Assignment Module 5

November 4, 2018
Abstract

In this assignment a topic suggested by Jeff Dewynne was chosen. The paper *The Black-Scholes Equation Revisited: Asymptotic Expansions and Singular Perturbations* by Widdicks et al. is summarized and discussed. Further equations and figures not originally present in the paper were added for discussion purposes.

For the topic of this assignment, I choose the task suggested by Jeff Dewynne, that is to write a summary and discussion of one of the papers provided in the lecture notes. I choose the paper labeled 11 in the section “General references on asymptotics in finance”. The title of the paper is *The Black-Scholes Equation Revisited: Asymptotic Expansions and Singular Perturbations* and the authors are Widdicks et al. (for full reference see [1]). Since in this assignment I will very often have to refer to the paper under discussion, I would like to mention that every time I use “the paper”, “the authors”, Widdicks et al. and similar formulations in this assignment, I always refer to [1].

As the title suggests the authors apply asymptotic expansions to the Black-Scholes equation while assuming that the volatility parameter $\sigma$ can be considered small. They apply these methods to European, American and Barrier Options. The authors separated their paper into five sections: an introduction, European Options, American Options, Barrier Options and a short conclusion. In this assignment I will adopt the same structure, summarize the content of the paper [1] and add remarks and discussions where appropriate.

1 Introduction

The Black-Scholes equation is a partial differential equation used to price Options. Since its first publication in 1973 [2], the Black-Scholes equation has been studied extensively. This also includes various different approaches to numerically solving it directly or simplifying the problem before and then solve it. One possible approach is using asymptotic expansions and singular perturbation theory. In the first part of the introduction the authors elaborate on how this technique has been used by other researchers before in order to transform the Black-Scholes equation into an integral equation. These include Green’s function procedures [3, 4], Fourier transforms [5] and Laplace transforms [6].

Unlike the previously mentioned papers, the article under discussion directly works with the Black-Scholes equation using asymptotic series. The volatility parameter $\sigma$ is taken to be the small parameter in which expansions are done. The authors claim that in some circumstances using a proper rescaling of dependent and independent variables will lead to accurately priced options for a whole range of $\sigma$. As starting point for their approach, the authors choose the European options where an exact solution is known. This makes it possible to easily compare the results of the approximation and determine the errors.

2 European Options

2.1 Perturbation Theory

The second section of the paper starts with a short introduction of the basic ideas of asymptotic expansion. For a rigorous mathematical treatment the authors refer to [7, 8]. Assuming there is
some kind of differential equation of which \( f(x; \epsilon) \) is the solution, Widdicks et al. explain that the main idea is to expand \( f(x; \epsilon) \) into some kind of series which eventually gets truncated. In the simplest form, an \( N \)-th order approximation of \( f \) can be written as:

\[
f(x, \epsilon) = \sum_{n=0}^{N} a_n \epsilon^n \phi_n(x)
\]

where successive terms are getting smaller and smaller. Plugging this approximation back into the original equation results in a hierarchy of equations for the \( \phi_n \) of equation (1). These have to be solved starting with \( \phi_0 \) and using the previous results to find the solution for the next higher term. In an optimal scenario the series in (1) drops sharply for higher values of \( n \) and an accurate approximation can be found requiring only few terms. Most commonly this can be achieved by having an appropriate small parameter with \( \epsilon << 1 \). However, as the authors point out these so called “regular perturbations” will often fail in common cases and it is “singular perturbations” which have to be dealt with. One problem that leads to a singular perturbation is for example when the highest order derivative is multiplied with the small parameter \( \epsilon \). This is indeed the case for the Black-Scholes equation if the small parameter is taken to be \( \sigma \). However, the authors state that a proper rescaling of the variables remedies this problem as will be shown later on.

Widdicks et al. only mention fluid dynamics as an application for this approach, however, I would like to emphasize how common this technique is in physics in general. From classical physics (e.g. correct planetary orbits) over quantum mechanics (e.g. Born series) to quantum field theory, perturbation theory is widely used, important and well established tool.

