Constructing smooth forward curves in electricity markets

Abstract

The intention of this essay is to give a mathematical framework for the construction of maximum smooth forward curves and its application to real market data. In addition to an introductory presentation of the motivation for the need of smooth forward curves and the typical characteristics of commodity markets, basics of risk-neutral forward and swap price modelling are depicted. After introducing the main assumption of the decomposition of the forward curve into a seasonal and an adjustment function, which is proposed by [Benth2008], we give a maximum smoothness criterion which results into a fourth order polynomial spline. Including various constraints (connectivity of spline function resp. its first two derivatives and the observed market prices) we formulate an unconstrained optimization problem using a Lagrange multiplier. Finally, we apply the proposed framework and algorithm to two examples of real electricity market data.
1 Introduction

Since the liberalisation of the energy markets in the late 90's many energy exchanges have been established. Market participants (e.g. banks, energy companies or local distributors) trade basic products such as swaps, forwards and futures which are written on commodities like energy, gas or oil. Main characteristics of energy markets are a high volatility, seasonality, mean-reversion and a spiky behaviour which occurs due to an imbalance between supply and demand as many commodities have very limited storage possibilities.\(^1\) Another typical characteristic of energy markets is a delivery of the underlying over a period (i.e. average based forward contracts) instead of a delivery on a single date. Due to the large volumes of financial contracts in the OTC market it is necessary to have a structure curve when pricing OTC products. Therefore a smooth forward curve is a nice and essential tool to price electricity futures having any settlement period.

The aim of this work is to present theoretical basics for risk-neutral swap price modelling and the construction of a maximum smooth forward curve. This includes a description of the algorithm and applications to real market electricity data based on the previous described methods and algorithms. This essay is mostly based on the work of [Benth2007] resp. [Benth2008], which is an extension of the first reference.

This essay is structured as follows. After the introductory section we give a short survey of the literature on fitting yield curves to market data in the first part of the second section, including the two main approaches which were commonly used – fitting a parametric function by regression and fitting all observed yield curve points using spline methods. In the second part of this section we introduce a mixture of the aforementioned two methods which is proposed by [Benth2008]. One key assumption is the decomposition of the forward curve into a seasonal and an adjustment function. Finally we introduce mathematical basics for risk-neutral forward and swap price modelling. In the subsequent section we describe the mathematical framework for calculation of a maximum smooth forward curve which is commonly defined in the literature as the minimum mean square value of the second deriva-

\(^1\)According to [EY2003] the volatility of gas prices (50% – 100%) or electricity prices (100% – 500%) is clearly above those of foreign exchange rates (10% – 20%), LIBOR rates (10% – 20%) or indices like the S&P500 (20% – 30%).
tives. The focus of the next part of this section is to introduce the constraints which are naturally assumed in order to obtain maximum smoothest forward curves which match the observed market data. Finally we formulate the constrained optimization problem as an unconstrained problem using a Lagrange multiplier and give therefore an algorithm to application purposes. In the penultimate section we give applications for the previously described mathematical framework resp. the introduced algorithm for two examples in the electricity markets. The first example considers – analogously to [Benth2008] – the Nordic electricity Nord Pool using market data for a particular day. The second example is taken from the ICE market where we calculate smooth forward curves for a time period instead of a single day in order to analyse the evolvement and shape behaviour of the forward curves. The last section finishes with a short summary and gives an outlook on further ideas for the construction of smooth forward curves.

2 Risk-neutral forward and swap price modelling

Before starting with the theoretical framework for risk-neutral forward price modelling and the construction of continuous forward curves, we give a short survey of the literature on fitting yield curves to market data. One of the first works in the field of fitting smooth curves to prices of securities and coupon rates was [MC1971] in the year 1971. According to [Benth2008] the common approaches in the literature can be divided into two classes: The first approach uses a regression to fit a parametric function to the yield curve and the second approach uses splines to fit all observed yield curve points. In this essay we use a mixture of these two approaches which is suggested by [Benth2008]. One of the main assumptions, which will be explained in more detail below is to decompose the forward curve into a seasonal and an adjustment function.

