears:
  $\lim_{C_u, t \to \infty} r^2 \hat{\chi} = \Sigma(u)$ (in (A) spacetimes) and
  $\lim_{C_u, t \to \infty} r \hat{\underline{\chi}} = \Xi(u)$.



Null Hypersurfaces

The important role of null hypersurfaces in general relativity.



Early studies by E.T. Newman and R. Penrose.

Foliation of Null Infinity

Future null infinity $\mathcal{I}^+$ is defined to be the endpoints of all future-directed null geodesics along which $r \to \infty$. It has the topology of $\mathbb{R} \times \mathbb{S}^2$ with the function u taking values in $\mathbb{R}$.

Thus a null hypersurface C_u intersects $\mathcal{I}^+$ at infinity in a 2-sphere $S_{\infty,u}$.

Define the tensor of projection from the tangent space of M to that of $S_{t,u}$

$$\Pi^{\mu\nu} = g^{\mu\nu} + \frac{1}{2}(e_4^\nu e_3^\mu + e_3^\nu e_4^\mu).$$

Definition

We define the null components of the Weyl curvature W as follows:

$$\begin{aligned}
\underline{\alpha}_{\mu\nu}(W) &= \Pi_\mu^{\ \rho}\, \Pi_\nu^{\ \sigma}\, W_{\rho\gamma\sigma\delta}\, e_3^\gamma\, e_3^\delta & (12)\\
\underline{\beta}_\mu(W) &= \frac{1}{2}\Pi_\mu^{\ \rho}\, W_{\rho\sigma\gamma\delta}\, e_3^\sigma\, e_3^\gamma\, e_4^\delta & (13)\\
\rho(W) &= \frac{1}{4} W_{\alpha\beta\gamma\delta}\, e_3^\alpha\, e_4^\beta\, e_3^\gamma\, e_4^\delta & (14)\\
\sigma(W) &= \frac{1}{4}\ {}^*W_{\alpha\beta\gamma\delta}\, e_3^\alpha\, e_4^\beta\, e_3^\gamma\, e_4^\delta & (15)\\
\beta_\mu(W) &= \frac{1}{2}\Pi_\mu^{\ \rho}\, W_{\rho\sigma\gamma\delta}\, e_4^\sigma\, e_3^\gamma\, e_4^\delta & (16)\\
\alpha_{\mu\nu}(W) &= \Pi_\mu^{\ \rho}\, \Pi_\nu^{\ \sigma}\, W_{\rho\gamma\sigma\delta}\, e_4^\gamma\, e_4^\delta\,. & (17)
\end{aligned}$$

Thus, capital indices taking the values $1, 2$, we have:

$$
\begin{aligned}
W_{A3B3} &= \underline{\alpha}_{AB} && (18)\\
W_{A334} &= 2\ \underline{\beta}_{A} && (19)\\
W_{3434} &= 4\ \rho && (20)\\
{}^{*}W_{3434} &= 4\ \sigma && (21)\\
W_{A434} &= 2\ \beta_{A} && (22)\\
W_{A4B4} &= \alpha_{AB} && (23)
\end{aligned}
$$

with

α, $\underline{\alpha}$: S-tangent, symmetric, traceless tensors

β, $\underline{\beta}$: S-tangent 1-forms

ρ, σ : scalars .

`Notation: Hodge duals` ${}^{*}W$ `and` W^{*} `defined as`

$$
\begin{aligned}
{}^{*}W_{\alpha\beta\gamma\delta} &= \frac{1}{2}\varepsilon_{\alpha\beta\mu\nu}W^{\mu\nu}{}_{\gamma\delta}\\
W^{*}_{\alpha\beta\gamma\delta} &= \frac{1}{2}W_{\alpha\beta}{}^{\mu\nu}\varepsilon_{\mu\nu\gamma\delta}
\end{aligned}
$$

We also work with the following frame. It is $T = \frac{1}{\Phi}\frac{\partial}{\partial t}$. Let $T = E_0$ and (E_1, E_2, E_3) an orthonormal frame field for H_t. Thus we have the frame field (E_0, E_1, E_2, E_3) for the spacetime M.

The Weyl tensor $W_{\alpha\beta\gamma\delta}$ is decomposed into its electric and magnetic parts, which are defined by

$$E_{ab} := W_{aTbT} \tag{24}$$

$$H_{ab} := \frac{1}{2}\varepsilon^{ef}{}_{a}W_{efbT} \tag{25}$$

Here ε_{abc} is the spatial volume element and is related to the spacetime volume element by $\varepsilon_{abc} = \varepsilon_{Tabc}$.
In particular, in our notation it is

$$E_{NN} \ = \ \rho \qquad , \qquad H_{NN} \ = \ \sigma \ .$$

Peeling Stops

Solve initial value problem for the EV equations for (A) initial data to obtain **spacetimes of type (A)**. The Weyl curvature components have the following behavior towards future null infinity.

$$\underline{\alpha} = O\left(r^{-1}\,\tau_-^{-\frac{5}{2}}\right) \quad (26)$$

$$\underline{\beta} = O\left(r^{-2}\,\tau_-^{-\frac{3}{2}}\right) \quad (27)$$

$$\rho = O\left(r^{-3}\right) \quad (28)$$

$$\rho - \bar{\rho} = O\left(r^{-3}\right) \quad (29)$$

$$\sigma = O\left(r^{-3}\,\tau_-^{-\frac{1}{2}}\right) \quad (30)$$

$$\sigma - \bar{\sigma} = O\left(r^{-3}\,\tau_-^{-\frac{1}{2}}\right) \quad (31)$$

$$\beta = o\left(r^{-\frac{7}{2}}\right) \quad (32)$$

$$\alpha = o\left(r^{-\frac{7}{2}}\right) \quad (33)$$

Here $\tau_- := \sqrt{1+u^2}$ for retarded time u.