### 2.2 Application to Black-Scholes Equation

The authors start the section with stating the problem under consideration, which is the Black-Scholes equation for a European Call option \( C(S, t) \):

\[
\frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} - rC = 0
\]

where \( S \) denotes the value of the underlying, \( \sigma \) its volatility and \( r \) the risk-free rate. This partial differential equation is subject to the following boundary conditions:

\[
C(S, t = T) = \max(S - X, 0) \quad (3)
\]

\[
C(0, t) = 0 \quad (4)
\]

\[
\lim_{S \to \infty} \frac{\partial C}{\partial S} = 1 \quad (5)
\]

with the exercise price of the option \( X \). At first the authors use a regular perturbation approach of type (1) in \( N \)-th order given by:

\[
C(S, t, \sigma) = \sum_{n=0}^{N} \sigma^n C_n(S, t)
\]

Inserting this into the Black-Scholes equation (2) gives rise to:
\[ \sigma^0 \left( rS \frac{\partial C_0}{\partial S} + \frac{\partial C_0}{\partial t} - rC_0 \right) + \sigma^1 \left( rS \frac{\partial C_1}{\partial S} + \frac{\partial C_1}{\partial t} - rC_1 \right) + \sigma^2 \left( \frac{1}{2} S^2 \frac{\partial^2 C_2}{\partial S^2} + rS \frac{\partial C_2}{\partial S} + \frac{\partial C_2}{\partial t} - rC_2 \right) + \sigma^3 \left( \frac{1}{2} S^2 \frac{\partial^2 C_3}{\partial S^2} + rS \frac{\partial C_3}{\partial S} + \frac{\partial C_3}{\partial t} - rC_3 \right) + O(\sigma^4) = 0 \tag{7} \]

This means that even and odd \( C_n \) are pairwise equal in this approximation. Therefore, one might prefer the expansion

\[ C(S,t,\sigma) = \sum_{n=0}^{N} \sigma^{2n} C_n(S,t) \tag{8} \]

instead of (6). However, this is not immediately obvious from the article since the authors decided to already truncate the equation at the 0-th order of \( \sigma \). The authors continue to discuss the solution for \( C_0 \) which is given in equation (2.6) and reads:

\[ C_0(S,t) = \begin{cases} 
S - X e^{-r(T-t)} & \text{if } S > X e^{-r(T-t)} \\
0 & \text{if } S < X e^{-r(T-t)} 
\end{cases} \tag{9} \]

which at \( T = t \) reproduces the payoff at maturity of the option. Furthermore, there is clearly a discontinuity along the line \( S = X e^{-r(T-t)} \) that poses a problem. The authors point out that due to this discontinuity the first and second derivative of \( C_0 \) with respect to \( S \) are large and cannot be simply ignored. Therefore, the approximation (9) becomes invalid in the vicinity of this line. To tackle this issue, the authors suggest to adopt a technique commonly employed in fluid dynamics, the separation into inner and outer regions. In addition to the two zones already present in equation (9) a shear layer shall be introduced to help smear out the discontinuity and remedy the problems associated with the discontinuity. This shear layer and the two other zones are illustrated in figure 2.1. of the paper.

I would like to add two remarks to the arguments presented in the paper. As a consequence of the structure of (7) \( C_1 \) is also given by (9). Hence, the solution up to first order in \( \sigma \) is simply \( (1 + \sigma)C_0 \). Additionally to substantiate the claim in the paper that the first and second derivative of \( C_0 \) with respect to \( S \) are discontinuous, one should explicitly state them. Using the Heaviside step function \( \Theta(x) \) equation (9) and the partial derivatives with respect to \( S \) can be expressed as:

\[ C_0(S,t) = \left( S - X e^{-r(T-t)} \right) \Theta \left( X e^{-r(T-t)} \right) \tag{10} \]
\[ \frac{\partial C_0(S,t)}{\partial S} = \Theta \left( X e^{-r(T-t)} \right) \tag{11} \]
\[ \frac{\partial^2 C_0(S,t)}{\partial S^2} = \delta \left( X e^{-r(T-t)} \right) \tag{12} \]

where \( \delta(x) \) denotes the Dirac delta function. In my opinion, this makes it very clear that they are indeed discontinuous and the simple regular perturbation ansatz becomes inconsistent around
\[ S = X e^{-rt(T-t)}. \]

