Handbook for the Undergraduate Mathematics Courses
Supplement to the Handbook
Honour School of Mathematics & Philosophy
Syllabus and Synopses for Part C 2012–2013
for examination in 2013

Contents

1 Foreword

This Supplement to the Mathematics Course Handbook specifies the Mathematics courses available for Part C in Mathematics & Philosophy in the 2013 examination. It should be read in conjunction with the Handbook for Mathematics & Philosophy for the academic year 2012–2013, to be issued in Michaelmas Term. The Handbook contains in particular information on the format and rubrics for written examination papers in Mathematics, and the classification rules applicable to Part C.

See the current edition of the *Examination Regulations* for the full regulations governing the examinations.

1.1 Part C of the Honour School of Mathematics & Philosophy

The following is reproduced from the *Examination Regulations* applicable to the 2013 examinations.

The examination for Part C shall consist of units in Mathematics and subjects in Philosophy. The schedule of units in *Mathematics* shall be published in a supplement to the Mathematics Course Handbook by the beginning of the Michaelmas Full Term in the academic year of the examination concerned. Each candidate shall offer a total of three units chosen in any combination from the lists for Mathematics and for Philosophy. Units in Mathematics are taken from the schedule of units and half units for Part C as listed below. No unit or half unit in Mathematics may be offered in both Part B and Part C.

Further information on the units in Mathematics is given below.

1.2 "Units" and "half-units" and methods of examination

Most courses in Mathematics are assessed by examination. Most subjects offered have a 'weight' of one unit, and will be examined in a 3-hour examination paper. In many of these subjects it will also be possible to take the first half, or either half, of the subject as a 'half-unit'. Where this is the case, a half-unit will be examined in an examination paper of $1\frac{1}{2}$ hours duration. Each half-unit paper will contain **3** questions.

1.3 The Schedule of Mathematics units and half-units for Mathematics & Philosophy

All units and half-units in the Schedule below are drawn from the list of Mathematics Department units and half-units and "Other" units and half-units available in Mathematics Part C.

In addition you may apply for special approval to be examined in Mathematics Department units and half-units not included in the Schedule; any such subject approved will then be treated as falling under the Schedule. For the procedure for seeking approval, see Subsection 1.4 below.

For the 2013 examination, the Schedule is as follows. (N.B. All topics listed are half-units unless otherwise stated).

Schedule

C1.1a Model Theory

C1.1b Godel's Incompleteness Theorems

C1.2a Analytic Topology

C1.2b Axiomatic Set Theory

C2.1a Lie Algebras

C2.1b Representation Theory of Symmetric Groups

C2.2a Commumtative Algebra

C2.2b Homological Algebra

C2.3b Infinite Groups

C3.1a Algebraic Topology

C3.2b Geometric Group Theory

C3.3b Differentiable Manifolds

C3.4a Algebraic Geometry

C3.4b Lie Groups

C4.1a Funtional Analysis

C4.1b Banach and C*-algebras

C9.1a Modular Forms

C9.1b Elliptic Curves

C10.1a Stochastic Differential Equations

C11.1a Graph Theory

CD Dissertations on a Mathematical Topic (half or whole unit)

CCS1a Categories, Proofs and Processes

CCS3b Quantum Computer Science

CCS4b Automata, Logic and Games

OD Dissertations on a Mathematically related Topic (half or whole unit)

And also

Any other whole unit or half-unit course from the list of Mathematics Department units for which special approval has been granted.

1.4 Procedure for seeking approval of additional options where this is required

You may, if you have the support of your Mathematics tutor, apply to the Chairman of the Joint Committee for Mathematics and Philosophy for approval of one or more other options from the list of Mathematics Department units and half-units for Part C. This list can be found in the Supplement to the Mathematics Course Handbook giving syllabuses and synopses for courses in for Mathematics Part C.

Applications for special approval must be made through the candidate's college and sent to the Chairman of the Joint Committee for Mathematics and Philosophy, c/o Academic Administrator, Mathematical Institute, to arrive by **Friday of Week 5 of Michaelmas Term**. Be sure to consult your college tutors if you are considering asking for approval to offer one of these additional options.

Given that each of these additional options, which are all in applied mathematics, presume facility with some or other results and techniques covered in first or second year Mathematics courses not taken by Mathematics & Philosophy candidates, such applications will be exceptional. You should also be aware that there may be a clash of lectures for specially approved options and those listed in the above Schedule and with lectures in Philosophy; see the section in The Mathematics Part C Supplement on lecture clashes.