(32)-(33) hold under smallness assumptions, whereas for large data the behavior becomes $O(r^{-3})$.

In particular, we also derive for (A)

$$\not\nabla \rho \quad = \quad O\left(r^{-4}\right) \tag{34}$$

whereas in (CK) by Christodoulou and Klainerman it is

$$\not\nabla \rho \quad = \quad O\left(r^{-4} \tau_-^{-\frac{1}{2}}\right) . \tag{35}$$

Note that we have for (A) spacetimes

$$\rho - \bar{\rho} \; = \; O\left(r^{-3}\right)$$

whereas in (CK)

$$\rho - \bar{\rho} \; = \; O\left(r^{-3} \; \tau_-^{-\frac{1}{2}}\right)$$

takes extra decay in τ_-. The reason for the latter is that for (A) the mass depends on the angles, whereas it is a constant for (CK).

Convention

Convention on the "retarded time" u.

By u we denote the optical function corresponding to *minus the retarded time* in Minkowski spacetime, and by $\underline{u}$ the corresponding *advanced time*. We refer to u just as the *retarded time* with this sign convention.

Therefore, $u \to +\infty$ corresponds to the limit at spacelike infinity, $u \to -\infty$ to the limit at future timelike infinity.

In the following, M denotes the Bondi mass and M^+ its limit for $u \to +\infty$, namely the ADM mass.

Spacetimes with Slow Fall-Off

Spacetimes with Slow Fall-Off and Inhomogeneous Matter and Energy Distributions:

Recall initial data (B) from above, where towards spatial infinity:
$\bar{g}_{ij} = \delta_{ij} \ + \ o_3 \ (r^{-\frac{1}{2}})$ and $k_{ij} = o_2 \ (r^{-\frac{3}{2}})$.

Then the spacetime curvature components behave as:

$$
\begin{aligned}
\underline{\alpha} \quad &= \quad O \ (r^{-1} \ \tau_{-}^{-\frac{3}{2}}) \\
\underline{\beta} \quad &= \quad O \ (r^{-2} \ \tau_{-}^{-\frac{1}{2}}) \\
\rho, \ \sigma, \ \alpha, \ \beta \quad &= \quad o \ (r^{-\frac{5}{2}})
\end{aligned}
$$

The components of the curvature that are not peeling have leading order terms that are non-dynamical (and do not attain corresponding limits at $\mathcal{I}^+$). Take off these pieces $\Rightarrow$ obtain dynamical parts of these (non-peeling) curvature components.

In particular, dynamical parts of ρ show no antipodal symmetry.

Spacetimes with Slow Fall-Off

Respectively, we can consider initial data, where towards spatial infinity: $\bar{g}_{ij} = \delta_{ij} \ + \ o_3 \ (r^{-\alpha})$ and $k_{ij} = o_2 \ (r^{-1-\alpha})$ for $0 < \alpha < 1$.

Then the spacetime curvature components behave as:

$$\begin{aligned} \underline{\alpha} \quad &= \quad O \ (r^{-1} \ \tau_{-}^{-1-\alpha}) \\ \underline{\beta} \quad &= \quad O \ (r^{-2} \ \tau_{-}^{-\alpha}) \\ \rho, \ \sigma, \ \alpha, \ \beta \quad &= \quad o \ (r^{-2-\alpha}) \end{aligned}$$

Gravitational Radiation and Memory

Gravitational waves propagating from their source to our detectors.

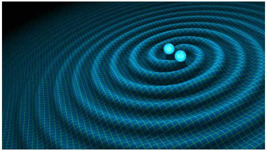
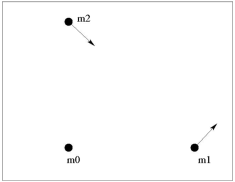


Photo: Courtesy of R. Hurt/Caltech-JPL.

Test masses will experience
• instantaneous displacements (while the wave packet is traveling through)
• permanent displacements (cumulative, stays after wave packet passed).
The memory effect of gravitational waves.



Memory - Permanent Displacement

Asymptotically Flat Spacetimes

The permanent displacement $\triangle x$ of test masses is related to the difference $(\Sigma^- - \Sigma^+)$ at $\mathcal{I}^+$:

$$\triangle x = -\frac{d_0}{r}\left(\Sigma^- - \Sigma^+\right) , \tag{36}$$

where d_0 denotes the initial distance between the test masses, and (in (A) spacetimes) the limit of the shear $\hat{\chi}$ is given as $\lim_{C_u, t\to\infty} r^2 \hat{\chi} = \Sigma(u)$.

Contributions to the permanent displacement $\triangle x$:

AF systems with $O(r^{-1})$ decay: Two types of memory. The ordinary memory is sourced by the change in the radial component of the electric part of the Weyl tensor. The null memory is sourced by F, the energy per unit solid angle radiated to infinity (including shear and component of energy-momentum tensor). No magnetic memory.

(B) spacetimes: Different tensors on the right hand side of (36). In addition, there is magnetic memory. All memories (electric and magnetic) diverge at rate $\sqrt{|u|}$. Additional structures. New memories.