As a next step, in order to closely examine the shear layer, the authors introduce new scaled variables given by

\[
\tau = T - t \\
\hat{C} = \frac{1}{\sigma} C \\
\hat{S} = \frac{1}{\sigma} (S - X e^{-\tau t})
\]

where it is still assumed that \( \sigma \) is a small parameter. Using these new variables the following transformation rules apply:

\[
\frac{\partial C}{\partial S} = \frac{\partial \hat{C}}{\partial \hat{S}} \\
\frac{\partial^2 C}{\partial S^2} = \frac{1}{\sigma} \frac{\partial^2 \hat{C}}{\partial \hat{S}^2} \\
\frac{\partial C}{\partial t} = -\frac{\partial \hat{C}}{\partial \tau} - r X e^{-\tau t} \frac{\partial \hat{C}}{\partial \hat{S}}
\]

which are given in equation (2.9) of the paper. However, there is a small mistake there. In the equation corresponding to (18) the term \( X e^{-\tau t} \) reads \( X^{-\tau t} \) which is probably simply a typo.

For what follows it is beneficial to rewrite the Black-Scholes equation (2) in the new variables and explicitly state it. For some reason the authors did not do that in their paper. Using the scaled variables leads to a modified Black-Scholes equation given by:

\[
\frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + r S \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} - r C = 0 \\
\frac{1}{2} \sigma (X e^{-\tau t} + \sigma \hat{S}) \frac{\partial^2 \hat{C}}{\partial \hat{S}^2} + r (X e^{-\tau t} + \sigma \hat{S}) \frac{\partial \hat{C}}{\partial \hat{S}} - \sigma \frac{\partial \hat{C}}{\partial \tau} - r X e^{-\tau t} \frac{\partial \hat{C}}{\partial \hat{S}} - r \sigma \hat{C} = 0 \\
\frac{1}{2} \left[ X^2 e^{-2\tau t} + 2 X e^{-\tau t} \sigma \hat{S} + \sigma^2 \hat{S}^2 \right] \frac{\partial^2 \hat{C}}{\partial \hat{S}^2} + r \hat{S} \frac{\partial \hat{C}}{\partial \hat{S}} - \frac{\partial \hat{C}}{\partial \tau} - r \hat{C} = 0
\]

From this equation it is directly clear that it will not suffer from the same problem as before since even if \( \sigma \to 0 \) there is still a second order derivative present. Therefore, using the same type of expansion as in (6) only now using the scaled variables

\[
\hat{C}(\hat{S}, \tau, \sigma) = \sum_{n=0}^{N} \sigma^n \hat{C}_n(\hat{S}, \tau)
\]

should lead to an improved result. Inserting the expansion (20) into the Black-Scholes equation (19) leads to
validity of the approximation also depend on \( r \). I have discussed or mentioned in any way how the results may depend on \( r \) if they were dimensionless which is clearly not the case. Furthermore, beside the volatility there is no point has it been clearly stated what kind of units are being used. The parameters are stated as it is also not possible to deduce it from that. Which is the second point I would like to make. At 2.3 shows the error as a function of the order of terms included in the approximation \( N \). 

The authors claim that this demonstrates that the approximation is accurate “even for relatively large values of the volatility”. Unfortunately, I do not agree with there assessment. This is due to several issues which I like to elaborate on. Firstly, it is not clear what the error actually is. It is implied that it is the difference between the analytical solution and the results of the approximation. This result can almost be simply read off from the scaled Black-Scholes equation (19) and contains the results given in the equations (2.11), (2.12) and (2.13) of the paper. Hence, writing down the scaled Black-Scholes equation (19) would have improved the readability of the paper.

In order to proceed with numerical calculations, the authors also transform the boundary conditions to scaled variables. The results are given in equation (2.14) and are used to directly numerically solve for the \( \hat{C}_1 \). As explained before this is done by solving for \( \hat{C}_0 \) first followed by the next higher \( n \) until the series is truncated at a certain \( N \). For the actual numerical calculation the authors use a Crank-Nicholson scheme in conjunction with second-order finite differencing.

