How robust is optimal execution when the temporary and permanent price impact parameters are stochastic?

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Abstract

The expectation and variance of the income of an agent following the Almgren-Chriss strategy was studied for the case that the temporary and permanent price impact parameters follow Ornstein-Uhlenbeck processes. Closed form solutions for the expected income and its variance are given for this model. The expected income is unchanged compared to the case of constant price impact parameters. The variance is larger when the price impact is stochastic and, in contrast to the constant price parameter case, it was found that the variance increases for large values of the impatience parameter, $\zeta$. This reduces the range of $\zeta$ in which a risk-averse agent would operate.

1 Introduction

An agent who executes a large order in the stock market will typically cause a change in the market price of the stock. The price change is normally detrimental to the agent: when they buy (sell) a large position they tend to increase (decrease) the price of the stock. Thus, an agent buying a large position will normally pay more than $NS_0$, where $N$ is the number of stocks to buy and $S_0$ is the stock price when the agent starts trading. Similarly, an agent liquidating a large position will typically obtain an income smaller than $NS_0$. This effect leads to the concept of optimal execution, which examines how an agent should trade in order to achieve a specified aim; for example, to maximise their expected income from liquidating a position.

Almgren and Chriss [1] studied optimal execution in a discrete time setting for the case that an agent has a permanent and temporary price impact. Temporary price impact arises when an agent executes a large market order against a limit order book (LOB). A large order will typically exceed the liquidity available at the best price, meaning that some of the order must be executed at less favourable prices. This is known as “walking the book”. Permanent price impact occurs because the other market participants react to the flow of market orders. For example, a string of large buy orders will tend to cause the other market participants to increase their prices. Optimal execution with temporary and permanent price impact was considered by Cartea et al. [2] in a continuous time setting. It was assumed that the permanent and temporary price impacts were linear in the agent’s trading speed and that these relationships had constant coefficients. However, data presented by Cartea and Jaimungal [3] suggest that this is not the case and that these coefficients are in fact stochastic.
This work examines the performance of the strategy proposed by Cartea et al. [2] for the case that the price impact parameters are stochastic. Closed form solutions are derived for the expectation of the income achieved from following the strategy and for its variance. These solutions are compared to estimates obtained from Monte Carlo simulation.

2 Optimal Liquidation with Permanent and Temporary Price Impact

In this section the optimal liquidation strategy is developed for a model with permanent and temporary price impact. The derivation follows that presented by Cartea et al. [2]. An agent wishes to liquidate a position of \( N \) shares within time \( T \) using only market orders. The agent’s actions have both a permanent and a temporary impact on the stock price. The evolution of the mid-point price of the stock is modelled using the equation

\[
dS_t = -g(v)v dt + \sigma_S dW^S,
\]

(1)

where \( g(v) \) describes the permanent price impact caused by the agent, \( v \) is the trading speed of the agent, \( \sigma_S \) is the volatility of the stock price and \( W^S \) is the Brownian motion driving the evolution of the stock price. In addition to the permanent price impact, the agent experiences a temporary price impact, which models the effect of “walking the book”. Therefore, the price at which the agent’s trades are executed is given by

\[
\hat{S}_t = S_t - f(v)v - \frac{1}{2}\Delta,
\]

(2)

where \( f(v) \) describes the temporary price impact and \( \Delta \) is the bid-ask spread. For simplicity it will be assumed that \( \Delta = 0 \) in the remainder of this work. The LOB is assumed to replenish immediately after a trade is executed. The agent’s income from selling their stock is given by

\[
X_t = \int_0^t v(q)\hat{S}_q dq.
\]

(3)

The performance criterion of the agent is

\[
H^v(t, x, S, q) = \mathbb{E}_{t,x,S,q}\left[ X_T + Q_T(S_T - \alpha Q_T) - \phi \int_0^T (Q_u)^2 du \right],
\]

(4)

which is the model chosen by Cartea et al. [2] and Cartea and Jaimungal [3]. Here \( Q_t \) is the agent’s inventory remaining at time \( t \). The first term is the income obtained up
to time $T$ and the second term represents the value of any remaining stock at time $T$ and contains a non-negative terminal inventory parameter, $\alpha$. The final term is an impatience term and penalises the remaining inventory along the duration of the strategy. $\phi$ is a non-negative impatience parameter. The agent’s value function is

$$H(t, x, S, q) = \max_{v \in A} H^*(t, x, S, q).$$ (5)

Applying the dynamic programming principle gives the following Hamilton-Jacobi-Bellman (HJB) equation for the value function

$$\frac{\partial H}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 H}{\partial S^2} - \phi q^2 + \max_v \left( v \left( S - f(v) \right) \frac{\partial H}{\partial x} - g(v) \frac{\partial H}{\partial S} - v \frac{\partial H}{\partial q} \right) = 0$$

(6)

$$H(T, x, S, q) = x + S q - \alpha q^2.$$ To simplify equation (6), it is assumed that both $f(v)$ and $g(v)$ are linear functions of $v$. Set $f(v) = kv$ and $g(v) = bv$, where $k$ and $b$ are finite constants with $k \geq 0$ and $b \geq 0$. Under these conditions, the optimal trading speed, $v^*$, can be found by maximising over $v$ and is given by

$$v^*(t) = \frac{1}{2k} S \frac{\partial H}{\partial x} - \frac{1}{2} \frac{\partial H}{\partial S} - \frac{\partial H}{\partial q}. \quad (7)$$

Substituting equation (7) into (6) yields an equation which can be solved for $H$:

$$H(t, x, S, q) = x + S q + q^2 \left( \sqrt{\phi k} \frac{1 + \zeta e^{2\gamma(T-t)}}{1 - \zeta e^{2\gamma(T-t)}} - \frac{1}{2} b \right). \quad (8)$$

where

$$\gamma = \sqrt{\phi k} \quad \text{and} \quad \zeta = \frac{\alpha - \frac{1}{2} b \sqrt{\phi k}}{\alpha - \frac{1}{2} b \sqrt{\phi k}}.$$

The optimal execution speed, $v^*(t)$, can be expressed in terms of state variables by substituting this expression for $H(t, x, S, q)$ into equation (7). This results in

$$v^*(t) = \gamma q \frac{\zeta e^{\gamma(T-t)} + e^{-\gamma(T-t)}}{\zeta e^{\gamma(T-t)} - e^{-\gamma(T-t)}}. \quad (9)$$

The relationship between $v$ and $q$ can be written as

$$dq = -v dt. \quad (10)$$

By evaluating equation (10) using (9), the optimal execution speed can be written in terms of $N$, the number of shares to be liquidated:

$$v^*(t) = \gamma \frac{\zeta e^{\gamma(T-t)} + e^{-\gamma(T-t)}}{\zeta e^{\gamma T} - e^{-\gamma T}} N. \quad (11)$$
Throughout the rest of this work equation (11) will be referred to as the Almgren-Chriss strategy. It specifies explicitly the speed at which the agent should trade to maximise equation (4). The aim of this work is to examine how robust this strategy is for the case that $b$ and $k$ are not constant but are driven by stochastic processes.

