Flowing to minimal surfaces

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Define a *good geometric flow* that changes maps $u : M^2 \to (N, g_N)$ into **minimal surfaces**.

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Result: A flow with elements in common with

- mean curvature flow (looking for minimal surfaces)
- harmonic map heat flow (known singularity structure)

Setting

Fix

- domain: M^2 closed, orientable surface (of genus $\gamma \in \mathbb{N}_0$)
- target: (N, g_N) compact manifold, w.l.o.g. $\hookrightarrow \mathbb{R}^k$

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For the two variables

- $u : M \to N$
- g Riemannian metric on domain M

consider

$$E(u, g) := \frac{1}{2} \int_M |du|_g^2 \ dv_g.$$

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Remark

$$\text{Area}(u) = \int_M |\partial_{x_1} u \wedge \partial_{x_2} u| \, dx_1 dx_2 \leq E(u, g)$$

with "=" iff u is conformal, i.e. iff $u^* g_N = \lambda \cdot g$, $\lambda \geq 0$.

Area $\leftrightarrow$ Energy

Remark

(u, g) critical point of $E(\cdot, \cdot)$

$$\Leftrightarrow$$

u critical point of *Area*, more precisely, a branched minimal immersion (or constant).

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Remark

(u,g) critical point of $E(\cdot,\cdot)$, i.e.

$$\begin{cases} 0 = \nabla_u E = -\tau_g(u) & \text{(harmonic)} \\ 0 = \nabla_g E = -\frac{1}{4}\mathrm{Re}(\Phi(u,g)) & \text{(conformal)} \end{cases}$$

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Here

- $\tau_g(u) = \text{tension} = \mathrm{tr}_g(\nabla_g du) = \Delta_g u + A_g(u)(\nabla u, \nabla u)$
- $\Phi(u,g) = \text{Hopf-differential} = \phi \cdot dz^2$
    - $z = x + iy$ complex coordinate of (M,g), $g = \lambda \cdot (dx^2 + dy^2)$
    - $\phi = |u_x|^2 - |u_y|^2 - 2i\langle u_x, u_y\rangle$

Definition of the flow

First definition of a flow

$$\partial_t u = -\nabla_u E = \tau_g(u), \quad \partial_t g = -\nabla_g E = \frac{1}{4}\mathrm{Re}(\Phi(u,g))$$

Definition of the flow

BAD definition of a flow

$$\partial_t u = -\nabla_u E = \tau_g(u), \quad \partial_t g = -\nabla_g E = \frac{1}{4}\mathrm{Re}(\Phi(u,g))$$

since metric component is not well controlled.

BAD definition of a flow

$$\partial_t u = -\nabla_u E = \tau_g(u), \quad \partial_t g = -\nabla_g E = \frac{1}{4}\mathrm{Re}(\Phi(u,g))$$

Instead evolve by

$$\begin{aligned} \partial_t u &= \tau_g(u) \\ \partial_t g &= \frac{1}{4}\mathrm{Re}(P_g^{\mathcal{H}}(\Phi(u,g))) \end{aligned} \tag{1}$$

for $P_g^{\mathcal{H}}$ the L^2 orthogonal projection

$P_g^{\mathcal{H}} : \{\phi dz^2 \text{ quad. differential } \} \to \mathcal{H}(M,g) = \{\text{holomorphic quad. diff.}\}$

Remark: $\dim(\mathcal{H}(M,g)) < \infty$

Why is this the right flow?

Claim

(1) $=$ gradient flow of E / symmetries.

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"Proof": Symmetries of E:

- conformal invariance, $E(u, g) = E(u, \lambda \cdot g)$
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$$\mathcal{A} = \{[(u,g)], u \in C^\infty(M,N), g \in \mathcal{M}_c\}.$$

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Use L^2 orthogonal splitting

$$T_g\mathcal{M}_c = \{L_x g = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} f_t^* g\} \oplus \mathrm{Re}(\mathcal{H}(g)).$$

Definition

Definition of our flow

Evolve a pair (u, g) of map $u : M \to N$ and metric $g \in \mathcal{M}_c$ by

$$\partial_t u = \tau_g(u)$$
$$\partial_t g = \frac{1}{4}\mathrm{Re}(P_g^{\mathcal{H}}(\Phi(u, g))) \qquad ((1))$$

describing the representative of the L^2 gradient flow of E on $\mathcal{A}$ chosen such that $t \mapsto g(t)$ has minimal L^2 length.

