A twistor approach to toric gravitational instantons

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- R. S. Ward. Stationary axisymmetric space–times: a new approach. GRG **15** (1983)
- N. M. J. Woodhouse, L. J. Mason. The Geroch group and non–Hausdorff twistor spaces. Nonlinearity **1** (1988) 73
- Paul Tod. Rod structures and patching matrices: a review. `arXiv:2411.02096`
- MD, Paul Tod. Twistor theory of the Chen–Teo gravitational instanton. `arXiv:2405.08170`, CQG **41**, 195008
- MD, Lionel Mason, Paul Tod (2026), in preparation

Roger Penrose's Twistor theory

- 1960s. Non–locality in $\mathcal{PT} = \mathbb{CP}^3 - \mathbb{CP}^1$. Penrose
  - space–time points $\longleftrightarrow$ twistor lines, twisting light rays $\longleftrightarrow$ points;
  - passive twistor correspondences:
    zero–rest–mass fields $\longleftrightarrow$ twistor functions (cohomology).

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  - Penrose, the non–linear graviton: ASD Ricci–flat metrics $\longleftrightarrow$ $\mathcal{O}(2)$ twistor functions = patching data for a deformed $\mathcal{PT}$.
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- This talk: Toric Ricci–flat gravitational instantons (neither SD nor ASD) from the Ward correspondence.

Axisymmetric Stationary Fields.

Twistor theory can be applied to several stationary axisymmetric field theories. I'd like to mention three of these: the 3-dim. Laplace equation, the Yang-Mills-Higgs equations, and Einstein's equations.

One can generate axisymmetric solutions of the 3-dim. Laplace equation by specializing the usual contour integral formula, and arriving at the following. Define γ and ζ by $\gamma = i\omega^0/\pi_{0'} - i\omega^1/\pi_{1'}$ and $\zeta = \pi_{0'}/\pi_{1'}$. Then

(A) $\varphi(x,y,z) = \frac{1}{2\pi i}\oint f(\gamma)\frac{d\zeta}{\zeta}$; f holomorphic,

is a solution of the Laplace equation, axisymmetric about the z-axis. (I think it's the general such solution.) The inverse twistor function appears to be extremely simple here, because of the fact that $\gamma = -2z$ on the z-axis,* so that $f(\gamma) = \varphi(0,0,-\frac{\gamma}{2})$. For example, the "Schwarzschild-Weyl" solution $\varphi = \frac{1}{2}\log\frac{r_+ + r_- - 2m}{r_+ + r_- + 2m}$ [1] is easily seen to arise from $f = \frac{1}{2}\log\frac{\gamma+2m}{\gamma-2m}$. Someone might like to investigate all this in more detail.

Notes: (i) φ is real if f satisfies the reality condition $\overline{f(\bar\gamma)} = f(\bar\gamma)$.

(ii) The formula (A) is essentially due to Whittaker [2].

Now the problem of finding stationary "multi-monopole" solutions of the Y-M-H equations (see [3] for details). Twistor methods give (at least implicitly)

*My conventions are $\omega^0 = i(t+z)\pi_{0'} + i(x-iy)\pi_{1'}$, $\omega^1 = i(x+iy)\pi_{0'} + i(t-z)\pi_{1'}$.

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the general such solution, as follows. Take a 2x2 unimodular matrix $F(\gamma,\zeta)$ of holomorphic functions, satisfying the reality condition $F(\bar\gamma, -\bar\zeta^{-1})$ = conjugate transpose of $F(\gamma,\zeta)$. This gives rise to a Y-M-H field in space-time which is stationary, real, and a solution of the field equations. The problem (not yet solved) is to ensure that this field has the correct global properties. The only explicit cases so far understood are axisymmetric ones, where the ζ-dependence of F is rather special. The two simplest ones are [4]

$$F = \begin{bmatrix} \gamma^{-1}(e^{\gamma}-e^{-\gamma}) & -\zeta^{-\gamma} \\ \zeta^{-1}e^{-\gamma} & \gamma e^{-\gamma}\end{bmatrix} \text{ and } F = \begin{bmatrix} H^{-1}(e^{\gamma}+e^{-\gamma}) & \zeta^{2}e^{-\gamma} \\ \zeta^{-2}e^{-\gamma} & He^{-\gamma}\end{bmatrix}$$

where $H = \gamma^2 + \pi^2/4$.