3 Distribution of Income Following the Almgren-Chriss Optimal Execution Strategy

In this section an expression for the distribution of the income achieved by following the Almgren-Chriss optimal execution strategy is developed. The price at which an agent is able to trade, considering both permanent and temporary price impact, is given by

$$\hat{S}_t = S_0 - v(t)k - b \int_0^t v(u)du + \sigma_s \int_0^t dW^S_u.$$ (12)

The income that the agent achieves by liquidating a position is

$$I_t = \int_0^t v(q) \hat{S}_q dq$$

$$= \int_0^t v(q) \left( S_0 - v(q)k - b \int_0^q v(u)du + \sigma_s \int_0^q dW^S_u \right) dq$$ (13)

Substituting equation (11) for $v$ and changing the order of integration for the stochastic integral gives

$$I_t = N \int_0^t S_0 \gamma \frac{\zeta e^{\gamma(T-t)} + e^{-\gamma(T-t)}}{\zeta e^{\gamma T} - e^{-\gamma T}} dq$$

$$- k \int_0^t \left( N \gamma \frac{\zeta e^{\gamma(T-t)} + e^{-\gamma(T-t)}}{\zeta e^{\gamma T} - e^{-\gamma T}} \right)^2 dq$$

$$- bN \int_0^t \gamma \frac{\zeta e^{\gamma(T-t)} + e^{-\gamma(T-t)}}{\zeta e^{\gamma T} - e^{-\gamma T}} \left( 1 - \frac{\zeta e^{\gamma(T-t)} - e^{-\gamma(T-t)}}{\zeta e^{\gamma T} - e^{-\gamma T}} \right) dq$$

$$+ N \sigma_s \int_0^t \gamma \frac{\zeta e^{\gamma(T-t)} + e^{-\gamma(T-t)}}{\zeta e^{\gamma T} - e^{-\gamma T}} dq dW^S_u.$$ (14)

The stochastic integral is a martingale and $I_t$ is normally distributed with expectation

$$\mathbb{E}(I_t) = N(S_0 - bN) \left( \frac{\zeta e^{\gamma(T-t)} - e^{-\gamma(T-t)}}{\zeta e^{\gamma T} - e^{-\gamma T}} + 1 \right)$$

$$- \frac{bN^2}{2} \zeta^2 e^{2\gamma(T-t)} + e^{-2\gamma(T-t)} - \zeta^2 e^{2\gamma T} - e^{-2\gamma T}$$

$$+ \frac{k \gamma N^2}{2} \zeta^2 e^{2\gamma(T-t)} - e^{-2\gamma(T-t)} - \zeta^2 e^{2\gamma T} + e^{-2\gamma T} - 4 \zeta \gamma t.$$ (15)
The variance of $I_t$ is the variance of the stochastic integral and can be computed using Itô’s Isometry to yield

$$
\mathbb{V}(I_t) = \left( \frac{\sigma S N}{\zeta e^{\gamma T} - e^{-\gamma T}} \right)^2 \left( (A^2(t) - 2\zeta)t - 2A(t) \left( \frac{\zeta e^{\gamma (T-t)} + e^{-\gamma (T-t)} - \zeta e^{\gamma T} - e^{-\gamma T}}{2} \right) \right)
$$

where

$$A(t) = -\zeta e^{\gamma (T-t)} + e^{-\gamma (T-t)}. \quad (16)$$

4 Stochastic $k$

Here the performance of the Almgren-Chriss optimal execution strategy is examined for the case that the temporary price impact, $k$, is stochastic. It is assumed that the dynamics of $k$ are governed by an Ornstein-Uhlenbeck process, such that

$$
dk_t = \theta_k (k_\infty - k_t) dt + \sigma_k dW^k. \quad (17)
$$

Here $\theta_k$ is the speed of mean reversion of $k$; $\sigma_k$ is the volatility of $k$; $k_\infty$ is the long-run mean value of $k$ and $W^k$ is the Brownian motion driving the evolution of $k$. It is assumed that $W^k$ is independent of the Brownian motion driving the evolution of the stock price. The solution to equation (17) can be written as

$$k_t = k_0 e^{-\theta_k t} + k_\infty (1 - e^{-\theta_k t}) + \sigma_k \int_0^t e^{-\theta_k (t-u)} dW^k_u. \quad (18)
$$

where $k_0$ is the initial value of $k$. In this work it is assumed that $k_0 = k_\infty$. Under these conditions equation (18) can be simplified to

$$k_t = k_\infty + \sigma_k \int_0^t e^{-\theta_k (t-u)} dW^k_u. \quad (19)
$$

With this model for $k$, the execution price is given by

$$\hat{S}_t = S_0 - v(t) \left( k_\infty + \sigma_k \int_0^t e^{-\theta_k (t-u)} dW^k_u \right) - b \int_0^t v(u) du + \sigma_S \int_0^t dW^S_u. \quad (20)
$$

The income from liquidating a position is

$$I_t = \int_0^q v(q) \left( S_0 - v(q) \left( k_\infty + \sigma_k \int_0^q e^{-\theta_k (q-u)} dW^k_u \right) - b \int_0^q v(u) du + \sigma_S \int_0^q dW^S_u \right) dq. \quad (21)$$
Comparing equations (13) and (21) it can be seen that they differ only by a stochastic integral term given by

\[ I^k_t = \sigma_k \int_0^t \int_q^q v^2(q) e^{-\theta_k(q-u)} dq dW^k_u. \]  

(22)

Exchanging the order of integration yields

\[ I^k_t = \sigma_k \int_0^t \int_u^t v^2(q) e^{-\theta_k(q-u)} dq dW^k_u. \]  

(23)

\( I^k_t \) is normally distributed with \( \mathbb{E}(I^k_t) = 0 \). Therefore, for the model proposed here, the expected income from following the Almgren-Chriss strategy is independent of the values of \( \theta_k \) and \( \sigma_k \). Furthermore, the expected income is equal to the expected income with constant \( k \) and is given by equation (15). The variance of \( I^k_t \) is calculated by applying Itô’s Isometry to equation (23) and can be expressed as

\[
\mathbb{V}(I^k_t) = \sigma^2_k \left( \frac{\gamma N}{\zeta e^{\gamma t} - e^{-\gamma t}} \right)^4 \left( Z^2(t) \frac{1 - e^{-2\theta_k t}}{2\theta_k} - 2CZ(t) \frac{1 - e^{-\theta_k t}}{\theta_k} \right.
\]

\[
- 2B(t)Z(t) \frac{1 - e^{-(\theta_k + 2\gamma) t}}{2\gamma + \theta_k} - 2A(t)Z(t) \frac{1 - e^{-t(\theta_k + 2\gamma)}}{\theta_k - 2\gamma}
\]

\[
+ A^2(t) \frac{1 - e^{4\gamma t}}{4\gamma} + B^2(t) \frac{1 - e^{-4\gamma t}}{4\gamma} + (2AB(t) + C^2)t
\]

\[
+ 2CB(t) \frac{1 - e^{-2\gamma t}}{2\gamma} - 2A(t)C \frac{1 - e^{2\gamma t}}{2\gamma} \right)
\]  

(24)

where

\[
A(t) = -\frac{\zeta e^{2\gamma(T-t)}}{2\gamma + \theta}
\]

\[
B(t) = \frac{e^{-2\gamma(T-t)}}{2\gamma - \theta}
\]

\[
C = \frac{2\zeta}{\theta}
\]

\[
Z(t) = A(t) + B(t) + C.
\]

For the model proposed here, the total variance for the Almgren-Chriss strategy with stochastic \( k \) can be found by summing the contributions from the uncertainty due to \( W^S \) and \( W^k \). That is the total variance is the sum of equations (16) and (24).