The specification for the construction of a maximum smooth forward curve (see section 3) was first used by [AD1994]. They defined a maximum smoothness criterion of the forward rate for the best fitting of a yield curve. Additionally, a derivation of a closed-form solution for their approach was given. A nice extension including
a correction of the work of [AD1994] is given by [LS2002]. In particular, they show that the smoothest forward curve can be produced by an unconstrained fourth-degree polynomial where the cubic term cannot be excluded.\(^2\)

In contrast to other (commodity) markets, electricity and gas futures are delivered over a time period and therefore will be treated in this work – analogously to Benth [2008] – as swap contracts. Contracts with a fixed maturity time of the underlying commodity will be treated as forwards. Assuming a long position in a forward contract at time \(t\) with a risk-free rate \(r\) and the underlying price dynamics \(S(t)\) we obtain for the delivery time \(\tau\), with \(0 \leq t \leq \tau < \infty\), and the agreed price upon delivery \(f(t, \tau)\) the payment

\[
S(\tau) - f(t, \tau).
\]

As it is costless to enter into such a contract we can extract the forward price from the following equation

\[
e^{-r(\tau-t)} \mathbb{E}_Q[S(\tau) - f(t, \tau)|\mathcal{F}_t] = 0,
\]

where \(Q\) is an equivalent martingale measure\(^3\), \(S\) an integrable random variable with respect to \(Q\) and \(f(\cdot, \tau)\) an adapted process. The fundamental pricing relation between the spot price and the forward price is then given by

\[
f(t, \tau) = \mathbb{E}_Q[S(\tau)|\mathcal{F}_t].
\]

Analogously the payoff off a swap contract at time \(t\) from a continuous flow of electricity is given by

\[
\int_{\tau_1}^{\tau_2} e^{-r(u-t)}(S(u) - F(t, \tau_1, \tau_2)) \, du,
\]

where \(F(t, \tau_1, \tau_2)\) is the electricity futures price at time \(t\) for the delivery period \([\tau_1, \tau_2]\) with \(t \leq \tau_1\).\(^4\) Analogously to (1) the risk-neutral price to enter an electricity futures contract is given by

\[
e^{-rt} \mathbb{E}_Q \left[ \int_{\tau_1}^{\tau_2} e^{-r(u-t)}(S(u) - F(t, \tau_1, \tau_2)) \, du | \mathcal{F}_t \right] = 0.
\]

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\(^2\) [AD1994] have an incorrect result regarding the shape of the fourth-order polynomial which gives a suboptimal solution.

\(^3\) The energy markets are incomplete such that the measure \(Q\) is not unique.

\(^4\) For mathematical convenience we use the integral definition instead of a summation of the payoff which would be actually more correct due to hourly spot prices from the electricity spot market.
For settlement at time $t$ with $F(t, \tau_1, \tau_2)$ being adapted we obtain
\[ F(t, \tau_1, \tau_2) = \mathbb{E}_Q \left[ \int_{\tau_1}^{\tau_2} \frac{r e^{-ru}}{e^{-\tau_1} - e^{-\tau_2}} S(u) \, du \big| \mathcal{F}_t \right]. \tag{5} \]

For settlement at the end of the delivery period $\tau_2$ the payoff is given by
\[ e^{-r\tau_2} \mathbb{E}_Q \left[ \int_{\tau_1}^{\tau_2} (S(u) - F(t, \tau_1, \tau_2)) \, du \big| \mathcal{F}_t \right] = 0, \tag{6} \]
which yields an electricity futures price (the term $F(t, \tau_1, \tau_2)$ is referred as swap price of an electricity or gas contract)
\[ F(t, \tau_1, \tau_2) = \mathbb{E}_Q \left[ \int_{\tau_1}^{\tau_2} \frac{1}{\tau_2 - \tau_1} S(u) \, du \big| \mathcal{F}_t \right]. \tag{7} \]

As a generalisation for different settlement days we introduce a weight function $w$ which is given by
\[ w(u, s, t) = \frac{\hat{w}(u)}{\int_s^t \hat{w}(v) \, dv}, \tag{8} \]
with $0 \leq u \leq s < t$. For $\hat{w}(u) = 1$ resp. $\hat{w}(u) = \exp(-ru)$ we obtain $w(u, s, t) = \frac{1}{t-s}$ resp. $w(u, s, t) = \frac{re^{-ru}}{e^{-ru} - e^{-rs}}$. Therefore the connection between a swap contract and the underlying spot process (which we assume is integrable) can be written in general as
\[ F(t, \tau_1, \tau_2) = \mathbb{E}_Q \left[ \int_{\tau_1}^{\tau_2} w(u, \tau_1, \tau_2) S(u) \, du \big| \mathcal{F}_t \right]. \tag{9} \]

An important result with the right side of equation (9) being integrable is
\[ F(t, \tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} w(u, \tau_1, \tau_2) f(t, u) \, du. \tag{10} \]

This term means that a swap contract is equal to a series of weighted continuous forward contracts. Therefore it is (approximately) sufficient to price a weighted series of forward contracts in order to price a swap contract.