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- genus $\gamma = 0$: $\mathcal{H}(g) = \{0\} \Rightarrow (1) =$harmonic map flow
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Relation to other flows

- genus $\gamma = 0$: $\mathcal{H}(g) = \{0\} \Rightarrow (1) =$harmonic map flow
  Global existence and asymptotic convergence (Struwe '85)
- $\gamma = 1$: Ding-Li-Liu'06: "Modified gradient flow" of

$$(u, a, b) \mapsto E(u, g_{a,b}), a, b \in \mathbb{R}$$

  agrees with (1)

Existence of solutions

Theorem 1 (R. '12)

To any $(u_0, g_0) \in H^1(M, N) \times \mathcal{M}_c$ there exists a (weak) solution (u, g) of

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defined on $[0,T)$ and $T<\infty$ only if $(M, g(t))$ degenerates in moduli space as $t \nearrow T$, i.e.

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This solution is smooth away from finitely many times

- at which finitely many harmonic spheres "bubble off"
- across which g remains $C_t^{0,1}C_x^\infty$.

A few words on the proof...

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Important to understand dependence on g of

$$P_g^{\mathcal{H}} : \{\text{quad. diff.}\} \to \{\text{holomorphic. quad. diff.}\}.$$

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- Alternative point of view

$$\mathrm{Re}(\mathcal{H}(M,g)) = \{k \in \mathrm{Sym}^{(0,2)} : \mathrm{tr}_g(k) = 0 = \mathrm{div}_g(k)\}$$

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- "Explicit" formula for P_g, given in terms of solutions of elliptic PDE's to be solved on the varying surfaces (M,g)
- Ideas from Teichmüller theory (slice theorem) about the structure of the Banachmanifold $\mathcal{M}^s_{-1}$ of H^s hyperbolic metrics.

A few words on the proof...

Key-lemma to prove existence of solutions

The map $g \mapsto P_g$ is locally Lipschitz-continuous on the Banachmanifold $\mathcal{M}^s_{-1}$ in the sense that for every tensor $k \in Sym^{(0,2)}$

$$\|(P_{g_1} - P_{g_2})(k)\|_{H^s} \leq C \cdot \|g_1 - g_2\|_{H^s} \cdot \|k\|_{L^1}.$$

Asymptotics

Theorem 2 (R.+Topping '12)

If the solution (u, g) of Theorem 1 satisfies

$$\inf_{t\in[0,\infty)} \ell(g(t)) > 0 \tag{2}$$

then there are $t_i \to \infty$ and diffeomorphisms f_i

$$f_i^*(u(t_i), g(t_i)) \to (u_\infty, g_\infty)$$

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where

- u_∞ is a branched minimal immersion or constant
- u_∞ has the same action on π_1 as u_0,

$$u_* : \pi_1(M) \ni [\sigma] \mapsto [u \circ \sigma] \in \pi_1(N)$$

Can solutions degenerate?

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Not if they are topologically non-degenerate:

Theorem 3 (R.+Topping '12)

If $(u_0)_*$ is injective (i.e. u_0 incompressible) then the flow (1) is global and

$$\ell(g(t)) \geq \delta(E_0, N) > 0$$

$\Rightarrow$ From Theorem 2 we recover result of Sacks-Uhlenbeck and Schoen-Yau on the existence of branched minimal immersions with given injective action on π_1.

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If the solution of Theorem 1 is *global*, but

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If the solution of Theorem 1 is *global*, but

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then there exist $t_i \to \infty$, diffeomorphisms $f_i : \Sigma \to M \setminus \cup_j \sigma_i^j$ such that

- $f_i^*(M, g(t_i))$ converges to a punctured hyperbolic surface (Σ, h)
- $u(t_i) \circ f_i$ converges to a limit map $u_\infty : \Sigma \to (N, g_N)$ which is, on each connected component of Σ, a branched minimal immersion (or constant)

A few words on the proof of the asymptotics

Energy decays according to

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Thus if solution is global, there is $t_i \to \infty$ such that

- $\tau_{g(t_i)}(u(t_i)) \to 0 \rightsquigarrow$ harmonic limit map u_∞
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BUT

maps converge only **weakly** in H^1 and Φ is quadratic in ∇u

$\Rightarrow$ $P_g(\Phi(u,g)(t_i)) \to 0$ does **not** imply $P_{g_\infty}(\Phi(u_\infty, g_\infty)) = 0$

One key lemma

Poincaré estimate for quadratic differentials (R.-Topping '12)

For

- a closed hyperbolic surface (M, g)
- every quadratic differential Ψ on (M, g)

$$\|\Psi - P_g^{\mathcal{H}}(\Psi)\|_{L^1(M,g)} \leq C \cdot \|\partial_{\bar{z}}\Psi\|_{L^1(M,g)}$$

One key lemma

Uniform Poincaré estimate for quadratic differentials (R.-Topping '12)

For any genus bound $\Gamma \in \mathbb{N}$ there exists $C < \infty$ such that for

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C depends on

- topology (i.e. genus)
- but **NOT** on the geometry (diameter, injectivity radius,..)

of the surface.