Introducing uncertainty to the value of \( k \) does not affect the expected income from following the Almgren-Chriss strategy. However, when \( k \) is stochastic, the optimal execution strategy is not given by simply replacing \( k \) with \( k_\infty \) in equation (11). An agent who is able to observe \( k \) is able to increase their expected income by adjusting their strategy based on \( k \). For example, when \( k < k_\infty \) the agent would increase \( v \) to take advantage of the lower penalty for trading faster.
5 Stochastic $b$

In this section the performance of the Almgren-Chriss strategy is examined for the case that the permanent price impact, $b$, is stochastic. It is again assumed that $b_t$ obeys an Ornstein-Uhlenbeck process with an initial value equal to the long-run mean, $b_\infty$. Therefore, $b_t$ can be expressed as

$$b_t = b_\infty + \sigma_b \int_0^t e^{-\theta_b(t-u)} dW^b_u. \quad (25)$$

Assuming that the temporary price impact, $k$, is constant, the execution price is

$$\hat{S}_t = S_0 - kv(t) - \int_0^t v(q) \left( b_\infty + \sigma_b \int_0^q e^{-\theta_b(q-u)} dW^b_u \right) dq + \sigma_S \int_0^t dW^S_u. \quad (26)$$

The income from liquidating a position is

$$I_t = \int_0^t v(z) \hat{S}_z dz$$

$$= \int_0^t v(z) \left( S_0 - kv(t) - \int_0^z v(q) \left( b_\infty + \sigma_b \int_0^q e^{-\theta_b(q-u)} dW^b_u \right) dq + \sigma_S \int_0^z dW^S_u \right) dz. \quad (27)$$

Comparing equations (13) and (27) it can be seen that they differ only by a stochastic integral term given by

$$I^b_t = \sigma_b \int_0^t \int_0^z \int_0^q v(z)v(q)e^{-\theta_b(q-u)} dW^b_u dqdz. \quad (28)$$

Exchanging the order of integration yields:

$$I^b_t = \sigma_b \int_0^t \int_0^t \int_0^z v(z)v(q)e^{-\theta_b(q-u)} dqdzdW^b_u. \quad (29)$$

It can be seen from equation (29) that $E(I^b_t) = 0$. Therefore the uncertainty in $b$ does not affect the expected income from following the Almgren-Chriss strategy. The variance due to uncertainty in $b$ can be calculated based on equation (29). Details are given in Appendix A.

For brevity, $k$ was assumed to be constant in equation (26). However, the effect of stochastic $k$ can be included in the same way as in equation (21). For the model proposed here, the uncertainty arising from $W^b$ acts independently to the uncertainty due to $W^S$ and $W^k$. This means that, when both $b$ and $k$ are stochastic, the total variance for the Almgren-Chriss strategy can be found by summing the contributions from the uncertainty due to $W^S$, $W^k$ and $W^b$. This is demonstrated in section 7.
6 Correlation Between $b$ and $k$

In sections 4 and 5 it was assumed that the Brownian motions driving $k$ and $b$ are independent. In reality this is unlikely to be the case for most stocks: Cartea and Jaimungal [3] measured the correlation between $b$ and $k$ for four stocks and found values between 0 and 0.72. Consider the case that the coefficient of correlation between $W^b$ and $W^k$ is $\rho_{bk}$, where $0 \leq \rho_{bk} \leq 1$. Correlation between $W^b$ and $W^k$ does not affect the expected income from following the Almgren-Chriss strategy, which is still given by equation (15). However; introducing correlation leads to an extra term for the variance of the income:

$$\mathbb{V}(I^{bk}_t) = 2\rho_{bk}\sigma_b\sigma_k \int_{q-u}^{q} \int_{t-u}^{t} v^3(z) v(q) e^{-\theta_b(q-u)-\theta_k(z-u)} dq dz du.$$  \hspace{1cm} (30)

The solution to equation (30) is given in Appendix B. Correlations between $W^S$ and $W^b$ and $W^k$ could be treated in a similar manner but will not be considered in this work.

7 Monte Carlo Simulations

This section reports the results of Monte Carlo simulations of the liquidation of $N = 1 \times 10^4$ shares following the Almgren-Chriss strategy. The results presented are based on $1 \times 10^4$ simulated paths and a timestep of $dt = 1 \times 10^{-3}$ was used to generate the paths using an Euler-Maruyama method. The code is given in Appendix C. Table 1 gives the parameter values used for the reference case. Unless otherwise stated, the results presented below use these parameter values.

Figure 1 shows the expected income from following the Almgren-Chriss strategy as a function of time. Data are shown for simulations of the reference case and for $\sigma_k = \sigma_b = 0$. Equation (15) is also shown. Figure 1(a) shows data for $\phi = 1 \times 10^{-3}$, whereas 1(b) shows data for $\phi = 0.01$. These plots indicate that, for the model proposed here, the expected income from following the Almgren-Chriss strategy does not depend on whether $k$ and $b$ are stochastic or fixed at their mean value. This is in agreement with the analysis presented above.

Figure 2 shows the variance of the income from following the Almgren-Chriss strategy as a function of time. Data are shown for 4 combinations of $\sigma_k$ and $\sigma_b$. Figure 2(a) shows the variance for the case that $b$ and $k$ are constant, i.e. $\sigma_k = \sigma_b = 0$. As expected, the variance is a monotonic increasing function of $t$. The estimate of $\mathbb{V}(I_t)$ obtained from Monte Carlo is in excellent agreement with equation (16).
Figure 1: Expected income achieved following the Almgren-Chriss strategy: Comparison of estimates obtained from Monte Carlo simulations with the analytical result (equation (15)).
Table 1: Simulation parameters

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<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
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<tr>
<td>$N$</td>
<td>Number of shares to sell</td>
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<td>$S_0$</td>
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<td>$\sigma_s$</td>
<td>Volatility of stock price</td>
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<td>$\alpha$</td>
<td>Terminal penalty parameter</td>
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<td>$\phi$</td>
<td>Impatience parameter</td>
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<td>$k_\infty$</td>
<td>Temporary price impact parameter</td>
<td>$1 \times 10^{-4}$</td>
</tr>
<tr>
<td>$b_\infty$</td>
<td>Permanent price impact parameter</td>
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<td>$\sigma_k$</td>
<td>Volatility of $k$</td>
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<td>$\theta_b$</td>
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</tr>
<tr>
<td>$T$</td>
<td>Time horizon</td>
<td>1</td>
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</tbody>
</table>

Figure 2(b) shows the variance for the case that $b$ is constant but $k$ is stochastic. Equations (16) and (24) and their sum are also plotted. It can be seen that there is excellent agreement between the estimate of $\mathbb{V}(I_t)$ obtained from Monte Carlo and the sum of equations (16) and (24). This is in agreement with the argument made in section 4 that the total variance can be found by summing the contributions from $W^k$ and $W^S$.

Figure 2(c) shows the variance for the case that $k$ is constant but $b$ is stochastic. Equations (16) and (32) and their sum are also plotted. Again, there is excellent agreement between the Monte Carlo simulation and the sum of (16) and (32). Finally, figure 2(d) shows the variance for the case that both $k$ and $b$ are stochastic. It can be seen that the total variance is the sum of the contributions from $W^b$, $W^k$ and $W^S$.

Figure 3 shows the expected income from the Almgren-Chriss strategy and its variance for 3 values of $\rho_{bk}$. The expectation of the income follows equation (15), that is, it is unchanged from the case of constant $b$ and $k$. Figure 3(b) demonstrates that a positive correlation between $b$ and $k$ increases $\mathbb{V}(I)$ compared to the case of no correlation. Cartea and Jaimungal [3] reported that $b$ and $k$ tend to be positively correlated for most stocks. This works against a risk averse agent since it increases $\mathbb{V}(I)$ for the same expected income. For $\rho_{bk} < 0$ the variance of income is lower than when $b$ and $k$ are uncorrelated. However, this case is unlikely to occur in practice.
It has been demonstrated above that the expected income from following the Almgren-Chriss strategy is not affected by the introduction of uncertainty in the values of $b$ and $k$. However, $V(I_t)$ is increased when the parameter values are stochastic. The impact on a risk-adverse agent can be thought of in terms of the Sharpe ratio, defined by

$$\lambda = \frac{\mathbb{E}(I_T)}{\sqrt{V(I_T)}}. \quad (31)$$

Introducing uncertainty into $b$ and $k$ has the effect of decreasing $\lambda$. A risk-averse agent could adjust their choice of $\phi$ to control $\lambda$. Figure 4(a) shows the expected income at time $T$ as a function of $\phi$. All other parameter values are those given in table 1. The expected income is a monotonic decreasing function of $\phi$. This occurs because, as $\phi$ is increased, the agent sells more quickly at the start of the trading
(a) Expected income for three values of $\rho_{bk}$.

