Applications of controlled paths

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Outline

I will exhibith various applications of the idea of a "controlled path".

- Rough path theory and controlled distributions
- Averaging by oscillations
- Non-linear PDEs with random dispersion
- Stochastic Burgers equation with derivative white noise perturbation

Rough differential equations

- A central theme of stochastic analysis is the study of stochastic differential equations

$$\mathrm{d}_t Y_t = \varphi(Y_t)\mathrm{d}X_t = \sum_i \varphi_i(Y_t)\mathrm{d}_t X_t^i$$

  where for example X is a Brownian motion (M-dimensional), $(V_i : \mathbb{R}^d \to \mathbb{R}^d)_{i=1,\dots,M}$ a collection of vector fields on $\mathbb{R}^d$ (smooth). Standard framework: Itô theory of stochastic integration:

$$Y_t = Y_0 + \int_0^t \varphi(Y_t)\mathrm{d}X_t$$

  The integral on the r.h.s. is defined as a limit in $L^2(\mathbb{P})$.
- Rough path theory (T. Lyons) is a way to give a meaning to the above integral path-wise: take a sample x of the Brownian motion X and try to solve the equation

$$y_t = y_0 + \int_0^t \varphi(y_t)\mathrm{d}x_t$$

  in the space of continuous functions: $y \in C([0,T];\mathbb{R}^d)$.

Problems

- What is the meaning of the integral $\int_0^t \varphi(y_t)\mathrm{d}x_t$?
- Fact: x is only $C^{1/2-}([0,T];\mathbb{R}^M)$. We expect the same regularity from y.
- Then $\varphi(y) \in C^{1/2-}$ and $\partial_t x \in C^{-1/2-}$ (Here for convenience $C^\gamma = B^\gamma_{\infty,\infty}$)
- The product $\varphi(y)\partial_t x$ is not well defined.

In Itô theory the product turn out to be defined (in some sense) due to the stochastic cancellations due to the independece of the increments of x (and the fact that y does not "look into the future").

What goes wrong?

Take $f \in C^{\gamma}(\mathbb{R})$, $g \in C^{\rho}(\mathbb{R})$, $\gamma, \rho \in (0,1)$

The problem is how to define

$$fDg$$

when f, g are Holder functions $(Dg(t) = g'(t))$.

- (Inhomogeneous) Littlewood-Paley decomposition

$$f = \sum_{i \geqslant -1} \Delta_i f$$

where $\Delta_i f$ contains the oscillations of f on the scale 2^i:

$$\|D^n \Delta_i f\|_{L^\infty} \lesssim 2^{(n-\gamma)i}$$

- Paraproduct

$$fDg = \sum_{i,j} \Delta_i f \Delta_j Dg = \pi_<(f, Dg) + \pi_\circ(f, Dg) + \pi_>(f, Dg)$$

with $\pi_<(f,g) = \sum_{i<j-1} \Delta_i f \Delta_j g$, $\pi_\circ(f,g) = \sum_{|i-j|\leqslant 1} \Delta_i f \Delta_j g$,
$\pi_>(f,g) = \pi_<(g,f)$.

Area

- Fact: $\pi_<(f, Dg)$ and $\pi_>(f, Dg)$ are always well defined:

$$\pi_<(f, Dg) \in C^{\rho-1}, \qquad \pi_>(f, Dg) \in C^{\gamma+\rho-1}$$

- The problem is here: $\pi_\circ(f, Dg)$. Well defined only if $\gamma + \rho > 0$ and in this case

$$\pi_\circ(f, Dg) \in C^{\gamma+\rho-1}$$

- Seems not enough for Brownian motion ($\gamma = \rho < 1/2$).

Area process

Take x, y two independent samples of Brownian motion, then it is possible to show that

$$\pi_\circ(x, Dy)$$

exists and belongs to C^{0-} almost surely. Again: stochastic cancellations.

So at least xDy well defined. What else?

Controlled Besov distributions

[Joint work with N. Perkowski and P. Imkeller]

Fix $1/3 < \gamma < 1/2$ and assume $x, y \in C^\gamma$ with $\pi_\circ(x, Dy) \in C^{2\gamma-1}$.

Let f be controlled by x in the following sense:

$$f = \pi_<(f', x) + f^\sharp$$

with $f' \in C^\gamma$ and $f^\sharp \in C^{2\gamma}$. (f looks like x in the small scales).

Commutator estimate

Set $R(f', x, Dy) = \pi_\circ(\pi_<(f', x), Dy) - f'\pi_\circ(x, Dy)$

$$\|R(f', x, Dy)\|_{3\gamma-1} \lesssim \|f'\|_\gamma \|x\|_\gamma \|Dy\|_{\gamma-1}$$

But now

$$fDy = \pi_<(f, Dy) + \underbrace{f'\pi_\circ(x, Dy + \pi_>(f, Dy)}_{C^{2\gamma-1}} + \underbrace{\pi_\circ(f^\sharp, Dy) + R(f', x, Dy)}_{C^{3\gamma-1}}$$

and all the objects in the r.h.s. are well defined.