First results are shown in figure 2.2 where the error as a function of volatility for \( N = 2 \) is plotted. The other parameters are \( \hat{S} = 0, X = 100, r = 0.06 \) and \( \tau = 0.5 \). The error seems to be exponentially increasing with the volatility and is almost indistinguishable from the axis around \( \sigma = 0.2 \) and lower. The authors claim that this demonstrates that the approximation is accurate “even for relatively large values of the volatility”. Unfortunately, I do not agree with there assessment. This is due to several issues which I like to elaborate on. Firstly, it is not clear what the error actually is. It is implied that it is the difference between the analytical solution and the results of the approximation. However, it is not clear if it is an absolute or a relative error. Since there are no units used at all it is also not possible to deduce it from that. Which is the second point I would like to make. At no point has it been clearly stated what kind of units are being used. The parameters are stated as if they were dimensionless which is clearly not the case. Furthermore, beside the volatility there is another parameter in the Black-Scholes equation (2) namely the risk-free rate \( r \). The authors do not discuss or mention in any way how the results may depend on \( r \). Since the volatility \( \sigma \) is assumed to be a “small” parameter, what does that mean in combination with the risk-free rate? Does the validity of the approximation also depend on \( r \) or only on \( \sigma \)? Lastly, it is not clear to me why a linear scale was used instead of a logarithmic one. The way figure 2.2 is plotted renders almost a third of the plot useless since the only thing one can conclude is that the error is small enough to be indistinguishable from the axis. For theses reasons I believe the authors could have made a much more compelling argument with minor adjustments. In my opinion figure 2.2 is not enough to support the claim the authors make.

Figure 2.3 shows the error as a function of the order of terms included in the approximation \( N \). The parameters are \( \hat{S} = 0, X = 100, r = 0.06, T = 0.5 \) and \( \sigma = 0.2 \). The error overall drops

\[
\begin{align*}
\sigma^0 & \left( \frac{X^2}{2} e^{-2r\tau} \frac{\partial^2 \hat{C}_0}{\partial S^2} + rS \frac{\partial \hat{C}_0}{\partial S} \frac{\partial \hat{C}_0}{\partial \tau} - \hat{r}\hat{C}_0 \right) \\
+ \sigma^1 & \left( \frac{X^2}{2} e^{-2r\tau} \frac{\partial^2 \hat{C}_1}{\partial S^2} + X e^{-r\tau} \hat{S} \frac{\partial^2 \hat{C}_0}{\partial S^2} + rS \frac{\partial \hat{C}_1}{\partial S} - \frac{\partial \hat{C}_1}{\partial \tau} - \hat{r}\hat{C}_1 \right) \\
+ \sigma^2 & \left( \frac{X^2}{2} e^{-2r\tau} \frac{\partial^2 \hat{C}_2}{\partial S^2} + X e^{-r\tau} \hat{S} \frac{\partial^2 \hat{C}_1}{\partial S^2} + \frac{\hat{S}^2}{2} \frac{\partial^2 \hat{C}_0}{\partial S^2} + \frac{\partial \hat{C}_2}{\partial S} - \frac{\partial \hat{C}_2}{\partial \tau} - \hat{r}\hat{C}_2 \right) \\
+ \ldots & \\
+ \sigma^n & \left( \frac{X^2}{2} e^{-2r\tau} \frac{\partial^2 \hat{C}_n}{\partial S^2} + X e^{-r\tau} \hat{S} \frac{\partial^2 \hat{C}_{n-1}}{\partial S^2} + \frac{\hat{S}^2}{2} \frac{\partial \hat{C}_{n-2}}{\partial S^2} + rS \frac{\partial \hat{C}_n}{\partial S} - \frac{\partial \hat{C}_n}{\partial \tau} - \hat{r}\hat{C}_n \right) \\
+ \ldots & = 0
\end{align*}
\]
and is apparently constant on a logarithmic scale as of $N = 4$. For $N = 1$ and $N = 3$ a slight increase is visible. The authors state that the error being constant for $N > 4$ is due to errors introduced by the numerical scheme dominating truncation errors. Therefore, only very few terms of the approximation have to be included. After this demonstration of how the approximation works and showing that it indeed works the authors move on to the more challenging problem of American options.