(b) Variance of income for three values of $\rho_{bk}$. The points show estimates from Monte Carlo simulation, whereas the lines show the sum of equations (16), (24), (32) and (33).

Figure 3: Expectation and variance of income achieved following the Almgren-Chriss strategy for three values of $\rho_{bk}$. 

\[ E(L_t) = \begin{cases} \text{Equation (15)} & \text{if } \rho_{bk} = -0.5 \\
\text{MC } \rho = 0 & \text{if } \rho_{bk} = 0 \\
\text{MC } \rho = 0.5 & \text{if } \rho_{bk} = 0.5 \end{cases} \]
window and incurs a larger price penalty from “walking the book”.

Figure 4(b) shows the variance of the income at time $T$ as a function of $\phi$. The contributions to $\nabla(I_T)$ from $W^b$ (equation (32)) and $W^S$ (equation (16)) decrease monotonically as $\phi$ is increased. This is because, as $\phi$ is increased, the agent trades faster at the start of the trading window. As a result, it is likely that most of the trading is done before $W^b$ and $W^S$ deviate significantly from their initial values. After a small initial decrease at small values of $\phi$, the contribution from $W^k$ (equation (24)) increases as $\phi$ is increased. From equation (13) it can be seen that $I$ depends on $v^2k$. Therefore, large values of $v$, which is the case for large $\phi$, result in a large sensitivity to $k$. For the parameter values chosen here, $\nabla(I_T)$ has a minimum at $\phi = 0.01$. In the range $0 \leq \phi \leq 0.01$ an agent can adjust $\phi$ to trade $\nabla(I_T)$ against $E(I_T)$. In this way the agent can control $\lambda$. The region $\phi > 0.01$ is of no practical interest because in this region the risk increases while the expected income decreases.

8 Conclusions

The effect of introducing randomness into the values of $k$ and $b$ was investigated for an agent following the Almgren-Chriss strategy. It was found that the expected income achieved was unchanged compared to the case that $k$ and $b$ are constant. However, the variance of the income increases when $k$ and $b$ are stochastic. For the case that the Brownian motions driving $k$ and $b$ are independent of each other and of the Brownian motion driving the stock price, the total variance of the Almgren-Chriss strategy can be found by summing the contributions from these three sources of uncertainty. When this is not the case cross terms must be considered.

The effect of the impatience parameter, $\phi$, was examined for the case that $k$ and $b$ are stochastic. The contributions to the variance of the Almgren-Chriss strategy from uncertainty in $S$ and $b$ decrease monotonically with $\phi$. In contrast, the contribution from $k$ increases with $\phi$ for large $\phi$. Consequently, for the parameter values used here, $\nabla(I_T)$ has a minimum at $\phi = 0.01$. Below this value a risk averse agent can trade expected income against variance by adjusting $\phi$. The region $\phi > 0.01$ would not be used in practice because the risk increases while the expected income decreases.
(a) Expected income as a function of $\phi$. Equation (16) and Monte Carlo estimation.

(b) Variance of income as a function of $\phi$. Equations (16), (24) and (32) show the contributions from $W^s$, $W^k$ and $W^b$, respectively.

Figure 4: Expectation and variance of income achieved following the Almgren-Chriss strategy as a function of the impatience parameter $\zeta$. 
References


Appendices

A Variance due to Stochastic $b$

Equation (29) can be used to calculate $\mathbb{V}(I_t^b)$ by applying Itô’s Isometry. This yields:

$$
\mathbb{V}(I_t^b) = \sigma_b^2 \left[ \frac{\gamma N}{\zeta \epsilon^T - e^{-\gamma T}} \right]^4 \left[ (E^2 + G^2 e^{4\gamma(T-t)} + K^2 e^{-4\gamma(T-t)}) \left(1 - e^{-2b\theta_b t}\right) \\
- \frac{D^2}{4\gamma} (e^{\gamma(T-t)} - e^{\gamma T}) + \frac{F^2}{4\gamma} (e^{-4\gamma(T-t)} - e^{-4\gamma T}) - \frac{H^2}{2\gamma} (e^{-4\gamma(T-t)} - e^{2\gamma(2T-t)}) \\
+ \frac{I^2}{2\gamma} (1 - e^{-2\gamma t}) - \frac{J^2}{2\gamma} (1 - e^{2\gamma t}) + \frac{L^2}{2\gamma} (e^{-4\gamma(T-t)} - e^{-2\gamma(2T-t)}) \\
+ 2 \left( \frac{DE}{\theta_b - 2\gamma} (e^{2\gamma(T-t)} - e^{2\gamma(T-\theta_b t)}) + (DF + HI e^{2\gamma(T-t)} + HL + IJ + JLe^{-2\gamma(T-t)})) \\
+ \frac{DG}{\theta_b - 2\gamma} (e^{2\gamma(T-t)} - e^{2\gamma(2T-t)-\theta_b t}) - \frac{DH}{3\gamma} (e^{2\gamma(T-t)} - e^{\gamma(4T-t)}) \\
- \frac{DI}{\gamma} (e^{2\gamma(T-t)} - e^{\gamma(2T-t)}) - \frac{DJ}{3\gamma} (e^{2\gamma(T-t)} - e^{\gamma(2T+t)}) \\
+ \frac{DK}{\theta_b - 2\gamma} (1 - e^{t(2\gamma-\theta_b)}) - \frac{DL}{\gamma} (1 - e^{-t}) \\
+ \frac{EF}{2\gamma + \theta_b} (e^{-2\gamma(T-t)} - e^{-2\gamma(2T-t)-\theta_b t}) + \frac{EG}{2\theta_b} e^{-2\gamma(T-t)}(1 - e^{-2b\theta_b t}) \\
+ \frac{EH}{\theta_b - \gamma} (e^{2\gamma(T-t)} - e^{\gamma(2T-t)-\theta_b t}) + \frac{EI}{\gamma + \theta_b} (1 - e^{-(\theta_b+\gamma)t}) + \frac{EJ}{\theta_b - \gamma} (1 - e^{t(\gamma-\theta_b)}) \\
+ \frac{EK}{2\theta_b} e^{-\gamma(T-t)}(1 - e^{-2b\theta_b t}) + \frac{EL}{\gamma + \theta_b} (e^{-2\gamma(T-t)} - e^{-\gamma(2T-t)-\theta_b t}) \\
+ \frac{FG}{2\gamma + \theta_b} (1 - e^{-t(2\gamma+\theta_b)}) + \frac{FH}{\gamma} (1 - e^{-t}) \\
+ \frac{FI}{3\gamma} (e^{-2\gamma(T-t)} - e^{-\gamma(2T+t)}) + \frac{FJ}{\gamma} (e^{-2\gamma(T-t)} - e^{\gamma(t-2T)}) \\
+ \frac{FK}{2\gamma + \theta_b} (e^{-4\gamma(T-t)} - e^{-2\gamma(2T-t)-\theta_b t}) + \frac{FL}{\gamma} (e^{-4\gamma(T-t)} - e^{-\gamma(4T-t)}) \\
+ \frac{GH}{\theta_b - \gamma} (e^{4\gamma(T-t)} - e^{\gamma(4T-3\gamma)-\theta_b t}) + \frac{GI}{\gamma + \theta_b} (e^{2\gamma(T-t)} - e^{\gamma(2T-3\gamma)-\theta_b t}) \\
+ \frac{GJ}{\theta_b - \gamma} (e^{2\gamma(T-t)} - e^{\gamma(2T-t)-\theta_b t}) + \frac{HK}{\theta_b - \gamma} (1 - e^{t(\gamma-\theta_b)}) + \frac{IK}{\gamma + \theta_b} (e^{-2\gamma(T-t)} - e^{-\gamma(2T-t)-\theta_b t}) \\
- \frac{HJ}{2\gamma} (e^{2\gamma(T-t)} - e^{2\gamma T}) + \frac{HK}{\theta_b - \gamma} (1 - e^{t(\gamma-\theta_b)}) + \frac{IK}{\gamma + \theta_b} (e^{-2\gamma(T-t)} - e^{-\gamma(2T-t)-\theta_b t}) \\
+ \frac{IL}{2\gamma} (e^{-2\gamma(T-t)} - e^{-2\gamma T}) + \frac{JK}{\theta_b - \gamma} (e^{-2\gamma(T-t)} - e^{-\gamma(2T-3\gamma)-\theta_b t}) \\
+ \frac{KL}{\theta_b + \gamma} (e^{-4\gamma(T-t)} - e^{-\gamma(4T-3\gamma)-\theta_b t}) \right] 
$$