Solving RDEs

Reconsider

$$f_t = f_0 + \int_0^t \varphi(f_s) Dx_s \mathrm{d}t$$

with x a sample from a M-dimensional Brownian motion. Then $x \in C^\gamma$ for some $1/3 < \gamma < 1/2$ and $\pi_\circ(x^i, Dx^j) \in C^{2\gamma-1}$ for all $i, j = 1, \ldots, M$.

We can now solve this equation in the space of f controlled by x:

- Paralinearization theorem: $\varphi(f) = \pi_<(\nabla\varphi(f), f)$ + smoother remainder
- Controlled hypothesis $f \simeq \pi_<(f', x)$ implies $\varphi(f) = \pi_<(\nabla\varphi(f) f', x)$ + smoother remainder
- Product: $\varphi(f) Dx = \pi_<(\varphi(f), Dx) + \nabla\varphi(f) f' \pi_\circ(x, Dx)$ + smoother remainder
- Integration: $\int \varphi(f) Dx = \pi_<(\varphi(f), x) + \nabla\varphi(f) f' \int \pi_\circ(x, Dx)$ + smoother remainder

So the map

$$\Gamma(f) = f_0 + \int_0^t \varphi(f_s) Dx_s \mathrm{d}t$$

remain in the space of controlled paths and we can set up a fixed point.

Averaging along a Brownian motion

Take a bounded function $b : \mathbb{R}^d \to \mathbb{R}^d$ and a d-dimensional Brownian motion (Bm) W. A. Davie has showed that the average of b along the Brownian trajectory w:

$$\sigma^w_{s,t}(b)(x) = \int_s^t b(w_r + x)\mathrm{d}r$$

satisfy

$$\mathbb{E}|\sigma^W_{s,t}(b)(y) - \sigma^W_{s,t}(b)(x)|^{2p} \lesssim_p \|b\|_{L^\infty}|x-y|^{2p}|t-s|^p$$

from which follows

$$|\sigma^w_{s,t}(b)(y) - \sigma^w_{s,t}(b)(x)| \lesssim_{w,b} |x-y||t-s|^{1/2}(1 + \log_+^{1/2}\frac{1}{|x-y|} + \log_+^{1/2}\frac{1}{|t-s|})$$

From this it is possible to deduce that the ODE (not SDE)

$$x_t = x + \int_0^t b(x_s)\mathrm{d}s + w_t$$

has a unique solution in $C(\mathbb{R}_+;\mathbb{R}^d)$ for almost every sample path w of the Brownian motion.

Fractional Brownian motion

To have the freedom to vary the regularity of the driving paths and retain many nice features of the Brownian motion (Gaussian, stationary increments, scaling) a convenient model for noise is the fractional Brownian motion (fBm) B^H of Hurst index $H \in (0,1)$.

$(B_t^H)_{t\in[0,T]}$ is a Gaussian process with stationary increments, zero mean and covariance

$$\mathbb{E}[(B_t^H - B_s^H)^2] = |t-s|^{2H}$$

Setting $H = 1/2$ gives Brownian motion back.
The fBm B^H has trajectories almost surely in any C^γ for any $\gamma < H$.

Averaging along an fBm

Let $\mathcal{F}L^\alpha$ the set of distribution $b : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$N_\alpha(b) = \int_{\mathbb{R}^d} (1+|\xi|)^\alpha |\hat{b}(\xi)| \mathrm{d}\xi < +\infty.$$

Then it is possible to show that if $(w_t)_{t\geqslant 0}$ is the sample path of a d-dim. fractional Brownian motion and $x \in Q^w_\gamma \subset C(\mathbb{R};\mathbb{R}^d)$ is *controlled* by w in the sense that

$$x_t - x_s = w_t - w_s + O(|t-s|^\rho)$$

for some $\rho > 1/2$, for all $b \in \mathcal{F}L^\alpha$ with $\alpha > 1 - 1/2H$ the integral

$$\lim_{n\to\infty} \int_0^t b_n(x_s)\mathrm{d}s =: \int_0^t b(x_s)\mathrm{d}s$$

is well defined for any sequence of smooth function $(b_n)_{n\geqslant 1}$ such that $N_\alpha(b - b_n) \to 0$ and independent of the sequence. Moreover the map $t \mapsto \int_0^t b(x_s)\mathrm{d}s$ is C^γ for some $\gamma > 1/2$.