A discrete version of equation (10) which is useful in practice due to discrete settlement times is given by:
\[ F(t, \tau_1, \tau_2) = \sum_{i=1}^{N} w(u_i, \tau_1, \tau_2) f(t, u_i) \Delta_i, \tag{11} \]
with settlement at $N$ points in time $u_1 < u_2 < \ldots < u_N$, $\tau_1 = u_1, \tau_2 = u_N$ and $\Delta_i = u_{i+1} - u_i$. The intention of this essay is the construction of $f(0, u)$ using
market quotes of swap contracts $F(0, \tau_1, \tau_2)$ with different delivery periods. In future sections we will abbreviate $f(0, u)$ with $f(u)$ resp. $F(0, \tau_1, \tau_2)$ with $F(\tau_1, \tau_2)$.

**Continuous seasonal forward curve**

Let $m$ denote the observed number of contracts at a given day, $\tau_s$ denotes the start of the settlement period for the contract with the shortest time to delivery and $\tau_e$ denotes the end of the settlement period for the contract with the longest time to delivery.

The function $f$ which holds the relationship for $m$ contracts in equation (10) resp. (11) will be decomposed into a seasonal term $\Lambda$ and into an adjustment function $\varepsilon$. In mathematical terms the decomposition of the forward price is expressed by the following equation:

$$f(u) = \Lambda(u) + \varepsilon(u), \quad u \in [\tau_s, \tau_e],$$

(12)

where $\Lambda(u)$ and $\varepsilon(u)$ are two continuous functions. According to [Benth2008] the adjustment function $\varepsilon$ which corresponds to the forward curve’s deviation from the seasonality can also be understood as a risk premium which has to be paid due to the lack of the possibility of hedging the forward contract. Therefore the function $\varepsilon$ which is a function of time to delivery can be interpreted as the market price of risk.

On the basis of the information of weather conditions or hydro reservoir fillings the future price information in the short end of the forward curve is well-known to the traders. In contrast, the information in the long end of the curve is difficult to determine as it can be several years ahead where the market’s view is less sensitive to time. Therefore the adjustment function $\varepsilon$ should be time-varying and flat in the long end such that we assume from now on

$$\varepsilon'(\tau_e) = 0.$$ 

(13)

In order to remain the seasonal patterns the smoothness is calculated on the adjustment function $\varepsilon(t)$ and not on the forward curve $f(t)$. For an explanation of the decomposition in equation (12) we refer to [Benth2008] section 7.1.2. In the next section follows a detailed explanation for the algorithm to compute forward curves using a maximum smoothness criterion.
3 Maximum smooth forward curve

Analogously to [AD1994] and [Benth2008] we use the mean square value of the second derivatives which is also common to define smoothness in engineering application to derive the smoothest possible forward curve. We will describe it in a mathematical framework as follows: We denote by $C^2_0([\tau_s, \tau_e])$ (real-valued) functions in $[\tau_s, \tau_e]$ which are twice continuously differentiable and have zero derivative in $\tau_e$. The smoothest possible function on an interval $[\tau_s, \tau_e]$ is defined as the one which minimises the term

$$\int_{\tau_s}^{\tau_e} [\varepsilon''(u)]^2 \, du,$$

over some subclass $C$ with $\varepsilon \in C$ being the minimising function. The smoothest forward curve can be interpreted as the one for which $\varepsilon$ solves the minimisation problem in (14). The subclass $C$ should be chosen that it is possible to take swap price data into account by exact matching of the prices. It can be shown that the maximum smoothness curve is a fourth-order polynomial spline. We do not give a proof for this theorem and refer instead to [AD1994], [Benth2007] and [Lim2002]. The single steps of this proof are straightforward as in the first step the Lagrange method is used to obtain an expression for the solution using partial integration. In the second step it has to be shown that the solution is a polynomial spline of order four.