Gravitational Wave Memory

The beginnings....

• Ordinary (formerly called "linear") effect
=> was known for a long time in the slow motion limit [Ya.B. Zel'dovich, A.G. Polnarev 1974]

• Null (formerly called "nonlinear") effect
=> was found by [D. Christodoulou 1991]

• Growing amount of work on memory by many authors.

The following F and F_T generate null memory.

For the Einstein vacuum equations (see D. Christodoulou), the energy radiated away per unit angle in a given direction is $F/4\pi$ with

$$F(\cdot) = \frac{1}{2} \int_{-\infty}^{+\infty} \mid \Xi(u, \cdot) \mid^2 du$$

and Ξ denoting the corresponding limit of the shear (i.e. news)

$$-\frac{1}{2} \lim_{C_u, t \to \infty} r \underline{\widehat{\chi}} = \Xi \left(u, \cdot\right) \ .$$

For the Einstein equations coupled to some other fields (see L. Bieri, P. Chen, D. Garfinkle, S.-T. Yau) the energy radiated away per unit angle in a given direction is $F_T/4\pi$ with

$$F_T(\cdot) = \frac{1}{2} \int_{-\infty}^{+\infty} \left(\mid \Xi(u, \cdot) \mid^2 \ + C \ \mathcal{T}_{33}(u, \cdot) \right) du$$

where C is a (positive) constant.

Bianchi Equations - Electric Memory

Einstein vacuum equations:

Consider the Bianchi equation for $\not{D}_3\rho$.

Notation $\rho_3 := \not{D}_3\rho \; + \; \frac{3}{2}tr\underline{\chi}\rho$.

In the Bianchi equation for $\not{D}_3\rho$

$$\not{D}_3\rho \; + \; \frac{3}{2}tr\underline{\chi}\rho \quad = \quad -\not{d}iv\,\underline{\beta} - \frac{1}{2}\hat{\chi}\underline{\alpha} \; + \; (\varepsilon - \zeta)\underline{\beta} \; + \; 2\underline{\xi}\beta \tag{37}$$

we focus on the higher order terms,

$$\rho_3 \; = \; - \quad \underbrace{\not{d}iv\,\underline{\beta}}_{=O(r^{-3}\tau_-^{-\frac{1}{2}})} \quad - \quad \underbrace{\frac{1}{2}\hat{\chi}\cdot\underline{\alpha}}_{=O(r^{-\frac{5}{2}}\tau_-^{-\frac{3}{2}})} \quad + \; l.o.t.$$

A short computation shows that

$$\rho_3 = - \underbrace{\operatorname{div}\!\!\!/\, \underline{\beta}}_{=O(r^{-3}\tau_-^{-\frac{1}{2}})} - \underbrace{\frac{\partial}{\partial u}(\hat{\chi} \cdot \hat{\underline{\chi}})}_{=O(r^{-\frac{5}{2}}\tau_-^{-\frac{3}{2}})} + \underbrace{\frac{1}{4} tr\chi |\hat{\underline{\chi}}|^2}_{=O(r^{-3}\tau_-^{-1})} + l.o.t.$$

Thus it is

$$\rho_3 + \frac{\partial}{\partial u}(\hat{\chi} \cdot \hat{\underline{\chi}}) = -\operatorname{div}\!\!\!/\, \underline{\beta} + \frac{1}{4} tr\chi |\hat{\underline{\chi}}|^2 = O(r^{-3}\tau_-^{-\frac{1}{2}}) \quad (38)$$

Structures:

For small data, ρ_3 as well as $\frac{\partial}{\partial u}(\hat{\chi} \cdot \hat{\underline{\chi}})$ take a well-defined limit at $\mathcal{I}^+$ when multiplied with r^3.

For large data, that is not the case, but many more terms of order $r^{-\frac{5}{2}}\tau_-^{-\frac{3}{2}}$ exist in ρ_3 as well as in $\frac{\partial}{\partial u}(\hat{\chi} \cdot \hat{\underline{\chi}})$ and potentially terms of order $r^{-\frac{5}{2}}\tau_-^{-1-\alpha}$ with $\alpha \geqslant 0$ in ρ_3. However, as a consequence of equation (38) all these terms on the left hand side of (38) cancel.

Limit at $\mathcal{I}^+$ of the left hand side of (38)

$\Rightarrow$ leading order term originates from ρ_3 and is of order $O(r^{-3}\tau_-^{-\frac{1}{2}})$.

Bianchi Equations - Magnetic Memory

Consider the Bianchi equation for $\not{D}_3\sigma$.

Notation $\sigma_3 = \not{D}_3\sigma + \frac{3}{2}tr\underline{\chi}\sigma$. In the Bianchi equation for σ_3

$$\sigma_3 \ = \ -c\not{u}rl\,\underline{\beta} - \frac{1}{2}\hat{\chi}\cdot\ {}^*\underline{\alpha} + \varepsilon\ {}^*\underline{\beta} - 2\zeta\ {}^*\underline{\beta} - 2\underline{\xi}\ {}^*\beta$$

we concentrate on the higher order terms

$$\sigma_3 \quad = \quad -c\not{u}rl\,\underline{\beta} - \frac{1}{2}\hat{\chi}\cdot\ {}^*\underline{\alpha} + l.o.t. \tag{39}$$

A short computation yields

$$\sigma_3 + \frac{\partial}{\partial u}(\hat{\chi}\wedge\hat{\underline{\chi}}) \quad = \quad -c\not{u}rl\,\underline{\beta} \ = \ O(r^{-3}\tau_-^{-\frac{1}{2}}) \tag{40}$$

For $\hat{\chi}\wedge\hat{\underline{\chi}}$ the orders of the terms are at the level of $\hat{\chi}\cdot\hat{\underline{\chi}}$ above.