Finally, stationary axisymmetric vacuum space-times. As pointed out by Witten [5], this problem can also in principle be solved by twistor methods: the matrices F above "encompass" all such space-times. The only case as yet understood (by me) is that of the Weyl solutions, which correspond to $F = \mathrm{diag}(e^{f(\gamma)}, e^{-f(\gamma)})$.

References 1. Kramer et al, Exact Solutions, p.201.
2. Whittaker & Watson §18·3.
3. Jaffe & Taubes, Vortices and Monopoles.
4. Ward, YMH Monopole of Charge 2, to appear in CMP.
5. L. Witten, Phys Rev D19, 718 (thanks to KPT for pointing this paper out to me)

Mystery: how are the twistor methods related to "Bäcklund" methods; cf. Forgács et al, Budapest preprint KFKI-1980-122; Cosgrove, JMP 21 2417.

Richard Ward,
Trinity College,
Dublin.

- Complete Riemannian four–manifold (M, g) with

$$\mathrm{Ric}_g = 0, \qquad \int_M |\mathrm{Riemann}_g|^2 \,\mathrm{vol} < \infty,$$

which asymptotically 'look like' flat space.

- Hawking's Euclidean quantum gravity (1977).
- Some are analytic continuations of Lorentzian black–holes (Kerr), others are not (hyper–Kähler, anti–self–duality).

Gravitational instantons

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- Hawking's Euclidean quantum gravity (1977).
- Some are analytic continuations of Lorentzian black–holes (Kerr), others are not (hyper–Kähler, anti–self–duality).
- Riemannian black hole uniqueness conjecture: Euclidean Schwarzschild and Kerr are the only asymptotically flat examples.
- *False* — Chen–Teo (2011, 2015), Li–Sun (2025), Teo (2026).
- So the problem becomes: classify them. Find explicit examples. More conjectures to disprove..

Toric instantons: the normal form

- Toric: an effective isometric T^2 action with at least one fixed point. Killing vectors K_1, K_2; lattice $\Lambda = \{K : \exp(2\pi K) = \mathrm{Id}\} \cong \mathbb{Z}^2$. Orbit space $\Sigma = M/T^2 = \mathbb{H}^2$.

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- Gram matrix $J_{AB} = g(K_A, K_B)$, $r^2 \equiv \det J$. Then r is harmonic on Σ; z is its conjugate, $dr = *_\Sigma dz$, and

$$g = \Omega^2(dr^2 + dz^2) + J_{AB}\, d\varphi^A d\varphi^B, \qquad K_A = \partial/\partial\varphi^A. \tag{1}$$

- $\mathrm{Ric}_g = 0 \iff$ the J–equation

$$r^{-1}\partial_r\big(r\, J^{-1}\partial_r J\big) + \partial_z\big(J^{-1}\partial_z J\big) = 0 \tag{2}$$

together with a quadrature for the conformal factor Ω.

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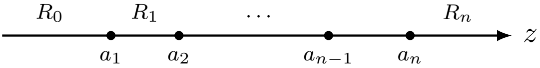
- Completely integrable: a Lax pair, and a symmetry reduction of anti–self–dual Yang–Mills (L. Witten, Ward).

Rod structure

The torus action degenerates on $r = 0$, where $\det J = 0$:

$$\operatorname{rank} J(0,z) = \begin{cases} 0 & \text{isolated turning points } = \text{nodes}, \\ 1 & \text{otherwise.} \end{cases}$$



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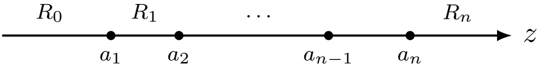

- n nodes, $n+1$ rods $R_0 = (-\infty, a_1)$, $R_i = (a_i, a_{i+1})$, $R_n = (a_n, \infty)$.
- Rod vector l_i: the Killing vector that vanishes on R_i, i.e. $J(0,z)\, l_i = 0$, constant in z.