Smooth forward curve constrained by closing prices

We denote by

$$S_p = \{(\tau_1^b, \tau_1^c), (\tau_2^b, \tau_2^c), \ldots, (\tau_m^b, \tau_m^c)\}$$

(15)
a list of the corresponding start and end dates for $m$ different swap contracts which are observable in the market. We construct another list $T$ which consists of $n$ different dates

$$T = \{\tau_0, \tau_1, \ldots, \tau_n\}.$$  

(16)

The list $T$ includes the start day of the contract with the nearest settlement day and each end date of the $m$ different swap contracts (i.e. especially $\tau_s = \tau_1^b = \tau_0$

\footnote{An alternative measure for the smoothness of the adjustment function $\varepsilon$ could be: $\min \int_{\tau_s}^{\tau_e} \frac{[\varepsilon''(u)]}{[\varepsilon''(u)] + 1}$, see [AD1994], p.55.}
and $\tau_e = \tau_{m}^e = \tau_n$). In order to take overlapping settlement periods into account we add additionally to the $m + 1$ dates respectively the first date of two overlapping periods to $\mathcal{T}$ (for a graphical illustration see Figure 1).

Figure 1: Splitting Overlapping Average-Based Forward Contracts

As previously described the adjustment function $\varepsilon$ consists of fourth-order polynomial splines which are twice continuously differentiable and have a zero derivative in the long end of the curve, i.e. $\varepsilon'(\tau_n) = 0$.

$$\varepsilon(u) = \begin{cases} a_1 u^4 + b_1 u^3 + c_1 u^2 + d_1 u + e_1, & u \in [\tau_0, \tau_1] \\
    a_2 u^4 + b_2 u^3 + c_2 u^2 + d_2 u + e_2, & u \in [\tau_1, \tau_2] \\
    \vdots & \vdots & \vdots \\
    a_n u^4 + b_n u^3 + c_n u^2 + d_n u + e_n, & u \in [\tau_{n-1}, \tau_n] \end{cases}$$

In order to find the parameters $x^T = [a_1 \ b_1 \ c_1 \ d_1 \ e_1 \ a_2 \ b_2 \ c_2 \ d_2 \ e_2 \ \ldots \ a_n \ b_n \ c_n \ d_n \ e_n]$ the following constrained convex programming problem has to be solved

$$\min_x \int_{\tau_0}^{\tau_n} [\varepsilon''(u; x)^2] \, du,$$

subject to the natural constraints in the connectivity of the spline function at the joints (18), continuity of the first-order differential of the spline function (19), continuity of the second-order differential of the spline function (20), for $j = 1, \ldots, n - 1$, boundary condition in $\tau_n$ (21) and fitting of the observed market prices $F_i^C$ (22) for
\( i = 1, ..., m \)
\[
(a_{j+1} - a_j)u_j^4 + (b_{j+1} - b_j)u_j^3 + (c_{j+1} - c_j)u_j^2 + (d_{j+1} - d_j)u_j + e_{j+1} - e_j = 0, \quad (18)
\]
\[
4(a_{j+1} - a_j)u_j^3 + 3(b_{j+1} - b_j)u_j^2 + 2(c_{j+1} - c_j)u_j + d_{j+1} - d_j = 0, \quad (19)
\]
\[
12(a_{j+1} - a_j)u_j^2 + 6(b_{j+1} - b_j)u_j + 2(c_{j+1} - c_j) = 0, \quad (20)
\]
\[
\varepsilon'(u_n, x) = 0, \quad (21)
\]
\[
F_C^i = \int_{\tau_i^a}^{\tau_i^b} w(u, \tau_i^b, \tau_i^e)(\varepsilon(u) + s(u)) \, du. \quad (22)
\]

Analogously to [Benth2008] we use for both applications in the subsequent section as weighting function in (8) \( w(u, \tau_1, \tau_2) = 1/(\tau_2 - \tau_1) \), i.e. \( \hat{w}(u) = 1 \). According to [LS2002], this is for reasonable levels of the interest rate (we assume an interest rate of zero) a very good approximation despite of marked-to-market of the contracts in the delivery period.