Electric Memory and Magnetic Memory

For spacetimes of slow fall-off such as (B) or data with $0 < \alpha < 1$.

Equation for the electric memory at future null infinity.

$$(\mathcal{P}^- - \mathcal{P}^+) - \int_{-\infty}^{+\infty} |\Xi|^2 \, du \ = \ di\not{v} \, di\not{v} \, (Chi^- - Chi^+) \tag{41}$$

Diverging terms in first term on left hand side sourced by electric Weyl curvature component. More contributions and structures in this term. Second term on left hand side finite for (B) spacetimes, but growing for even slower decay.

Equation for the magnetic memory at future null infinity.

$$(\mathcal{Q}^- - \mathcal{Q}^+) \ = \ c\not{u}rl \, di\not{v} \, (Chi^- - Chi^+) \tag{42}$$

Diverging terms on left hand side sourced by magnetic Weyl curvature component. More contributions and structures in this term.

Memory in Spacetimes with Slow Fall-Off

For the more general spacetimes of slow decay we found (B 2020):

1. There is the magnetic memory effect growing with $|u|^{\frac{1}{2}}$, respectively growing with $|u|^{1-\alpha}$ for $0 < \alpha < 1$, sourced by $\mathcal{Q}$ and finite contributions from both $\mathcal{Q}$ and other structures.

2. $\mathcal{Q}$ has further diverging terms at lower order.

3. There is the electric memory. This electric part is growing with $|u|^{\frac{1}{2}}$, respectively growing with $|u|^{1-\alpha}$ for $0 < \alpha < 1$, sourced by $\mathcal{P}$, further lower-order growing terms and finite contributions from $\mathcal{P}$ and from F (the latter start growing for systems of decay $O(r^{-\frac{1}{2}})$ and slower).

4. $c\cancel{u}rl\ di\cancel{v}\,(Chi^- - Chi^+)$ being non-trivial allows for the magnetic structures to appear in gravitational radiation and to enter the permanent changes of the spacetime. Thus, these more general spacetimes generate memory of magnetic type.

Points 1, 2, 4 were established in (B 2020).

Point 3, the leading order behavior as well as the null memory were established in (B 2018).

Adding Neutrinos

(B 2020) Einstein-null-fluid equations describing neutrino radiation:

$$R_{\mu\nu} \;=\; 8\pi \, T_{\mu\nu} \; .$$

Describe the neutrinos in this equation, represented via the energy-momentum tensor given by

$$T^{\mu\nu} \;=\; \mathcal{N} K^{\mu} K^{\nu} \tag{43}$$

with K being a null vector and $\mathcal{N} = \mathcal{N}(\theta_1, \theta_2, r, \tau_-)$ a positive scalar function depending on r, τ_-, and the spherical variables θ_1, θ_2.

When coupled to the Einstein equations in the most general settings, the energy-momentum tensor $T^{\mu\nu}$ obeys those loose decay laws. No symmetry nor other restrictions imposed.

In particular, we do not have stationarity outside a compact set, but instead a distribution of neutrinos decaying very slowly towards infinity.

"Geometric terms": same growth rate as in EV case.

"T" terms: growing at rate $\sqrt{|u|}$.

"Geometric terms": same growth rate as in EV case.

"T" terms: growing at rate $\sqrt{|u|}$.

In particular:

For data as in (B) as well as for data with exact $r^{-\frac{1}{2}}$ decay in the remainder of the metric, there is a contribution from the neutrinos to the electric memory growing at rate $\sqrt{|u|}$.

For data with exact $r^{-\frac{1}{2}}$ decay in the remainder of the metric, in addition, we find the following contribution from the neutrinos to the magnetic memory: Fix u_0, then the integral $\int_{u_0}^{u} \left(c\!\!/url\, T \right)^{*}_{34_3} du$ diverges like $\sqrt{|u|}$ as $|u| \to \infty$.

Solve the corresponding Hodge system on S^2 to derive the full changes of the spacetime.

In the special class of Einstein-neutrino spacetimes with slow fall-off (that is $O(r^{-\frac{1}{2}})$ decay of the remainder of the metric) we find that the angular momentum radiated away caused by the matter is

$$\mathcal{A}_T(\cdot) = 4\pi \int_{-\infty}^{+\infty} \left(c\not{u}rl\, T\right)^*_{34_3}(u,\cdot)du \ .$$

Hereby, $\mathcal{T}_{33}$ and $\left(c\not{u}rl\, T\right)^*_{34_3}$ denote corresponding limits generated by the stress-energy tensor of matter, that is the null fluid describing the neutrino distribution.

Summary

Summary

• Spacetimes decaying like $O(r^{-\alpha})$ for $0 < \alpha < 1$ cause magnetic memory of the above types diverging at $|u|^{1-\alpha}$.

• The corresponding electric memories diverge at the same rate.

• Neutrinos contribute to the electric memory growing at rate $\sqrt{|u|}$.

• A non-trivial curl of neutrino stress-energy starts occurring at $O(r^{-\frac{1}{2}})$.