1.5 Registration for Part C courses 2012–2013

CLASSES Students will have to register in advance for the classes they wish to take. Students will have to register by Friday of Week 10 of Trinity Term 2012 using the online registration system which can be accessed at https://www.maths.ox.ac.uk/courses/registration/.

2 Schedule

2.1 C1.1a: Model Theory — Prof Zilber — 16MT

Level: M-level **Method of Assessment:** Written examination.

Weight: Half-unit (OSS paper code 2B60).

Recommended Prerequisites

This course presupposes basic knowledge of First Order Predicate Calculus up to and including the Soundness and Completeness Theorems. A familiarity with (at least the statement of) the Compactness Theorem would also be desirable.

Overview

The course deepens a student's understanding of the notion of a mathematical structure and of the logical formalism that underlies every mathematical theory, taking B1 Logic as a starting point. Various examples emphasise the connection between logical notions and practical mathematics.

The concepts of completeness and categoricity will be studied and some more advanced technical notions, up to elements of modern stability theory, will be introduced.

Learning Outcomes

Students will have developed an in depth knowledge of the notion of an algebraic mathematical structure and of its logical theory, taking B1 Logic as a starting point. They will have an understanding of the concepts of completeness and categoricity and more advanced technical notions.

Synopsis

Structures. The first-order language for structures. The Compactness Theorem for first-order logic. Elementary embeddings. Löwenheim–Skolem theorems. Preservation theorems for substructures. Model Completeness. Quantifier elimination.

Categoricity for first-order theories. Types and saturation. Omitting types. The Ryll Nardzewski theorem characterizing aleph-zero categorical theories. Theories with few types. Ultraproducts.

Reading

1. D. Marker, *Model Theory: An Introduction* (Springer, 2002).
2. W. Hodges, *Shorter Model Theory* (Cambridge University Press, 1997).

3. J. Bridge, *Beginning Model Theory* (Oxford University Press, 1977). (Out of print but can be found in libraries.)

Further reading

1. All topics discussed (and much more) can also be found in W. Hodges, *Model Theory* (Cambridge University Press, 1993).

2.2 C1.1b: Gödel's Incompleteness Theorems — Dr Paseau — 16HT

Level: M-level **Method of Assessment:** Written examination.

Weight: Half-unit (OSS paper code 2A60)

Recommended Prerequisites

This course presupposes knowledge of first-order predicate logic up to and including soundness and completeness theorems for a formal system of first-order predicate logic (B1 Logic).

Overview

The starting point is Gödel's mathematical sharpening of Hilbert's insight that manipulating symbols and expressions of a formal language has the same formal character as arithmetical operations on natural numbers. This allows the construction for any consistent formal system containing basic arithmetic of a 'diagonal' sentence in the language of that system which is true but not provable in the system. By further study we are able to establish the intrinsic meaning of such a sentence. These techniques lead to a mathematical theory of formal provability which generalizes the earlier results. We end with results that further sharpen understanding of formal provability.

Learning Outcomes

Understanding of arithmetization of formal syntax and its use to establish incompleteness of formal systems; the meaning of undecidable diagonal sentences; a mathematical theory of formal provability; precise limits to formal provability and ways of knowing that an unprovable sentence is true.

Synopsis

Gödel numbering of a formal language; the diagonal lemma. Expressibility in a formal language. The arithmetical undefinability of truth in arithmetic. Formal systems of arithmetic;

arithmetical proof predicates. Σ_0-completeness and Σ_1-completeness. The arithmetical hierarchy. ω-consistency and 1-consistency; the first Gödel incompleteness theorem. Separability; the Rosser incompleteness theorem. Adequacy conditions for a provability predicate. The second Gödel incompleteness theorem; Löb's theorem. Provable Σ_1-completeness. Provability logic; the fixed point theorem. The ω-rule. The Bernays arithmetized completeness theorem; formally undecidable Δ^0_2-sentences of arithmetic.

Reading

1. Lecture notes for the course.

Further Reading

1. Raymond M. Smullyan, *Gödel's Incompleteness Theorems* (oup, 1992).
2. George S. Boolos and Richard C. Jeffrey, *Computability and Logic* (3rd edition, Cambridge University Press, 1989), Chs 15, 16, 27 (pp 170–190, 268-284).
3. George Boolos, *The Logic of Provability* (Cambridge University Press, 1993), pp. 44–49.