[joint work with R. Catellier]

Regularization by oscillations

If $\alpha > 2 - 1/2H$ the averaging map

$$\sigma^x_{s,t}(b)(y) = \int_s^t b(x_r + y)\mathrm{d}r$$

is Lipshitz:

$$\left|\sigma^x_{s,t}(b)(y) - \sigma^x_{s,t}(b)(z)\right| \lesssim_{x,w} N_\alpha(b)|y-z||t-s|^\gamma.$$

The previous results allows to study the the ODE in $\mathbb{R}^d$

$$x_t = x_0 + \int_0^t b(x_s)\mathrm{d}s + w_t$$

where $b \in \mathcal{F}L^\alpha$.

- Existence in Q^w_γ for $\alpha > 1 - 1/2H$
- Uniqueness in Q^w_γ for $\alpha > 2 - 1/2H$ + Lipshitz flow.
- If b is not random we can have uniqueness for $\alpha > 1 - 1/2H$.

Nonlinear PDEs with random dispersion

[joint work with K. Chouk]

Consider (Stratonovich-) stochastic nonlinear PDEs of the form

$$\partial_t \phi_t = A\phi_t \partial_t B_t + N(\phi_t)$$

for $\phi : [0,T] \times \mathbb{T} \to \mathbb{C}$ or $\mathbb{R}$ where B is a (1d) Brownian motion.

Various cases:

- NSE: ϕ complex, $A = i\partial_\xi^2$ and $N(\phi) = \pm i|\phi|^2\phi$
- ∂NSE: ϕ complex, $A = i\partial_\xi^2$ and $N(\phi) = \pm i\partial_\xi(|\phi|^2\phi)$
- KdV: ϕ real, $A = \partial_\xi^3$ and $N(\phi) = \partial_\xi \phi^2$

Recent work of [Debussche–De Bouard] on randomly modulated NSE in $\mathbb{T}$ (motivated by dispersion management in optical fibers)

Spaces

$$|\phi|_\alpha = \|(1+|\xi|^2)^{\alpha/2}\hat{\phi}(\xi)\|_{L^2_\xi}$$

where $\hat{\phi}$ is the space Fourier transform of ϕ.

Almost sure results (with a universal exceptional set):

- NSE: Global unique solution in L^2 + Lipshitz flow map
- KdV: Local unique solution in H^{-1+} + Lipshitz flow map

Formulation of the equation

Let $U_t = e^{AB_t}$ so that

$$\partial_t U_t = A U_t \partial_t B_t$$

then ϕ should solve

$$\phi_t = U_t(\phi_0 + \int_0^t U_s^{-1} N(\phi_s) \mathrm{d}s).$$

The path $\phi \in C([0,T], H^\alpha)$ is controlled if

$$\phi_t = U_t \psi_t$$

with $\psi_t \in C^\rho([0,T], H^\alpha)$ for some $\rho > 1/2$.

Introduce the map $X_{s,t} : H^\alpha \to H^\alpha$ given by

$$X_{s,t}(\psi) = \int_s^t U_r^{-1} N(U_r \psi) \mathrm{d}r$$

Key estimate

$$\|X_{s,t}(\psi) - X_{s,t}(\psi')\|_\alpha \lesssim F(\|\psi\|_\alpha + \|\psi'\|_\alpha)|t-s|^\gamma \|\psi - \psi'\|_\alpha$$

for some $\gamma > 1/2$.

Formulation as a controlled path problem

The mild equation take the form

$$\psi_t = \psi_0 + \int_0^t U_s^{-1} N(U_s \psi_s) \mathrm{d}s = \psi_0 + \int_0^t \left[\frac{\mathrm{d}}{\mathrm{d}s} X_{0,s} \right] (\psi_s)$$

$$= \psi_0 + \int_0^t X_{\mathrm{d}s}(\psi_s) = \psi_0 + \lim \sum_i X_{t_i, t_{i+1}}(\psi_{t_i})$$

The key estimate implies

$$t \mapsto \int_0^t X_{\mathrm{d}s}(\psi_s) = \int_0^t U_s^{-1} N(\phi_s) \mathrm{d}s$$

is in $C^\gamma([0,T]; H^\alpha)$ for any controlled path ϕ and coincide with the limit

$$\lim_{n \to \infty} \int_0^t U_s^{-1} N(P_n \phi_s) \mathrm{d}s = \int_0^t X_{\mathrm{d}s}(\psi_s)$$

(P_n is the projector on the Fourier modes $|k| \leqslant n$) and is γ-Hölder in time for some $\gamma > 1/2$ and locally Lipshitz in ϕ (in the controlled path norm).

By standard fixed-point argument we get a (unique) local solution to the PDE.
In the NSE case the L^2 conservation law allow to extend the solution to a global one.

Thanks