• The integral $\int_u \frac{\partial}{\partial u}(\hat{\chi} \cdot \underline{\hat{\chi}})\, du$ as well as $\int_u \frac{\partial}{\partial u}(\hat{\chi} \wedge \underline{\hat{\chi}})\, du$ generates finite electric (former), respectively finite magnetic (latter) memory.

Velocity-Coded Memory

B and A. Polnarev 2024:
Results on Velocity-Coded Memory

Scenario: A supermassive black hole surrounded by a large accretion disk. A less massive black hole moves perpendicular to the plane of the disk and intersects it.

After crossing the disk $\Rightarrow$ smaller black hole experiences a jump of acceleration.
$\Rightarrow$ Acceleration jump is seen as a jump in curvature, which happens in a very short time interval.

At the detector, this burst arrives and lasts for the short time $\triangle u$. After this short time $\triangle u$, the velocity of the test masses stays constant over a very long time interval δu. $\Rightarrow$ Velocity-Coded Memory

Electromagnetic Memory

Consider the Maxwell equations.

EM Memory (L. Bieri and D. Garfinkle 2013). The electromagnetic memory is a residual velocity (i.e. kick) of test charges.

It consists of ordinary kick and null kick .

The kick points in the direction of S^A and has a magnitude of

$$\Delta v = \frac{q}{mr}|S^A| \tag{44}$$

- ordinary kick due to difference between the early and late time values of the radial component of the electric field E_r
- null kick due to charge radiated to infinity, that is F giving the amount of charge radiated to infinity per unit solid angle.

Experiment to Measure Electromagnetic Memory

B and D. Garfinkle 2023, 2024:
Experiment to Measure Electromagnetic Memory

Electromagnetic memory $\Rightarrow$ requires a source whose charges are not confined to any bounded spatial region.

Experiment: Create a situation of unbound charges for a short time.

Measure the memory in the far field region.

We also found the following:

Other fields behaving like that:

The stress-energy tensor of the fields gets out to null infinity for

- a field that is both **charged and massless** being the analog for electromagnetism of fields whose stress-energy gets out to null infinity (**Maxwell equations with massless charge, linear**),
- Maxwell-Klein-Gordon system for a charged, massless scalar field (**nonlinear**),
- **charged null dust** (**nonlinear**, can be derived from [BG] result on null fluids).

(A) and (A*) Spacetimes: Limits at Future Null Infinity $\mathcal{I}^+$

Limits at Future Null Infinity $\mathcal{I}^+$ for (A) Spacetimes

$$\begin{aligned}
\lim_{C_u, t\to\infty} r^3\rho &= P(u,\theta,\phi) \\
\bar{P} &= \bar{P}(u) \\
(P-\bar{P})(u,\theta,\phi) &: \text{ does not decay in } |\, u \,| \text{ as } |\, u \,| \to \infty, \\
& \quad \text{leading order term is dynamical, i.e. depends on } u, \\
& \quad \text{and also depends on the angles } \theta,\ \phi \\
\lim_{u\to+\infty} P(u,\theta,\phi) &= P^+(\theta,\phi)
\end{aligned}$$

We see that $P = P(u,\theta,\phi)$ is a function on $R \times S^2$, and $P^+ = P^+(\theta,\phi)$ is a function on S^2. Thus, in particular, as $u \to +\infty$, the quantity $P(u,\theta,\phi)$ tends to a function $P^+(\theta,\phi)$ on S^2, not a constant.

For (CK) spacetimes it is

$$\begin{aligned}
P-\bar{P} &= O(|\, u \,|^{-\frac{1}{2}}) \\
\lim_{u\to+\infty} P &= P^+ = \lim_{u\to+\infty} \bar{P} = \bar{P}^+ = -2M^+_{ADM} = constant
\end{aligned}$$

Limits at Spacelike Infinity

Limits at Spacelike Infinity for (A) Spacetimes

Consider ρ.

Denote by $P_{H_0}(\theta,\phi)$ the limit of $r^3\rho$ at spacelike infinity. The following limits obey

$$P_{H_0}(\theta,\phi) \neq P^+(\theta,\phi) \quad \text{in general} \ ,$$

however,

$$\int_{S^2} P_{H_0}(\theta,\phi) = \int_{S^2} P^+(\theta,\phi) \ .$$

$\mathbf{P_{H_0}}(\theta,\phi)$, respectively $\mathbf{P}^+(\theta,\phi)$, do not have any $l=1$ modes, but they have all the other modes $l=0$ and $l \geqslant 2$.

Limits at Future Null Infinity

Recall first the limit at spacelike infinity:

$\mathbf{P}_{\mathbf{H_o}}(\theta, \phi)$ does not have any $l = 1$ modes, but it has all the other modes $l = 0$ and $l \geqslant 2$..

At future null infinity: First take the limit at future null infinity and then the limit as retarded time $u \to \infty$ (past):

$\mathbf{P}^{+}(\theta, \phi)$ does not have any $l = 1$ modes, but it has all the other modes $l = 0$ and $l \geqslant 2$.