2.3 C1.2a: Analytic Topology — Dr Suabedissen — 16MT

Level: M-level **Method of Assessment:** Written examination.

Weight: Half-unit (OSS paper code 2A61)

We find it necessary to run Analytic Topology in MT. Classes for Analytic Topology will be run both in MT and HT, so that students who find themselves overburdened in MT will have the opportunity to attend classes on Analytic Topology in HT.

Recommended Prerequisites

Part A Topology; a basic knowledge of Set Theory, including cardinal arithmetic, ordinals and the Axiom of Choice, will also be useful.

Overview

The aim of the course is to present a range of major theory and theorems, both important and elegant in themselves and with important applications within topology and to mathematics as a whole. Central to the course is the general theory of compactness and Tychonoff's theorem, one of the most important in all mathematics (with applications across mathematics and in mathematical logic) and computer science.

Synopsis

Bases and initial topologies (including pointwise convergence and the Tychonoff product topology). Separation axioms, continuous functions, Urysohn's lemma. Separable, Lindelöf and second countable spaces. Urysohn's metrization theorem. Filters and ultrafilters. Tychonoff's theorem. Compactifications, in particular the Alexandroff One-Point Compactification and the Stone–Čech Compactification. Connectedness and local connectedness. Components and quasi-components. Totally disconnected compact spaces, Boolean algebras and Stone spaces. Paracompactness (brief treatment).

Reading

1. S. Willard, *General Topology* (Addison–Wesley, 1970), Chs. 1–8.
2. N. Bourbaki, *General Topology* (Springer-Verlag, 1989), Ch. 1.

2.4 C1.2b: Axiomatic Set Theory — Prof. Zilber — 16HT

Level: M-level **Method of Assessment:** Written examination.

Weight: Half-unit (OSS paper code 2B61).

Recommended Prerequisites

This course presupposes basic knowledge of First Order Predicate Calculus up to and including the Soundness and Completeness Theorems, together with a course on basic set theory, including cardinals and ordinals, the Axiom of Choice and the Well Ordering Principle.

Overview

Inner models and consistency proofs lie at the heart of modern Set Theory, historically as well as in terms of importance. In this course we shall introduce the first and most important of inner models, Gödel's constructible universe, and use it to derive some fundamental consistency results.

Synopsis

A review of the axioms of ZF set theory. The recursion theorem for the set of natural numbers and for the class of ordinals. The Cumulative Hierarchy of sets and the consistency of the Axiom of Foundation as an example of the method of inner models. Levy's Reflection Principle. Gödel's inner model of constructible sets and the consistency of the Axiom of Constructibility ($V = L$). The fact that $V = L$ implies the Axiom of Choice. Some advanced cardinal arithmetic. The fact that $V = L$ implies the Generalized Continuum Hypothesis.

Reading

For the review of ZF set theory:

1. D. Goldrei, *Classic Set Theory* (Chapman and Hall, 1996).

For course topics (and much more):

1. K. Kunen, *Set Theory: An Introduction to Independence Proofs* (North Holland, 1983) (now in paperback). Review: Chapter 1. Course topics: Chapters 3, 4, 5, 6 (excluding section 5).

Further Reading

1. K. Hrbacek and T. Jech, *Introduction to Set Theory* (3rd edition, M Dekker, 1999).

2.5 C2.1a: Lie Algebras — Dr McGerty — 16MT

Level: M-level **Method of Assessment:** Written examination.

Weight: Half-unit (OSS paper code 2A62)

Recommended Prerequisites

Part B course B2a. A thorough knowledge of linear algebra and the second year algebra courses; in particular familiarity with group actions, quotient rings and vector spaces, isomorphism theorems and inner product spaces will be assumed. Some familiarity with the Jordan–Hölder theorem and the general ideas of representation theory will be an advantage.

Overview

Lie Algebras are mathematical objects which, besides being of interest in their own right, elucidate problems in several areas in mathematics. The classification of the finite-dimensional complex Lie algebras is a beautiful piece of applied linear algebra. The aims of this course are to introduce Lie algebras, develop some of the techniques for studying them, and describe parts of the classification mentioned above, especially the parts concerning root systems and Dynkin diagrams.