(32)
where

\[ D = \frac{\zeta^2}{\gamma + \theta_b} \left( \frac{1}{\gamma} - \frac{1}{(2\gamma + \theta_b)} \right) \]

\[ E = \frac{\zeta}{\theta_b} \left( \frac{1}{\gamma + \theta_b} - \frac{1}{\gamma - \theta_b} \right) \]

\[ F = \frac{1}{\gamma - \theta_b} \left( \frac{1}{\gamma} - \frac{1}{2\gamma - \theta_b} \right) \]

\[ G = \frac{\zeta^2}{(\gamma + \theta_b)(2\gamma + \theta_b)} \]

\[ H = -\frac{\zeta^2}{(\gamma + \theta_b)\gamma} \]

\[ I = \frac{\zeta}{\gamma(\gamma - \theta_b)} \]

\[ J = \frac{\zeta}{\gamma(\gamma + \theta_b)} \]

\[ K = \frac{1}{(\gamma - \theta_b)(2\gamma - \theta_b)} \]

\[ L = \frac{1}{\gamma(\theta_b - \gamma)} \]
B Variance due to Correlation between $b$ and $k$

Equation (30) can be used to calculate $\mathbb{V}(I_t^{bk})$.

\[
\mathbb{V}(I_t^{bk}) = 2^b \sigma_b \sigma_k \left( \frac{\gamma N}{\zeta e^{\gamma T} - e^{-\gamma T}} \right)^4 \left( \frac{AD}{\theta_k - 2\gamma} (e^{\gamma(T-t)} - e^{2\gamma(2T-t) - \theta_k t}) \right) \\
+ \left( (AE + CG) e^{2\gamma(T-t)} + AGe^{\gamma(T-t)} + AK + CE + BG + CK e^{-2\gamma(T-t)} \right) \\
+ BK e^{-4\gamma(T-t)} + BKe^{-2\gamma(T-t)} \left( \frac{1 - e^{-t(\theta_k + \theta_b)}}{\theta_k + \theta_b} \right) \\
+ \frac{AF}{2\gamma + \theta_k} (1 - e^{-t(\theta_k + 2\gamma)}) + \frac{AH}{\theta_k - \gamma} \left( e^{2\gamma(T-t)} - e^\gamma(4T-3t) - \theta_k t \right) \\
+ (AI e^{2\gamma(T-t)} + AL + CI + CLe^{-2\gamma(T-t)} - e^{-t(\theta_k + \gamma)} \right) \\
+ (AJ e^{2\gamma(T-t)} + CJ + BJe^{-2\gamma(T-t)} - e^{-t(\theta_k - \gamma)}) \right) \\
\left( \frac{AD e^{2\gamma(T-t)} - e^{\gamma T}}{4\gamma} - \frac{AE e^{2\gamma(T-t)} - e^{2\gamma(2T-t) - \theta_k t}}{\theta_b - 2\gamma} - \frac{AF - AG e^{2\gamma(T-t)} - e^{2\gamma(2T-t) - \theta_b t}}{\theta_k - 2\gamma} \right) \\
+ \frac{AH}{3\gamma} + \frac{AF e^{2\gamma(T-t)} - e^\gamma(2T-t)}{\gamma} + \frac{AJ e^{2\gamma(T-t)} - e^\gamma(2T+t)}{3\gamma} \\
- AK \frac{1 - e^{-t(\theta_k - 2\gamma)}}{\theta_k - 2\gamma} + (AL + CJ) \frac{1 - e^{\gamma t}}{\gamma} + \frac{CD e^{2\gamma(T-t)} - e^{2\gamma T - \theta_k t}}{\theta_k - 2\gamma} \\
+ CF \frac{e^{2\gamma(T-t)} - e^{2\gamma T - \theta_k t}}{\theta_k + 2\gamma} + \frac{CD e^{2\gamma(T-t)} - e^{2\gamma T}}{2\gamma} \\
- C(E + Ge^{2\gamma(T-t)} + Ke^{-2\gamma(T-t)} - e^{-t(\theta_k - 2\gamma)}) \right) \\
+ CH \frac{e^{2\gamma(T-t)} - e^\gamma(2T-t)}{\theta_b - 2\gamma} - (CI + BH) \frac{1 - e^{-t\theta_k}}{\gamma} - \frac{CL e^{2\gamma(T-t)} - e^\gamma(2T-t)}{\gamma} \\
+ BD \frac{1 - e^{(2\gamma - \theta_k)}}{\theta_k - 2\gamma} + BF \frac{e^{2\gamma(T-t)} - e^{2\gamma(2T-t) - \theta_k t}}{\theta_k + 2\gamma} \\
+ BH \frac{1 - e^{-t(\theta_k - \gamma)}}{\theta_k - 2\gamma} + BI e^{2\gamma(T-t)} \frac{1 - e^{-t(\theta_k + \gamma)}}{\theta_k + \gamma} \\
+ BL e^{-4\gamma(T-t)} - e^{-\gamma(4T-3t) - \theta_k t} - \frac{BDt - BL e^{2\gamma(T-t)} - e^{-2\gamma T - \theta_k t}}{\theta_b + 2\gamma} \\
- BF \frac{e^{-4\gamma(T-t)} - e^{-4\gamma T}}{4\gamma} - BG \frac{1 - e^{-t(\theta_b + 2\gamma)}}{\theta_b + 2\gamma} \\
- BI e^{-2\gamma(T-t)} - e^{-\gamma(2T+t)} - BJ e^{2\gamma(T-t)} - e^{-\gamma(2T-t)} \\
- BK e^{-4\gamma(T-t)} - e^{-2\gamma(2T-t) - \theta_k t} - BL e^{-4\gamma(T-t)} - e^{-\gamma(4T-t)} \right)
\] (33)
where

\[ D = \frac{\zeta^2}{\gamma + \theta_b} \left( \frac{1}{\gamma} - \frac{1}{(2\gamma + \theta_b)} \right) \]

\[ E = \frac{\zeta}{\theta_b} \left( \frac{1}{\gamma + \theta_b} - \frac{1}{\gamma - \theta_b} \right) \]

\[ F = \frac{1}{\gamma - \theta_b} \left( \frac{1}{\gamma} - \frac{1}{2\gamma - \theta_b} \right) \]

\[ G = \frac{\zeta^2}{(\gamma + \theta_b)(2\gamma + \theta_b)} \]

\[ H = -\frac{\zeta^2}{(\gamma + \theta_b)\gamma} \]

\[ I = \frac{\zeta}{\gamma(\gamma - \theta_b)} \]

\[ J = \frac{\zeta}{\gamma(\gamma + \theta_b)} \]

\[ K = \frac{1}{(\gamma - \theta_b)(2\gamma - \theta_b)} \]

\[ L = \frac{1}{\gamma(\theta_b - \gamma)} \]

and

\[ A = -\frac{\zeta^2}{2\gamma + \theta_k} \]

\[ B = \frac{1}{2\gamma - \theta_k} \]

\[ C = -\frac{2\zeta}{\theta_k} \]