Theorem [L. Bieri (2022)]

For (A) spacetimes, the normalized curvature components $r\underline{\alpha}$, $r^2\underline{\beta}$, $r^3\rho$, $r^3\sigma$ have limits on C_u as $t \to \infty$:

$$\lim_{C_u, t\to\infty} r\underline{\alpha} = A(u,\cdot)\,, \qquad \lim_{C_u, t\to\infty} r^2\underline{\beta} = \underline{B}(u,\cdot)\ ,$$
$$\lim_{C_u, t\to\infty} r^3\rho = P(u,\cdot)\ , \qquad \lim_{C_u, t\to\infty} r^3\sigma = Q(u,\cdot)$$

where the limits are on S^2 and depend on u. These limits satisfy

$$|A(u,\cdot)| \leqslant C(1+|u|)^{-5/2} \qquad |\underline{B}(u,\cdot)| \leqslant C(1+|u|)^{-3/2}$$
$$|Q(u,\cdot)| \leqslant C(1+|u|)^{-1/2}$$

whereas $P(u,\cdot)$, $(P(u,\cdot) - \bar{P}(u))$ do not decay in $|u|$.

Moreover, the following limits exist

$$\lim_{C_u, t\to\infty} r^2\widehat{\chi} =: \Sigma(u,\cdot) \tag{45}$$
$$-\frac{1}{2}\lim_{C_u, t\to\infty} r\underline{\widehat{\chi}} = \lim_{C_u, t\to\infty} r\hat{\eta} =: \Xi(u,\cdot) \tag{46}$$

Limits

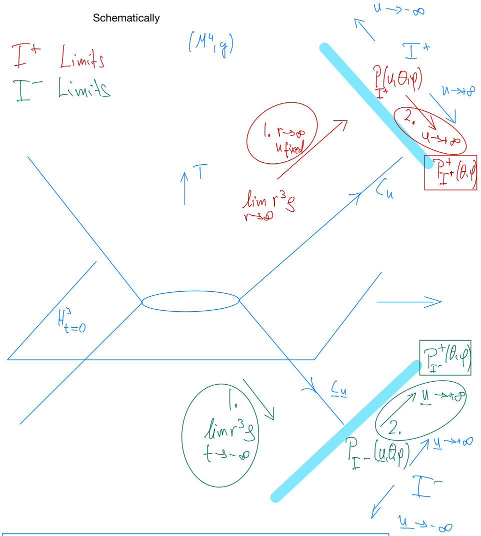
At future null infinity $\mathcal{I}^+$, let the retarded time u tend to the past (that is to $+\infty$ in our convention), respectively the future (that is to $-\infty$ in our convention).

(CK) :
$(P - \overline{P})^+ = (P - \overline{P})^- = 0.$

Binary merger with slow initial velocities :
$(P - \overline{P})^+ = 0 \ , \quad (P - \overline{P})^- \neq 0.$

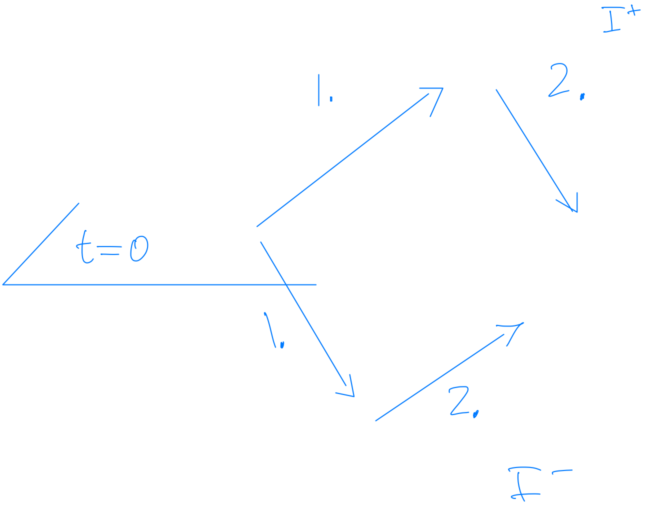
Binary merger with large initial velocities : general case :
$(P - \overline{P})^+ \neq 0 \ , \quad (P - \overline{P})^- \neq 0.$

Taking Limits



Compare $P^+_{I^+}(\theta,\varphi)$ to $P^+_{I^-}(\theta,\varphi)$.

Taking Limits



$P^{+}_{\mathcal{I}^+}(\theta, \varphi)$ and $P^{+}_{\mathcal{I}^-}(\theta, \varphi)$ are functions on S^2.

Simple Situations with Antipodal Symmetry

Simple Situations with Antipodal Symmetry:

Schwarzschild spacetime:

$\rho = -\frac{2m}{r_s^3} \quad \Rightarrow \quad$ trivially holds for corresponding limits.

Boosted Schwarzschild:

Compute corresponding limits for ρ:

$$P^*_{\mathcal{I}^+} = 2m \cdot a \quad , \quad P^*_{\mathcal{I}^-} = 2m \cdot b$$

where the factors a and b include the boost parameter γ and the velocity v, and a is antipodally symmetric to b. That is, if p, q are antipodal points on S^2, then $a(p) = b(q)$.

Therefore: $P^*_{\mathcal{I}^+}(p) = P^*_{\mathcal{I}^-}(q)$.