We obtain from (17) a minimisation problem with a total of \( 3n + m - 2 \) constraints which can be transformed to a quadratic form by inserting \( \varepsilon''(u) \) into (17) and integrating the expression such that we have:

\[
\min_x \int_{\tau_0}^{\tau_n} [\varepsilon''(u;x)^2] \, du = \min_x \int_{\tau_{i-1}}^{\tau_i} (12a_iu^2 + 6b_iu + 2c_iu)^2 du = \min_x x^T H x,
\]

with \( H = \begin{bmatrix} h_1 & 0 & \cdots & 0 \\ 0 & h_n \end{bmatrix} \), \( h_j = \begin{bmatrix} 144 \Delta_j^5 & 18 \Delta_j^4 & 8 \Delta_j^3 & 0 & 0 \\ 18 \Delta_j^5 & 12 \Delta_j^3 & 6 \Delta_j^2 & 0 & 0 \\ 8 \Delta_j^5 & 6 \Delta_j^2 & 4 \Delta_j^1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \)

and \( \Delta_j^l = \tau_j^l - \tau_j^{l-1} \) for \( l = 1, ..., 5, \)

where the vector \( x \) is \( 5n \times 1 \) dimensional and \( H \) a symmetric matrix of dimension \( 5n \times 5n \). Due to the fact that all constraints in (18) - (22) are linear with respect to \( x \), the constraints can be written in the matrix form \( Ax = b \), where \( A \) has dimension \( (3n + m - 2) \times 5n \) and \( b \) is a \( (3n + m - 2) \) vector. Introducing the corresponding Lagrange multiplier vector \( \lambda = [\lambda_1, \lambda_2, ..., \lambda_{3n+m-2}] \) to the constraints the objective function becomes

\[
\min_{x, \lambda} Z(x, \lambda) = \min_{x, \lambda} x^T H x + \lambda^T (Ax - b).
\]

\( ^6 \)We have \( n - 1 \) constraints for each equation in (18)-(20), one constraint in (21) and finally \( m \) market price constraints in (22).
If \([x^*, \lambda^*]\) is a solution, the following two equations have to be satisfied

\[
\frac{\partial}{\partial x} Z(x, \lambda)|_{x^*, \lambda^*} = 2Hx^* + A^T\lambda^* = 0 \tag{23}
\]

\[
\frac{\partial}{\partial \lambda} Z(x, \lambda)|_{x^*, \lambda^*} = Ax^* - b = 0 \tag{24}
\]

or

\[
\begin{bmatrix} 2H & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ b \end{bmatrix}. \tag{25}
\]

The dimension of the left matrix is \((8n + m - 2) \times (8n + m - 2)\) and the dimension of the solution vector resp. the rightmost vector is \((8n + m - 2)\). In the next section we give two examples for the introduced methodology of the construction of smooth forward curves.

4 Applications

In this section we give two examples for the construction of smooth forward curves using the described algorithm. In the first example we use analogously to [Benth2008] electricity future price data which was collected from Nord Pool\(^7\) for a particular day. In the second example we use data from the ICE (Intercontinental Exchange) to construct forward curves for a particular time period.

4.1 Nord Pool electricity exchange contracts

In this example we construct a smooth yield curve for a particular day, 4th May 2005, using a simple trigonometric seasonality function and a zero seasonality function. An overview of the market data which was observable on this particular day is given in the appendix (see Table 1).\(^8\)

The quarterly listed market quotes for 2006 and 2007 show a strong seasonality which high prices in the winter months and low prices in the summer months. More

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\(^7\)Nord Pool is a Nordic power exchange which offers day-ahead and intraday markets. For a detailed description of the Nord Pool market and its traded commodities resp. contracts we refer to [Benth2008], section 1.

\(^8\)The prices are denoted in Norwegian Krone (NOK) using the mid between ask and bid prices. The quarterly and yearly contracts which are commonly traded in EUR were converted to NOK using the NOK/EUR currency rate on this day, see Benth[2008].
precisely, the first (Q1) and fourth (Q4) quarters have the highest prices, whereby the second (Q2) and third quarters (Q3) show for both years the lowest prices. To take the seasonality into account we use two different specifications for the construction of the smooth forward curves. In the first specification we have a zero seasonality, i.e. \( \Lambda(u) = 0 \) in equation (12). In the second specification we use a simple trigonometric function which is given by\(^9\)

\[ \Lambda(u) = 145.732 + 29.735 \times \cos \left( (u + 6.691) \frac{2\pi}{365} \right). \quad (26) \]

For the first specification we ignore the seasonality resp. set the seasonality equal to zero. This is especially appropriate for commodities which do not have strong seasonality structures like the oil market. Electricity and gas markets commonly show a strong seasonal behaviour which can also be seen at Nord Pool for our used electricity prices. Nevertheless we use a zero seasonality for comparison reasons with the trigonometric seasonal function. The second specification is a trigonometric seasonality function which was proposed by [LS2002] for their one-factor arithmetic model.