### 3 American Options

Using a similar approach as before, the authors cover American put options in section 3. Since American options may be exercised at any time, they are at least worth $X - S$. Therefore, equation (9) becomes

$$P_0(S,t) = \begin{cases} 
S - X & \text{if } S > X \\
0 & \text{if } S < X 
\end{cases} \quad (22)$$

instead (equation (3.2) in the paper). Hence, the shear layer of interest will be around the line $S = X$. Additionally, the authors introduce a new parameter $R = r/\sigma$ given by equation (3.1) and assumed to be of $O(1)$. Hence, the volatility and the risk free-rate are assumed to be comparable in size. As before scaled variables are introduced by:

$$\tau = T - t \quad (23)$$

$$\hat{P} = \frac{1}{\sigma} P \quad (24)$$

$$\hat{S} = \frac{1}{\sigma} (S - X) \quad (25)$$

leading to similar transformation rules as before

$$\frac{\partial P}{\partial S} = \frac{\partial \hat{P}}{\partial \hat{S}} \quad (26)$$

$$\frac{\partial^2 P}{\partial S^2} = \frac{1}{\sigma^2} \frac{\partial^2 \hat{P}}{\partial \hat{S}^2} \quad (27)$$

$$\frac{\partial P}{\partial t} = -\sigma \frac{\partial \hat{P}}{\partial \tau} \quad (28)$$

While the transformation rules are not stated in the paper since one might consider them obvious, it would still be beneficial to state the transformed Black-Scholes equation. However, as before the authors do not write it down explicitly. It is given by:

$$\frac{1}{2} \sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + r S \frac{\partial P}{\partial S} + \frac{\partial P}{\partial t} - r P = 0$$

$$\frac{1}{2} \sigma (X + \sigma \hat{S})^2 \frac{\partial^2 \hat{P}}{\partial \hat{S}^2} + r (X + \sigma \hat{S}) \frac{\partial \hat{P}}{\partial \hat{S}} - \sigma \frac{\partial \hat{P}}{\partial \tau} - r \sigma \hat{P} = 0$$

$$\frac{1}{2} \left[ X^2 + 2X\sigma \hat{S} + \sigma^2 \hat{S}^2 \right] \frac{\partial \hat{P}}{\partial \hat{S}} + R (X + \sigma \hat{S}) \frac{\partial \hat{P}}{\partial \hat{S}} - \frac{\partial \hat{P}}{\partial \tau} - R \sigma \hat{P} = 0 \quad (29)$$
and inserting an expansion of the type (20) results in

$$\sigma \frac{X^2 \partial^2 \hat{P}_0}{\partial S^2} + RX \frac{\partial \hat{P}_0}{\partial S} \frac{\partial \hat{P}_0}{\partial \tau} + \frac{\partial^2 \hat{P}_1}{\partial S^2} + RX \frac{\partial \hat{P}_1}{\partial S} \frac{\partial \hat{P}_0}{\partial \tau} - \frac{\partial \hat{P}_1}{\partial \tau} - \frac{\partial \hat{P}_0}{\partial \tau}$$

$$+ \frac{\partial^2 \hat{P}_2}{\partial S^2} + \frac{\partial \hat{P}_2}{\partial S} \frac{\partial \hat{P}_0}{\partial \tau} - \frac{\partial \hat{P}_2}{\partial \tau} - \frac{\partial \hat{P}_1}{\partial \tau}$$

$$+ \ldots = 0 \quad (30)$$

In the paper, the authors state and use the terms up to first order of (30) which are given in equation (3.5) and (3.7). Up to this point the approach is similar to the European option case. Widdicks et al. state in their paper that compared to before a major complication is the existence of an early exercise boundary $S_f(t)$. The transformed value is again given by $\hat{S}_f(t) = (S_f(t) - X)/\sigma$ and it may be expanded as follows:

$$\hat{S}_f(t) = \sum_{n=0}^{N} \sigma^n \hat{S}_{f_n}(t) \quad (31)$$

In order to guarantee a continuous delta, the authors enforce the boundary conditions given in equation (3.6) on $\hat{P}_0$. Using Taylor expansion and the smooth pasting condition further boundary conditions for $\hat{P}_1$ involving $S_{f_0}$ and $\hat{S}_{f_1}$, are found. Using the first two terms of the asymptotic approximation and the appropriate boundary conditions the authors are now ready to numerically calculate results. As a next step they explain the numerical scheme they use to do so.