Continue: Simple Situations with Antipodal Symmetry

Sums of Boosted Schwarzschild

Situations with “strong fall-off of initial data”:

$\Rightarrow$ Corresponding limits $(P - \bar{P})^*_{\mathcal{I}^+} = 0 = (P - \bar{P})^*_{\mathcal{I}^-}$

General Case: No Antipodal Symmetry

General Case: No Antipodal Symmetry (B., Zhongshan An 2022)
At **future null infinity** $\mathcal{I}^+$ we have

$$P = -\mathrm{d\!/\!iv}\,\mathrm{d\!/\!iv}\,\Sigma + \frac{1}{2}\triangle\!\!\!/\, H - N - \Sigma \cdot \Xi$$

Take the limit as $u \to \infty$:

$$P^+_{\mathcal{I}^+}(\theta, \phi) := \lim_{u \to +\infty} P_{\mathcal{I}^+}(u, \theta, \phi) = -\mathrm{d\!/\!iv}\,\mathrm{d\!/\!iv}\,\Sigma^+_{\mathcal{I}^+} + \frac{1}{2}\triangle\!\!\!/\, H^+_{\mathcal{I}^+} - N^+_{\mathcal{I}^+}$$

At **past null infinity** $\mathcal{I}^-$ we have

$$P_{\mathcal{I}^-} = \mathrm{d\!/\!iv}\,\mathrm{d\!/\!iv}\,\Xi - \frac{1}{2}\triangle\!\!\!/\, \underline{H} - N_{\mathcal{I}^-} - \Sigma_{\mathcal{I}^-} \cdot \Xi_{\mathcal{I}^-}$$

Taking the limit as $\underline{u} \to \infty$:

$$P^+_{\mathcal{I}^-}(\theta, \phi) := \lim_{\underline{u} \to +\infty} P_{\mathcal{I}^-}(\underline{u}, \theta, \phi) = \mathrm{d\!/\!iv}\,\mathrm{d\!/\!iv}\,\Xi^+_{\mathcal{I}^-} - \frac{1}{2}\triangle\!\!\!/\, \underline{H}^+_{\mathcal{I}^-} - N^+_{\mathcal{I}^-}$$

Σ and Ξ are the corresponding limits of $\hat{\chi}$, respectively $\underline{\hat{\chi}}$.

There is no "special" relation between Σ and Ξ.

There is no "special" relation between the corresponding terms in $\hat{\chi}$ and $\underline{\hat{\chi}}$. We can specify data freely.

$\Rightarrow$ **No antipodal symmetry in the curvature** component limits $P^{+}_{\mathcal{I}^+}(\theta, \phi)$ and $P^{+}_{\mathcal{I}^-}(\theta, \phi)$.

Situations with antipodal symmetry (B., Zhongshan An 2022):

- Boosted Schwarzschild
- Sums of Boosted Schwarzschild
- Scenarios described by a mass term expanded in spherical harmonics with only $l = even$ modes

Situations without antipodal symmetry (B., Zhongshan An 2022/25):

- General case above
- Asymmetric matter constellations
- Scenarios described by a mass term expanded in spherical harmonics with both $l = even$ and $l = odd$ modes

Concrete Examples

Concrete examples (B., David Garfinkle, James Wheeler, 2024/25)

Brill waves are solutions of the Einstein vacuum equations that are axisymmetric and time-symmetric, found by Dieter Brill (1959). Defined by the initial spatial metric on $H = \mathbb{R}^3$ given in cylindrical coordinates $\{\rho, z, \varphi\}$ by

$$g = \Psi^4(e^{2q}(d\rho^2 + dz^2) + \rho^2 d\varphi^2) \ .$$

Function $q = q(\rho, z)$ chosen, subject to mild conditions, conformal factor Ψ determined by the constraint equations. The time-symmetry implies that the extrinsic curvature vanishes.

Brill wave data with antipodal symmetry:

Mild conditions and behavior towards spatial infinity given by

$$q = O(r^{-\frac{3}{2}}) \quad , \quad \Psi = 1 + \frac{M}{2r} + O(r^{-\frac{3}{2}})$$

Brill wave data without antipodal symmetry:

Mild conditions and behavior towards spatial infinity given by

$$q = O(r^{-\frac{1}{2}}) \quad , \quad \Psi = 1 + O(r^{-\frac{1}{2}})$$

Angular Momentum

Angular Momentum at $\mathcal{I}^+$

in (A) spacetimes

Classical definition of angular momentum at $\mathcal{I}^+$:

$$J^k \quad := \quad \int_{S^2} \varepsilon^{AB} \nabla_B \tilde{X}^k (N_A - \frac{1}{4} C_A{}^D \nabla^B C_{DB}) \ \ , \ \ k = 1, 2, 3.$$

Bondi-Sachs coordinates.

$\tilde{X}^k$ for $k = 1, 2, 3$: standard coordinate functions in $\mathbb{R}^3$ restricted to S^2,

N_A: angular momentum aspect,

C_{AB}: shear tensor,

ε_{AB}: volume form of the standard round metric σ_{AB} of S^2.

Further, in the Bondi-Sachs notation, N_{AB} is the news tensor and m the mass aspect.

Christodoulou-Klainerman and Bondi-Sachs

We use (CK) notation.

Relate the Christodoulou-Klainerman notation to the Bondi-Sachs coordinate system. The left hand side is given in the (CK) notation:

$$\begin{aligned} B_A &= -N_A \\ \underline{B}_A &= \nabla\!\!\!/^{B} N_{AB} \\ \underline{A}_{AB} &= -2\partial_u N_{AB} \\ \Sigma_{AB} &= -\frac{1}{2} C_{AB} \\ \Xi_{AB} &= -\frac{1}{2} N_{AB} \ . \end{aligned}$$

In (A) spacetimes the limit B_A may not exist. Nevertheless, we can define angular momentum, because the involved $l = 1$ modes behave better.

Obtained a conservation of angular momentum for (A) spacetimes.