In order to construct smooth forward curves we took \( m = 24 \) contracts which have been quoted in the market on this particular day. Using the described construction for overlapping settlement periods we obtain a spline consisting of \( n = 32 \) polynomials. In Figure 2 the forward curves using the two different specifications with the corresponding seasonality function are depicted. The figure shows that the impact of the seasonality function varies depending on the maturity of the forward curve. At the short end and in the middle of the forward curves both specifications produce a nearly identical forward curve. According to [Benth2008] this observation is trivial due to the short term contracts as the marking-to-market constraint dominates the deterministic seasonality part of the forward curve. In the long end of the curve (roughly) between 900 and 1250 days to maturity the two forward curves differ in shape.

Overall, we realize that the choice of the seasonal functions (constant vs. trigonometric) has little influence on the curve shape in the short end resp. in the middle

\(^9\)For the calibration of the parameters we refer to [LS2002] resp. Benth[2008].
of the curve. In the long end of the curve where only a few contracts are available in the market the shapes differs.

![Forward and Electricity Curves](image)

Figure 2: The figure shows the forward curves using zero seasonality (solid blue line) and using the trigonometric function which is given in (26) (solid red line). Additionally to the forward curves the corresponding seasonality functions are illustrated (dashed blue line) resp. (dashed red line).

### 4.2 ICE Electricity contracts

In our second example we construct 40 forward curves (using ICE contracts) for a particular time period. In contrast to the previous example we use only a zero seasonality curve as the trigonometric function which was proposed by [LS2002] was calibrated to market data in the Nord Pool market and therefore cannot be used in this case. The chosen market data was chosen from the ICE for the time period between 4th January 2006 and 28th February 2006. For this time period are 22 future contracts on electricity traded which are split into monthly contracts for the next 12 months, quarterly contracts for the next 6 quarters and semi-annually contracts for the next 4 half-years.\(^\text{10}\)

All forward curves which are depicted in Figure 3 have a similar shape with strong

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\(^{10}\)For the 26th January 2006 are only 16 contracts available which are split into 12 monthly, 3 quarterly and 1 semi-annually contract.
seasonality effects. In the short end of the curves (for January and beginning of February 2006) the forward prices are high (up to 80 €) and decrease until the summer months in 2006. Afterwards the forward curves increase with spiky winter months (December 2006 and January 2007). The similar behaviour can be observed for the subsequent year.

The forward curves which are observed later in the market (e.g. 28th February 2006) have a flatter behaviour in the long end of the forward curve compared with the forward curves at start of the observation (e.g. 4th January 2006).

![Figure 3: Forward curves for different market dates](image)

5 Conclusions and Outlook

In this essay we have introduced an algorithm to construct maximum smooth forward curves from observed market data. We used a mixture (proposed by [Benth2008]) of the two approaches which are commonly used in practice for fitting yield curves – the fitting of a parametric function by regression and fitting of the observed yield curves by spline methods. Apart from basics of risk-neutral forward and swap price modelling we introduced an adjustment function which corresponds to the forward curve’s deviation from the seasonality and can be interpreted as the market price of risk. In the presented approach the forward curve is decomposed into a seasonal and
an adjustment function whereby the maximum smoothing which becomes a fourth order polynomial spline is done on the adjustment function. After incorporating the constraints of the connectivity of the spline function resp. its first two derivatives and of the observed market data constraints we formulate an unconstrained optimization problem using a Lagrange multiplier.

We applied our framework to two examples of electricity market data. In the first example we constructed a smooth forward curve including a trigonometric seasonality function which is proposed by [LS2002] using market data of a particular day. In the second example however we utilise the smoothing process for electricity data from the ICE for a particular time period (with a zero seasonality function). In this example the seasonality effect which is a main characteristic for commodity markets can be clearly seen for the constructed forward curves.

An extension of this work could be to analyse the volatility structure of the forward curves using different seasonal functions. [Benth2008] show that the volatility structure is dependent on the specification of the seasonality function. In this work we compared only two seasonal functions regarding this topic, a trigonometric seasonal function which is proposed by [LS2002] and a zero seasonality function. Another specification for the seasonality function is a spot forecast from a bottom-up model. For an example of forward price curves using a bottom-up model we refer to [FL2002].
References


## Appendix - Market Data from Nord Pool

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Table 1: Market data from Nord Pool, 4th May 2005