According to the authors the most challenging part is the unknown early exercise boundary. In order to tackle this problem they use the mapping

$$S^* = \hat{S} - \hat{S}_{f_0}(\tau) \quad (32)$$

which fixes the unknown boundary to $S^* = 0$. These so called “body-fitted” coordinates have been successfully used before [9]. The scheme the authors use to numerically solve the problem is as follows. A Crank-Nicholson method is employed with the current location of the early exercise boundary treated as an unknown. Newton iteration is used to solve the resulting algebraic system. The resulting tridiagonal plus one full column system is inverted using Gaussian elimination. In figure 3.1 the authors present a comparison between a full numerical solution of the problem and the results of the asymptotic approximation using the first two terms as a function of the volatility $\sigma$. The parameters are $S = X = 100$, $r = \sigma$ and $T = 0.5$. Widdicks et al. seem to be content with the fact that both lines are indistinguishable in the plot. However, if they want to showcase how well the approximation works they should have plotted the difference of the two curves and used an logarithmic scale instead. In my opinion, that would have been a much more informative figure.

In the next subsection the authors apply another rescaling using:
\[ (\bar{S}_{f_0}, \bar{S}, \bar{P}_0) = \frac{R}{X} (\hat{S}_{f_0}, \hat{S}, \hat{P}_0), \quad \bar{\tau} = R^2 \tau, \quad \bar{P}_1 = \frac{R^2}{X} \hat{P}_1 \] (33)
making it possible to produce a universal set of results in \( \bar{S}, \bar{\tau} \) space. Hence, the valuation of American options can be done by using a set of look-up tables and doing linear interpolation. The look-up tables are provided in table 3.1 and 3.2. The authors explain in detail how they are to be used and give example cases. To get a better idea of how the structure of data provided looks like, I decided to plot them. The plots are shown in the figures 1, 2 and 3. The first figure shows the data for \( \hat{P}_0(\bar{S}^*, \sqrt{\bar{\tau}}) \). In a large area \( \hat{P}_0(\bar{S}^*, \sqrt{\bar{\tau}}) \) is either zero or very small. Towards the value of \( \bar{S}^* = 0 \) the values increase considerably, however, in a rather smooth fashion. Therefore, interpolation errors are probably quite small as stated by the authors.

The second figure shows the early exercise boundary \( S_{f_0}(t = 0) \) as given in the bottom of table 3.1. Again, the graph is smooth and can be well approximated using linear interpolation.

The third figure shows \( \hat{P}_1(\bar{S}^*, \sqrt{\bar{\tau}}) \). For small values of \( \sqrt{\bar{\tau}} \), \( \hat{P}_1 \) quickly vanishes. For large values of \( \sqrt{\bar{\tau}} \), there are two distinctive peaks. One of them being positive at large values of \( \bar{S}^* \) and a second negative one for small values of \( \bar{S}^* \). Overall \( \hat{P}_1(\bar{S}^*, \sqrt{\bar{\tau}}) \) displays a certain symmetry and even though the structure is considerably more complex compared to the other two figures it should still be possible to accurately interpolate missing values with basic techniques.

These look-up tables are one of the main results of the paper. Using them and very basic calculations it is possible to value an American option. Compared to solving the full Black-Scholes equation with an early exercise boundary the use of the look-up tables is quite a strong simplification. The error due to the approximation is smaller than a penny and therefore still acceptable.

Figure 1: Graphic representation of the values for \( \hat{P}_0(\bar{S}^*, \sqrt{\bar{\tau}}) \) as given in table (3.1) of [1].
Figure 2: Graphic representation of the values for $S_f(t = 0)$ as given in table (3.1) of [1].