(34)
clear all
close all

% Variables for MC
MCloops = 10^4;

N = 10000; % Shares to sell
T = 1; % Available time in hours
n = 1000; % timesteps
dt = T/n;
S0 = 100; % Initial price
b_ = zeros(n+1,MCloops); % Permanent price impact (stochastic)
k_ = zeros(n+1,MCloops); % Temporary price impact (stochastic)
S = zeros(n+1,MCloops); % Mid point price
S_ = zeros(n+1,MCloops); % Execution price
X = zeros(n+1,MCloops); % Running cash total
t = linspace(0, T, n+1); % Time in hours
SS = zeros(1, n+1); % Expected mid price
XX = zeros(1, n+1); % Expected cash value
vari = zeros(1, n+1); % Expected variance of cash value
vv = zeros(1, n+1); % Sample variance of cash value
II = zeros(1, n+1); % Sample mean of cash value

% Almgren-Chriss parameters
b = 1e-4; % Permanent price impact
k = 1e-4; % Temporary price impact
phi = 0.001; % Impatience parameter
alpha = 0.1; % Terminal liquidation penalty parameter
sigma = 0.005; % Volatility
sigma_bk = 0.0; % Correlation between k and b

mat = [1, sigma_bk; sigma_bk, 1]; % Covariance matrix
L = chol(mat, 'lower');
WW = sqrt(dt)*randn(2,MCloops*(n+1));
WW = L*WW;
WW = reshape(WW,2,n+1,MCloops);
W = sqrt(dt)*randn(n+1,MCloops);

%Make b stochastic
%Wb = sqrt(dt)*randn(n+1,MCloops);
Wb= reshape(WW(1,:,:),n+1,MCloops);
theta_b = 0.010; %Reversion speed
sigma_b = 1e-6; %Volatility of b

%Make k stochastic
%Wk = sqrt(dt)*randn(n+1,MCloops);
Wk= reshape(WW(2,:,:),n+1,MCloops);
theta_k = 0.10; %Reversion speed
sigma_k = 2e-7; %Volatility of k

gamma = sqrt(phi/k);
psi = alpha - b/2 - sqrt(k*phi);
psi = (psi + 2*sqrt(k*phi))/psi;

%Calculate the AC optimal speed
denom = psi*exp(gamma*T) - exp(-gamma*T);
v = gamma*(T-t);
Q= (psi*exp(v) - exp(-v))/denom*N; %Optimal position
%Stock price following optimal
SS= S0-b*N+(psi*exp(v) - exp(-v))*b*N/denom ;

%Income following optimal
XX = (S0-b*N)*N*((-psi*exp(v) + exp(-v))/denom+1) - ... 
0.5*b*(N/denom)^2*(psi*2*exp(2*v)+exp(-2*v)) - ... 
psi*2*exp(2*gamma*T) - exp(-2*gamma*T)) ... 
- 0.5*gamma*k*(N/denom)^2*(-psi*2*exp(2*v)+exp(-2*v) ...
\[ + 4 \psi t \gamma + \psi^2 \exp(2 \gamma T) - \exp(-2 \gamma T) \];

\[ A = (-\psi \exp(v) + \exp(-v)); \]

\[ \text{vari} 2 = (A \cdot A - 2 \psi \exp(v) + \exp(-v)) \cdot \gamma; \]

\[ \text{var} 2 = \text{var} 2 \cdot (\sigma^2 N/\text{denom})^2; \%\text{Variance following AC} \]

\[ %\text{Variance from k} \]
\[ A = -\psi^2/(2 \gamma + \theta_k); \]
\[ B = -A; \]
\[ C = -2 \psi \theta_k; \]
\[ D = -C; \]
\[ E = 1/(2 \gamma - \theta_k); \]
\[ F = -E; \]

\[ A^2 = A^2/2/\theta_k \exp(4v) \cdot (2 \psi \theta_k - \psi^2 T - t); \]
\[ B^2 = B^2/4/\gamma^2 \exp(4v) - \exp(4 \gamma T); \]
\[ C^2 = C^2/2/\theta_k \cdot (2 \psi \theta_k - t); \]
\[ D^2 = D^2 \cdot t; \]
\[ E^2 = E^2/2/\theta_k \exp(-4v) \cdot (2 \psi \theta_k - t); \]
\[ F^2 = F^2/4/\gamma^2 \exp(-4v) - \exp(4 \gamma T); \]

\[ AB = A \cdot B/(\theta_k - 2 \gamma) \cdot (\exp(4v) - \exp(2 \gamma T - t) - t); \]
\[ AC = A \cdot C/2/\theta_k \exp(2v) \cdot (2 \psi \theta_k - t); \]
\[ AD = A \cdot D/\theta_k \exp(2v) \cdot (2 \psi \theta_k - t); \]
\[ AE = A \cdot E/2/\theta_k \exp(2v) \cdot (2 \psi \theta_k - t); \]
\[ AF = A \cdot F/(2 \gamma + \theta_k) \cdot (2 \psi \theta_k - t); \]

\[ BC = B \cdot C/(\theta_k - 2 \gamma) \cdot (\exp(2v) - \exp(2 \gamma T - t)); \]
\[ BD = B \cdot D/\gamma^2 \exp(2v) - \exp(2 \gamma T); \]
\[ BE = B \cdot E/(\theta_k - 2 \gamma) \cdot (2 \psi \theta_k - t); \]
\[ BF = B \cdot F \cdot t; \]

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\[ CD = C \frac{D}{\theta_k} (1 - \exp(-\theta_k t)) \]
\[ CE = C \frac{E}{2 \theta_k} \exp(-2v) (1 - \exp(-2\theta_k t)) \]
\[ CF = C \frac{F}{(\theta_k + 2\gamma) \exp(-2v)} (1 - \exp(-2\gamma T - \theta_k t)) \]
\[ DE = D \frac{E}{\theta_k} \exp(-2v) (1 - \exp(-\theta_k t)) \]
\[ DF = D \frac{F}{2\gamma} \exp(-2v) (1 - \exp(-2\gamma T)) \]
\[ EF = E \frac{F}{(\theta_k + 2\gamma) \exp(-4v)} (1 - \exp(-2\gamma (2T - t) - \theta_k t)) \]
\[ \text{vari} = A^2 + B^2 + C^2 + D^2 + E^2 + F^2 \]
\[ \text{vari} = \text{vari} + 2(AB + AC + AD + AE + AF + BC + BD + BE + BF + CD + CE + CF + DE + DF + EF) \]
\[ \text{vari} = \text{vari} \sigma_k^2 \left( \frac{\gamma N}{\text{denom}} \right)^4 \]

\[ \% \text{stochastic b} \]
\[ D_\star = -\psi^2 \frac{1}{(\gamma + \theta_b)} (-1/\gamma + 1/(2\gamma + \theta_b)) \]
\[ E_\star = \psi/\theta_b \left( \frac{1}{(\theta_b + \gamma)} - 1/(\gamma - \theta_b) \right) \]
\[ F_\star = 1/(\gamma + \theta_b) (-1/(2\gamma + \theta_b) + 1/\gamma) \]
\[ G = \psi^2 \frac{1}{(\gamma + \theta_b)} (2\gamma + \theta_b) \]
\[ H = -\psi^2 \frac{1}{(\gamma + \theta_b)} \gamma \]
\[ I = \psi/\gamma \left( \frac{1}{(\gamma + \theta_b)} \right) \]
\[ J = \psi/\gamma \left( \frac{1}{(\gamma + \theta_b)} \right) \]
\[ K = 1/(\gamma + \theta_b) (2\gamma + \theta_b) \]
\[ L = -1/(\gamma + \theta_b) \]
\[ D2 = -D_\star^2/4 \gamma \exp(4v) - \exp(4\gamma T) \]
\[ E2 = E_\star^2/2 \theta_b (1 - \exp(-2\theta_b t)) \]
\[ F2 = F_\star^2/4 \gamma \exp(-4v) - \exp(-4\gamma T) \]
G2 = \frac{G_2}{G} / \theta_b / 2 * (\exp(4*v) - \exp(4*v - 2*\theta_b * t)) ;
H2 = \frac{H_2}{H} / \gamma / 2 * (\exp(4*v) - \exp(2*\gamma / (2*T - t))) ;
I2 = 1 / (1 - \exp(-2*\gamma / (2*T - t))) ;
J2 = -J / (J / 2 * \gamma / (1 - \exp(2*\gamma / t))) ;
K2 = K / \theta_b / 2 * (\exp(-4*v) - \exp(4*v - 2*\theta_b * t)) ;
L2 = L / (L / 2 * \gamma / (1 - \exp(2*\gamma / t))) ;