Use the Bianchi equation for $\not{D}_3\beta$

$$\begin{aligned}\not{D}_3\beta \ + \ tr\underline{\chi}\beta &= \not{\mathcal{D}}_1^*(-\rho,\sigma) + 2\hat{\chi}\underline{\beta} \ + \ 3\zeta\rho + 3^*\zeta\sigma + \underline{\nu}\beta + \underline{\xi}\alpha \\ &= \not{\nabla}\rho + \varepsilon_{AB}\not{\nabla}^B\sigma + 2\hat{\chi}\underline{\beta} + l.o.t. \qquad (47)\end{aligned}$$

The right hand side of (47) obeys good fall-off behavior.

$\Rightarrow$ Multiply by r^4 and take the limit on a given C_u as $r \to \infty$.

Each of the components on the right hand side has a well-defined limit at $\mathcal{I}^+$. Therefore, it follows that the left hand side tends to a well-defined limit at $\mathcal{I}^+$.

This yields the limiting equation at $\mathcal{I}^+$

$$R = \nabla\!\!\!/ \, P + {}^*\nabla\!\!\!/ \, Q + 2\Sigma \cdot \underline{B} \ .$$

with

$$\lim_{C_u, r \to \infty} r^4 (D\!\!\!/_3 \beta + tr\underline{\chi}\beta) =: R(u, \theta, \phi) \ .$$

Compute from there, take the $l = 1$ modes and integrate to obtain

$$\begin{aligned} \int_{u_1}^{u_2} R_{[1]} du &= \int_{u_1}^{u_2} \nabla\!\!\!/ \, P_{l=1} du + \int_{u_1}^{u_2} {}^*\nabla\!\!\!/ \, Q_{l=1} du \\ &\quad + 2 \int_{u_1}^{u_2} (\Sigma \cdot \underline{B})_{[1]} du \end{aligned} \tag{48}$$

Each term on the right hand side is integrable(!). Not obvious, but can prove it.

Subscript [1] denotes projection on the sum of the 1st and 0th eigenspaces of $\Delta\!\!\!/$.

Recall from above that

$$R = \not\nabla P + {}^*\not\nabla Q + 2\Sigma \cdot \underline{B} \ .$$

The "interesting" term on the right hand side is $\not\nabla P$ because $\not\nabla \rho = O\ (r^{-4})$ in *(A) spacetimes*.

As a comparison, recall that in *(CK) spacetimes* it is $\not\nabla \rho = O\ (r^{-4}\tau_-^{-\frac{1}{2}})$.

Therefore, in (A) spacetimes, we obtain a behavior uR, and correspondingly for β a behavior like $r^{-4}|u|^{+1}$. Thus, β has less decay and the leading order term is dynamical.

Peeling Stops

From the Bianchi equations we derive the limiting equations at future null infinity $\mathcal{I}^+$ for the limits P, respectively Q. They read

$$\frac{\partial P}{\partial u} = \frac{1}{2} \not{div}\, \underline{B} - \Sigma \cdot \frac{\partial \Xi}{\partial u} \tag{49}$$

$$\frac{\partial Q}{\partial u} = \frac{1}{2} \not{curl}\, \underline{B} - \Sigma \wedge \frac{\partial \Xi}{\partial u} \tag{50}$$

$P(\theta, \phi, u)$ at highest order does not have any power law decrease nor increase in u as $|u| \to \infty$, but it depends on u and changes with u. Then by these equations it must hold that

$$\frac{\partial P}{\partial u} = o(|u|^{-\frac{3}{2}}) \ .$$

If we assume more decay, then we could also have

$$\frac{\partial P}{\partial u} = O(|u|^{-2}) \ .$$

Leading Behavior

Note that the (A) spacetimes are more general than the situation studied by Demetrios Christodoulou (2000), where $\log$ terms show up for β.

In particular, Christodoulou finds $r^{-4} \log r$ behavior for β.

In (A) spacetimes, peeling of the Weyl curvature components at future null infinity stops at the order r^{-3} respectively $r^{-4}|u|^{+1}$.

$\log$ terms are naturally present at lower order, but the leading order terms show less decay due to the more general behavior.

Summary

For (A) spacetimes the following hold:

• There are natural contributions from $(P - P_{[1]})$ and F to the gravitational wave memory effect.

• Peeling of the Weyl curvature components at future null infinity stops.

• The limit $\lim_{C_u, t\to\infty} r^3\rho = P(u,\theta,\phi)$ tends to a function $P^+(\theta,\phi)$ on S^2 when the retarded time $u \to +\infty$. In (CK) the corresponding limit is a constant.

• $\rho - \bar{\rho}$, respectively $P - \bar{P}$, does not decay in retarded time u. (Here, $\bar{\rho}$ means the mean value of ρ on $S_{t,u}$, and $\bar{P}$ the mean value of P on S^2.)

• Energy and momenta at future null infinity are well-defined. In particular, angular momentum can be defined and is finite despite the slow decay for β and its derivatives.

Outlook

- Gravitational wave sources where an extended neutrino halo is present: Expect to see these structures.
- Dark matter of certain types may behave as described here.
- Study various examples of systems with non-isotropic mass.
- Couple Einstein equations to other types of matter-energy to investigate similar questions.
- Intersections of gravitational waves and related problems.
- Use above and new results to obtain better understanding of gravitational wave and memory patterns for different sources.
  $\Rightarrow$ Read off new phenomena from these results and future observations.
  $\Rightarrow$ Find new physics.

Thank you!