Surface gravity and the lattice condition

- $d\rho^2 + \kappa^2\rho^2 d\varphi^2$ is smooth iff φ has period $2\pi/\kappa$. With $u \in \ker J(0, z)$ the surface gravity $\kappa(u)^2 = \lim_{r\to 0} J(u, u)/(\Omega^2 r^2)$ is constant along the rod, and $l = u/\kappa(u)$ is 2π–periodic rod vector. Note: κ needs Ω.

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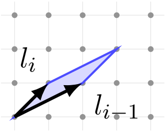
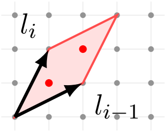
- No orbifold points: $|\det(l_{i-1}, l_i)| = 1$, equivalently the lattice condition

$$l_{i+1} = -l_{i-1} + p_i\, l_i, \qquad p_i \in \mathbb{Z},$$

and $(p_1, \ldots, p_{n-1})$ is the fan.



$|\det| = 1$: smooth, $\mathbb{C}^2$

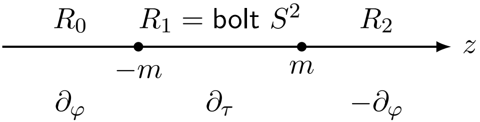


$|\det| = 3$: orbifold $\mathbb{C}^2/\mathbb{Z}_3$

Example: Euclidean Schwarzschild

$$g = \Big(1 - \frac{2m}{\rho}\Big) d\tau^2 + \Big(1 - \frac{2m}{\rho}\Big)^{-1} d\rho^2 + \rho^2 \big(d\theta^2 + \sin^2\theta \, d\varphi^2\big), \qquad \rho \geq 2m,$$

$$J = \mathrm{diag}\Big(1 - \frac{2m}{\rho}, \ \rho^2 \sin^2\theta\Big), \quad r = \sqrt{\rho(\rho - 2m)} \sin\theta, \quad z = (\rho - m)\cos\theta.$$



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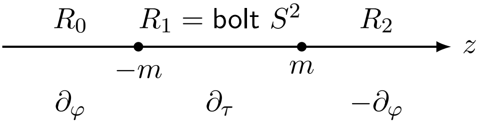

- $\kappa(\partial_\varphi)=1$, $\kappa(\partial_\tau)=1/(4m)$, so $l_0=\partial_\varphi$, $l_1 = 4m\,\partial_\tau$, $l_2 = -l_0 + 0\cdot l_1$: $p_1 = 0$.
- $\tau \sim \tau + 8\pi m$; $1/(8\pi m)$ is the Hawking temperature. $M = \mathbb{R}^2\times S^2$; in general $\chi(M) = n$.

ALF, AF and ALE

Definition

(M, g) is ALF if it approaches an S^1–bundle over S^2:
$g \sim (d\tau + 2N\cos\theta\, d\varphi)^2 + d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$.

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- $N = 0$: trivial bundle, $g \sim S^1 \times \mathbb{R}^3$ — AF (Schwarzschild, Kerr).

$$\text{AF}: \; l_0 = \pm l_n, \qquad \text{ALF}: \; |\det(l_0, l_n)| = N \neq 0.$$

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- e.g. $\mathbb{R}^4$, Kronheimer's Kronheimer's hyper–Kähler quotients.
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  $d\rho^2 + \frac{1}{4}\rho^2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)$.
- Volume growth: ALE ρ^4, ALF ρ^3.

Pick a rod R and a Killing vector K with $K \neq 0$ on R. The twist potential is $*(K \wedge dK) = d\psi$.

The Ernst potential

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- The matrix

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  also solves the J–equation (Bäcklund).