DE = D * E / (\theta_b - 2*\gamma) / (\exp(2*v) - \exp(2*\gamma / (2*T - \theta_b * t))) ;
DF = D * F / t ;
DG = D / G / (\theta_b - 2*\gamma) / (\exp(4*v) - \exp(2*\gamma / (2*T - \theta_b * t))) ;
DH = D / H / (4*v) / (\exp(4*v) - \exp(4*v - 4*v / (2*T - \theta_b * t))) ;
DI = D / I / (2*v) / (\exp(2*v) - \gamma / (2*T - t)) ;
DJ = D / J / (3 / \gamma) / (\exp(2*v) - \gamma / (2*T + t)) ;
DK = D / K / (\theta_b - 2*\gamma) / (1 - \exp(t / (2*\gamma - \theta_b * t))) ;
DL = D / L / (2*T - \theta_b * t) / (1 - \exp(\gamma / (2*T - \theta_b * t))) ;

EF = E / (2*\gamma / (\theta_b - \gamma)) / (\exp(-2*v) - \exp(-2*\gamma / (2*T - \theta_b * t))) ;
EG = E / G / (\theta_b * (\exp(2*v) - \exp(2*v - 2*\theta_b * t))) ;
EH = E / H / (\theta_b - \gamma) / (\exp(2*v) - \exp(\gamma / (2*T - \theta_b * t))) ;
EI = E / I / (\gamma / (\theta_b + \gamma)) / (1 - \exp(\theta_b + \gamma / (2*T - t))) ;
EJ = E / J / (\gamma / (\theta_b + \gamma)) / (1 - \exp(t / (\gamma - \theta_b * t))) ;
EK = E / K / (2*\gamma) / (\exp(-2*v) - \exp(-2*v - 2*\theta_b * t)) ;
EL = E / L / (\gamma / (\theta_b + \gamma)) / (\exp(-2*v) - \exp(-\gamma / (2*T - \theta_b * t))) ;

FG = F / G / (2*\gamma / (\theta_b + \gamma)) / (1 - \exp(-t / (2*\gamma + \theta_b * t))) ;
FH = F / H / (1 - \exp(-\gamma / t)) ;
FI = F / I / (\gamma / 3 / t) / (\exp(-2*v) - \exp(-\gamma / (2*T + t))) ;
FJ = F / J / (\gamma / (t - 2*T)) / (\exp(-2*v) - \exp(\gamma / (t - 2*T))) ;
FK = F / K / (2*\gamma) / (\exp(-4*v) - \exp(-2*\gamma / (2*T - \theta_b * t))) ;
FL = F / L / (\gamma / 3 / t) / (\exp(-4*v) - \exp(-\gamma / (4*T - t))) ;

GH = G / H / (\theta_b - \gamma) / (\exp(4*v) - \exp(\gamma / (4*T - 3*t - \theta_b * t))) ;
GI = G / I / (\gamma / (\theta_b + \gamma)) / (\exp(2*v) - \exp(\gamma / (2*T - 3*t - \theta_b * t))) ;
GJ = G / J / (\theta_b - \gamma) / (\exp(2*v) - \exp(\gamma / (2*T - t - \theta_b * t))) ;
GK = G / K / (2*\gamma) / (1 - \exp(-2*\theta_b * t)) ;

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GL = G*L/(theta_b+gamma)*(1 - exp(-t*(gamma+theta_b)));

HI = H*I*t.*exp(2*v);
HJ = H*J/2/gamma*(exp(2*v) - exp(2*gamma*T));
HK = H*K/(theta_b - gamma)*(1 - exp(t*(gamma-theta_b)));
HL = H*L*t;

IJ = I*J*t;
IK = I*K/(gamma+theta_b)*exp(-2*v) - exp(-gamma*(2*T-t) - theta_b*t);
IL = I*L/2/gamma*(exp(-2*v) - exp(-2*gamma*T));

JK = J*K/(theta_b+gamma)*(exp(-2*v) - exp(-gamma*(2*T-3*t) - theta_b*t));
JL = J*L*exp(-2*v).*t;
KL = K*L/(theta_b+gamma)*(exp(-4*v) - ... exp(-gamma*(4*T-3*t) - theta_b*t));

vari3 = D2+E2+F2+G2+H2+I2+J2+K2+L2;
vari3 = vari3 + 2*(DE+DF+DG+DH+DI+DJ+DK+DL... +EF+EH+EI+J+K+EL... +FG+FH+FI+J+K+FL... +GH+GI+GJ+GL... +HI+HJ+HK+HL... +IJ+IK+IL... +JK+JL... +KL);

vari3 = vari3*sigma_b^2*(gamma*N/denom)^4;