Figure 3: Graphic representation of the values for $P_1(S^*, \sqrt{\tau})$ as given in table (3.2) of [1].
4 Barrier Options

In the last section before the conclusion the authors consider barrier options for vanilla European puts. While the problem is similar to the case discussed in section 2 the barrier results in some novel behavior. Widdicks et al. distinguish between two different cases. In the first one the line $S = X e^{-r\tau}$ and the barrier are distinct and do not touch. In the second case a collision between the two occurs. For the first scenario the value above the shear layer is practically zero and below it is $P \approx X e^{-r\tau} - S$. However, that does not satisfy $P = 0$ on $S = B$. Therefore, the authors introduce another thin layer to remedy this problem. The analogous term in fluid dynamics would be a boundary layer. This new layer, however, is only of thickness $O(\sigma^2)$. The different layers are schematically depicted in figure 4.1. Next, the authors introduce a new scaled variable by:

$$S^* = \frac{S-B}{\sigma^2}$$

leading to

$$\frac{1}{2}B^2 \frac{\partial^2 P}{\partial S^*^2} + rB \frac{\partial P}{\partial S^*} = 0$$

for the Black-Scholes equation. The authors give the analytic solution for this equation using the boundary conditions given by equation (4.4) as:

$$P(S^*,t) = (X e^{-r(T-t)} - B)(1 - e^{-2rS^*/R})$$

The fact that the value of the option drops sharply to zero in a region that is only of thickness $O(\sigma^2)$ makes it usually quite difficult to calculate deltas close to the barrier. However, using this asymptotic approximation approach the authors are able to give a value for delta which reads

$$\Delta_{\text{barrier}} = \frac{2r}{\sigma^2 B^2}$$

and is given in equation (4.6) of the paper. The delta around the shear layer is less troublesome since the thickness is $O(\sigma)$ and the option is low in value.

For the second case the authors assume the barrier collides with the line $S = X e^{-r\tau}$ at some point before expiry. The time of collision is given by:

$$t_c = T + \frac{1}{r} \log \left(\frac{B}{X}\right).$$

For times $t > t_c$ the behavior is as in the case before. In order to investigate what happens around the collision, the following scaled variables are introduced:

$$\tau_1 = \frac{t - t_c}{\sigma^2}$$

$$S^* = \frac{S-B}{\sigma^2}$$

$$P^* = \frac{P}{\sigma}$$

This results in the modified Black-Scholes equation given in equation (4.8) reading

$$\frac{\partial P^*}{\partial \tau_1} + \frac{B^2}{2} \frac{\partial^2 P^*}{\partial S^*^2} + rB \frac{\partial P^*}{\partial S^*} = 0.$$
Using the ansatz

\[ P^*(S^*, \tau_1) = B\sqrt{\tau_1} \hat{P}(\eta) \]  

(43)

with

\[ \eta = \frac{S^* - Br\tau_1}{B\sqrt{\tau_1}} \]  

(44)

the authors derive the solution

\[ P^*(S^*, \tau_1) = B\sqrt{\tau_1}[-\eta N(-\eta) + N'(-\eta)] \]  

(45)

where \( N(\eta) \) is the cumulative distribution function. Therefore, at times after the collision a boundary layer with \( S = B \) and a shear layer along \( S = Br\tau_1 \) appear. The different zones are depicted schematically in figure 4.3 of the paper. As a last step the authors provide a composite solution given by equation (4.13) which contains all the different zones for \( \tau_1 >> 1 \) and reads

\[ P_{comp} = Br\tau_1[1 - \exp(-2rS^*/B)] + B\sqrt{\tau_1}[-\eta N(-\eta) + N'(-\eta)] - Br\tau_1. \]  

(46)

As a last result the authors show the options delta along \( S^* = 0 \) with \( B = r = 1 \) and the value of the option itself in figure 4.4.

5 Conclusion

In the conclusion, Widdicks et al. give a brief overview of the findings in the paper. They demonstrated the usefulness of singular perturbation theory with volatility assumed to be a small parameter. Using this technique, European options, American options and barrier options have been covered. The focus of this paper is the regions in which option prices vary rapidly where straightforward numerical schemes may run into difficulties.

While the above statements by the authors are true, there are still some points which could be improved. As explained earlier I find the paper would have benefited from a discussion of the risk-free rate and its role in this approximation as well as some improvements to the figures 2.2 and 3.1.

References