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- $J = (|K|^2, \omega \equiv J_{12}/J_{11})$ and $E = (|K|^2, \psi)$ related by $\frac{r}{|K|^4} *_2 d\psi = d\omega$. Twistor theory gives E; one quadrature gives J.

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- Gibbons–Hawking: $V(r, z)$ harmonic on $\mathbb{R}^3$, $K = \partial_\tau$

$$g = V\big(dr^2 + r^2 d\varphi^2 + dz^2\big) + V^{-1}\big(d\tau + A\big)^2, \qquad dV = *_{\mathbb{R}^3} dA,$$

$$\psi = V^{-1}, \qquad E = \begin{pmatrix} V & -1 \\ -1 & 0 \end{pmatrix}, \qquad \det E = -1.$$

- Restrict to the axis and analytically continue:

  $$J \to E \to P(z) = E(r=0,z) \to P(w), \qquad w = z - \frac{r}{2}\left(\zeta - \zeta^{-1}\right).$$

  $P(w)$ is *rational*, with simple poles and rank–one residues.

The patching matrix

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- Euclidean Schwarzschild ($K = \partial_\tau$ on R_2, so $\psi = 0$):

$$P = \begin{pmatrix} \frac{w+m}{w-m} & 0 \\ 0 & -\frac{w-m}{w+m} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{2m\, e_1 \otimes e_1}{w-m} + \frac{2m\, e_2 \otimes e_2}{w+m}.$$

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- The two asymptotic models on a semi–infinite rod:

$$P(\infty) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ (ALF)}, \qquad P \sim \begin{pmatrix} \frac{1}{2w} & 0 \\ 0 & -2w \end{pmatrix} \text{ (ALE)}.$$

Anti–self–dual Yang–Mills (ASDYM)

- Complexified Minkowski space $M_{\mathbb{C}} = \mathbb{C}^4$, coordinates $(W, Z, \widetilde{W}, \widetilde{Z})$

$$ds^2 = 2(dZ\, d\widetilde{Z} - dW\, d\widetilde{W}), \qquad \text{vol} = dW \wedge d\widetilde{W} \wedge dZ \wedge d\widetilde{Z}.$$

- $A \in \Lambda^1(M_{\mathbb{C}}) \otimes \mathfrak{sl}(2)$, $F = dA + A \wedge A$. ASDYM: $F = - \star F$, i.e.

$$F_{WZ} = 0, \qquad F_{\widetilde{W}\widetilde{Z}} = 0, \qquad F_{W\widetilde{W}} - F_{Z\widetilde{Z}} = 0.$$

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$$A = J^{-1}\partial_{\widetilde{W}} J\, d\widetilde{W} + J^{-1}\partial_{\widetilde{Z}} J\, d\widetilde{Z}, \qquad J = J(W, Z, \widetilde{W}, \widetilde{Z}) \in SL(2, \mathbb{C}),$$

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- Symmetry reduction to the J–equation:

$$Z = t + z, \quad \widetilde{Z} = t - z, \quad W = re^{i\theta}, \quad \widetilde{W} = re^{-i\theta}, \quad J = J(r, z).$$

Twistor construction

- ASDYM $\iff$ A is flat on α–planes in $M_{\mathbb{C}}$

$$\alpha = W + \zeta \widetilde{Z}, \qquad \beta = Z + \zeta \widetilde{W} \qquad \text{(I)}$$

Twistor space $\mathcal{PT} = \mathbb{CP}^3 \setminus \mathbb{CP}^1$, affine coordinates (α, β, ζ).

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- Theorem (Ward 1977). $1-1$ correspondence between gauge equivalence classes of A and holomorphic rank–2 bundles $E \to \mathcal{PT}$, trivial on twistor lines.
- Cover $\mathcal{PT}$ by U $(\zeta \neq \infty)$ and $\widetilde{U}$ $(\zeta \neq 0)$. Patching matrix $P_{U\widetilde{U}} = P(\alpha, \beta, \zeta)$; on a twistor line $P_{U\widetilde{U}} = P_U \, P_{\widetilde{U}}^{-1}$.