%Variance from correlation
AD = A*D_/(theta_k-2*gamma)*(exp(4*v) - exp(2*gamma*(2*T-t) - theta_k*t));
AE = A*E_/(theta_k+gamma)*(1 - exp(-t*(theta_k+gamma)));
AF = A*F_/(2*gamma+theta_k)*(1 - exp(-t*(theta_k+2*gamma)));
AG = A*G/(theta_k+gamma)*(1 - exp(-t*(theta_k+2*gamma)));
AH = A*H/(theta_k-gamma)*(exp(4*v) - exp(gamma*(4*T-3*t) - theta_k*t));
AI = A*I/(theta_k+gamma)*exp(2*v).*((1 - exp(-t*(theta_k+gamma))));
\(\text{AJ} = A \ast J / (\theta \ast k - \text{gamma}) \ast \exp (2 \ast v) \ast (1 - \exp (-t \ast (\theta \ast k - \text{gamma})))\);
\(\text{AK} = A \ast K / (\theta \ast k + \theta \ast b) \ast (1 - \exp (-t \ast (\theta \ast k + \theta \ast b)))\);
\(\text{AL} = A \ast L / (\theta \ast k + \text{gamma}) \ast (1 - \exp (-t \ast (\theta \ast k + \text{gamma})))\);
\(\text{BD} = B \ast D_1 / 4 \ast \text{gamma} \ast (\exp(4 \ast v) - \exp(4 \ast \text{gamma} \ast T))\);
\(\text{BE} = B \ast E_1 / (-2 \ast \text{gamma} + \theta \ast b) \ast (\exp(2 \ast v) - \exp(2 \ast \text{gamma} \ast T - \theta \ast b \ast t))\);
\(\text{BF} = B \ast F_1 \ast t\);
\(\text{BG} = B \ast G_1 / (\theta \ast b - 2 \ast \text{gamma}) \ast (\exp(4 \ast v) - \exp(2 \ast \text{gamma} \ast (2 \ast T - t - t \ast \theta \ast b)))\);
\(\text{BH} = B \ast H_1 / 3 \ast \text{gamma} \ast (\exp(4 \ast v) - \exp(\text{gamma} \ast (4 \ast T - t)))\);
\(\text{BI} = B \ast I_1 / \text{gamma} \ast (\exp(2 \ast v) - \exp(\text{gamma} \ast (2 \ast T - t)))\);
\(\text{BJ} = B \ast J_1 / 3 \ast \text{gamma} \ast (\exp(2 \ast v) - \exp(\text{gamma} \ast (2 \ast T + t)))\);
\(\text{BK} = B \ast K_1 / (\theta \ast b - 2 \ast \text{gamma}) \ast (1 - \exp(-t \ast (\theta \ast b - 2 \ast \text{gamma})))\);
\(\text{BL} = B \ast L_1 / \text{gamma} \ast (1 - \exp(t \ast \text{gamma}))\);
\(\text{CD} = C \ast D_2 / (\theta \ast k - 2 \ast \text{gamma}) \ast (\exp(2 \ast v) - \exp(2 \ast \text{gamma} \ast T - \theta \ast k \ast t))\);
\(\text{CE} = C \ast E_2 / (\theta \ast k + \theta \ast b) \ast (1 - \exp(-t \ast (\theta \ast k + \theta \ast b)))\);
\(\text{CF} = C \ast F_2 / (\theta \ast k + 2 \ast \text{gamma}) \ast (\exp(-2 \ast v) - \exp(-2 \ast \text{gamma} \ast T - \theta \ast k \ast t))\);
\(\text{CG} = C \ast G_2 / (\theta \ast k + \theta \ast b) \ast \exp(2 \ast v) \ast (1 - \exp(-t \ast (\theta \ast k + \theta \ast b)))\);
\(\text{CH} = C \ast H_2 / (\theta \ast k - \text{gamma}) \ast (\exp(2 \ast v) - \exp(\text{gamma} \ast (2 \ast T - t - \theta \ast k \ast t)))\);
\(\text{CI} = C \ast I_2 / (\theta \ast k + \text{gamma}) \ast (1 - \exp(-t \ast (\theta \ast k + \text{gamma})))\);
\(\text{CJ} = C \ast J_2 / (\theta \ast k - \text{gamma}) \ast (1 - \exp(-t \ast (\theta \ast k - \text{gamma})))\);
\(\text{CK} = C \ast K_2 / (\theta \ast k + \theta \ast b) \ast \exp(-2 \ast v) \ast (1 - \exp(-t \ast (\theta \ast k + \theta \ast b)))\);
\(\text{CL} = C \ast L_2 / (\theta \ast k + \text{gamma}) \ast (\exp(-2 \ast v) - \exp(-\text{gamma} \ast (2 \ast T - t - \theta \ast k \ast t)))\);
\(\text{DD} = D \ast D_3 / 2 \ast \text{gamma} \ast (\exp(2 \ast v) - \exp(2 \ast T \ast \text{gamma}))\);
\(\text{DE} = D \ast E_3 / \theta \ast b \ast (1 - \exp(-t \ast \theta \ast b))\);
\(\text{DF} = D \ast F_3 / 2 \ast \text{gamma} \ast (\exp(-2 \ast v) - \exp(-2 \ast T \ast \text{gamma}))\);
\(\text{DG} = D \ast G_3 / \theta \ast b \ast \exp(2 \ast v) \ast (1 - \exp(-\theta \ast b \ast t))\);
\(\text{DH} = D \ast H_3 / \text{gamma} \ast (\exp(2 \ast v) - \exp(\text{gamma} \ast (2 \ast T - t)))\);
\(\text{DI} = D \ast I_3 / \text{gamma} \ast (1 - \exp(-\text{gamma} \ast t))\);
\(\text{DJ} = D \ast J_3 / \text{gamma} \ast (1 - \exp(\text{gamma} \ast t))\);
\(\text{DK} = D \ast K_3 / \theta \ast b \ast \exp(-2 \ast v) \ast (1 - \exp(-\theta \ast b \ast t))\);
\(\text{DL} = D \ast L_3 / \text{gamma} \ast (\exp(-2 \ast v) - \exp(-\text{gamma} \ast (2 \ast T - t)))\);
\(\text{ED} = E \ast D_4 / (\theta \ast k - 2 \ast \text{gamma}) \ast (1 - \exp(t \ast (2 \ast \text{gamma} - \theta \ast k)))\);
\(\text{EE} = E \ast E_4 / (\theta \ast k + \theta \ast b) \ast \exp(-2 \ast v) \ast (1 - \exp(-t \ast (\theta \ast k + \theta \ast b)))\);
EF = E * F / (2 * gamma + theta_k) * (exp(-4 * v) - exp(-2 * gamma * (2 * T - t) - theta_k * t));
EG = E * G / (theta_k + theta_b) * (1 - exp(-t * (theta_k + theta_b)));
EH = E * H / (theta_k - gamma) * (1 - exp(-t * (theta_k - gamma)));
EI = E * I / (theta_k + gamma) * (exp(-4 * v) - exp(-gamma * (4 * T - 3 * t) - theta_k * t));

FD = F * D * t;
FE = F * E / (theta_b + 2 * gamma) * (exp(-2 * v) - exp(-2 * gamma * T - theta_b * t));
FF = F * F / 4 / gamma * (exp(-4 * v) - exp(-4 * gamma * T));
FG = F * G / (theta_b + 2 * gamma) * (1 - exp(-t * (theta_b + 2 * gamma)));
FH = F * H / gamma * (1 - exp(-gamma * t));
FI = F * I / 3 / gamma * (exp(-2 * v) - exp(-gamma * (2 * T + t)));
FJ = F * J / gamma * (exp(-2 * v) - exp(-gamma * (2 * T - t)));
FK = F * K / (theta_b + 2 * gamma) * (exp(-4 * v) - exp(-2 * gamma * (2 * T - t) - theta_b * t));
FL = F * L / 3 / gamma * (exp(-4 * v) - exp(-gamma * (4 * T - t)));

vari4 = ...
+ AD + AE + AF + AG + AH + AI + AJ + AK + AL ...
+ BD + BE + BF + BG + BH + BI + BJ + BK + BL ...
+ CD + CE + CF + CG + CH + CI + CJ + CK + CL ...
+ DD + DE + DF + DG + DH + DI + DJ + DK + DL ...
+ ED + EE + EF + EG + EH + EI + EJ + EK + EL ...
+ FD + FE + FF + FG + FH + FI + FJ + FK + FL;

vari4 = vari4 * 2 * sigma_bk * sigma_b * sigma_k * (gamma * N / denom) ^ 4;

% Speed from AC
v = (psi * exp(v) + exp(-v)) / denom * N * gamma;

% Estimate the execution cost from AC
S(1,:) = S0;
b.(1,:) = b;
k.(1,:) = k;
for i = 1:n
\[ b_{\_}(i+1,\_)=b_{\_}(i,\_)+\text{theta} \_b * (b_{\_}(i,\_)-b_{\_}(i,\_)) * dt+\text{sigma} \_b * W_{b}(i,\_); \]

\[ k_{\_}(i+1,\_)=k_{\_}(i,\_)+\text{theta} \_k * (k_{\_}(i,\_)-k_{\_}(i,\_)) * dt+\text{sigma} \_k * W_{k}(i,\_); \]

\[ S(i+1,\_)=S(i,\_)-0.5 * (v(i)+v(i+1)) * b_{\_}(i,\_)* dt+\text{sigma} * W_{s}(i,\_); \]

\[ S_{\_}(i+1,\_)=S(i+1,\_)-k_{\_}(i,\_)*0.5 * (v(i)+v(i+1)); \]

\[ X(i+1,\_)=X(i,\_)+S_{\_}(i+1,\_)*v(i)*dt; \]

\text{end}

\text{income = X;}
\text{for } i=1:n
\text{vv (i) = var (income(i,\_));}
\text{II (i) = mean (income(i,\_));}
\text{end}