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After the reduction (Woodhouse–Mason 1988) $\mathcal{PT}$ is a *non–Hausdorff* complex curve with coordinate w, and twistor line $w = z - \frac{r}{2}(\zeta - \zeta^{-1})$.

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4. regularity: normalise by surface gravities. Impose the lattice condition.

Generating all patching matrices

Definition

$\mathcal{C}_n$: real symmetric rational $P(w)$ with $\det P = -1$, n simple poles at $a_1 < \cdots < a_n$, and $P(\infty) = \mathrm{diag}(1, -1)$.

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Conventions: $\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\mathrm{adj}A = -\epsilon A^T \epsilon$.

Proposition (node insertion)

Let $P \in \mathcal{C}_n$, $a \neq a_j$, $\mu \in \mathbb{R}$, and let $u = u^\pm$ be null for $M = \mathrm{adj}P(a)$, i.e. $(u^\pm)^T M\, u^\pm = 0$, so $u^\pm = \begin{pmatrix} -M_{12} \pm 1 \\ M_{11} \end{pmatrix}$. Then $P^{new} = B\, P\, B^T \in \mathcal{C}_{n+1}$, where $B = \mathbf{1} + \frac{\mu}{w-a}\, u\, u^T \epsilon$.

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Theorem (D.–Mason–Tod 2026)

Every ALF patching matrix has the form $P = \mathrm{diag}(1, -1) + \sum_i \frac{\varepsilon_i \, v_i \otimes v_i}{w - a_i}$, and every such matrix is obtained by iterating the Proposition from $\mathrm{diag}(1, -1)$.

- non–GH branch: $2n$ real parameters (a_j, μ_j), minus one translation in w, minus $2n-2$ lattice conditions

  1 parameter for ALF, 2 for AF.

Counting and branching

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- Everything else is a non–GH branch. At $n = 2, 3$ these are Kerr–NUT, Euclidean Plebiański–Demiański, and Chen–Teo.
- Every three–node ALF patching matrix is "2–centre Gibbons–Hawking $+$ one insertion". Chen–Teo is the *general* three–node ALF solution. (Patching matrix, D–Tod 2024).

Recursive Riemann–Hilbert splitting

Theorem (D.–Mason–Tod 2026)

Let $P = F\widetilde{F}^{-1}$ *be the normalised splitting, and let* $\zeta^{\mp}$ *be the roots of* $w(\zeta) = a$, *with* $|\zeta^{-}| \leq 1 \leq |\zeta^{+}|$. *Then* $P^{new} = BPB^{T}$ *splits as*

$$F^{new} = B\,F\,\widetilde{\chi}, \qquad \widetilde{F}^{\,new} = B^{-T}\,\widetilde{F}\,\chi, \qquad \chi = \mathbf{1} + \frac{\chi_{+}}{\zeta - \zeta^{-}} + \frac{\chi_{-}}{\zeta - \zeta^{+}},$$

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where $\chi_{\pm}$ *are rank one and given in closed form.*

- Explicit solution $E^{new} = F^{new}(\zeta = 0)$: all instanton solutions generated by algebra via twistors.

Catalogue

n	ALE $(2n-2)$	ALF $(2n-1)$
1	AE, $\mathbb{R}^4$	$n=1$ GH (ASD Taub–NUT)
2	Eguchi–Hanson, or orbifold (GH)	GH, or Kerr–NUT orbifold. Regular: Kerr (AF), Taub–bolt (ALF)
3	A_3 GH, or Euclidean Plebiański–Demiański (orbifolds)	triple Taub–NUT (GH), or the 5–parameter Chen–Teo (2–dimensional family of instantons)
4	GH, or ??. *Nakajima conjecture*: all ALE instantons are hyper–Kähler, hence, if toric, GH	GH, Teo–Li–Sun instanton (AF)
$n>4$	GH, ??	existence (Li–Sun) for AF. ALF ?

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Happy birthday Roger!