Learning Outcomes

Students will learn how to utilise various techniques for working with Lie algebras, and they will gain an understanding of parts of a major classification result.

Synopsis

Definition of Lie algebras, small-dimensional examples, some classical groups and their Lie algebras (treated informally). Ideals, subalgebras, homomorphisms, modules.

Nilpotent algebras, Engel's theorem; soluble algebras, Lie's theorem. Semisimple algebras and Killing form, Cartan's criteria for solubility and semisimplicity.

The root space decomposition of a Lie algebra; root systems, Cartan matrices and Dynkin diagrams. Classification of irreducible root systems. Description (with few proofs) of the classification of complex simple Lie algebras; examples.

Reading

1. J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory,* Graduate Texts in Mathematics 9 (Springer-Verlag, 1972, reprinted 1997). Chapters 1–3 are relevant and part of the course will follow Chapter 3 closely.
2. B. Hall, *Lie Groups, Lie Algebras, and Representations. An Elementary Introduction,* Graduate Texts in Mathematics 222 (Springer-Verlag, 2003).
3. K. Erdmann, M. J. Wildon, *Introduction to Lie Algebras* (Springer-Verlag, 2006), ISBN: 1846280400.

Additional Reading

1. J.-P. Serre, *Complex Semisimple Lie Algebras* (Springer, 1987). Rather condensed, assumes the basic results. Very elegant proofs.
2. N. Bourbaki, *Lie Algebras and Lie Groups* (Masson, 1982). Chapters 1 and 4–6 are relevant; this text fills in some of the gaps in Serre's text.

2.6 C2.1b: Representation Theory of Symmetric Groups — Prof. Henke —16HT

Level: M-level **Method of Assessment:** Written examination.

Weight: Half-unit (OSS paper code 2B62).

Recommended Prerequisites

A thorough knowledge of linear algebra and the second year algebra courses; in particular familiarity with the symmetric groups, (symmetric) group actions, quotient vector spaces, isomorphism theorems and inner product spaces will be assumed. Some familiarity with basic representation theory from B2 (group algebras, simple modules, reducibility, Maschke's theorem, Wedderburn's theorem, characters) will be an advantage.

Overview

The representation theory of symmetric groups is a special case of the representation theory of finite groups. Whilst the theory over characteristic zero is well understood, this is not so over fields of prime characteristic. The course will be algebraic and combinatorial in flavour, and it will follow the approach taken by G. James. One main aim is to construct and parametrise the simple modules of the symmetric groups over an arbitrary field. Combinatorial highlights include combinatorial algorithms such as the Robinson–Schensted–Knuth correspondence. The final part of the course will discuss some finite-dimensional representations of the general linear group $\mathrm{GL}_n(\mathbb{C})$, and connections with representations of symmetric groups. In particular we introduce tensor products, and symmetric and exterior powers.

Synopsis

Counting standard tableaux of fixed shape: Young diagrams and tableaux, standard-tableaux, Young–Frobenius formula, hook formula. Robinson–Schensted-Knuth algorithm and correspondence.

Construction of fundamental modules for symmetric groups: Action of symmetric groups on tableaux, tabloids and polytabloids; permutation modules on cosets of Young subgroups. Specht modules, and their standard bases. Examples and applications.

Simplicity of Specht modules in characteristic zero and classification of simple S_n-module over characteristic zero. Characters of symmetric groups, Murnaghan–Nakayama rule.

Submodule Theorem, construction of simple S_n-modules over a field of prime characteristic. Decomposition matrices. Examples and applications.

Some finite-dimensional $\mathrm{GL}_n(\mathbb{C})$-modules, in particular the natural module, its tensor powers, and its symmetric and exterior powers. Connections with representations of S_n over $\mathbb{C}$.

Reading

1. W. Fulton, *Young Tableaux,* London Mathematical Society Student Texts 35 (Cambridge University Press, 1997). From Part I and II.

2. D. Knuth, *The Art of Computer Programming, Volume 3* (Addison–Wesley, 1998). From Chapter 5.

3. B. E. Sagan, *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions,* Graduate Texts in Mathematics 203 (Springer–Verlag, 2000). Chapters 1 – 2.

Additional Reading

1. W. Fulton, J. Harris, *Representation Theory: A first course,* Graduate Texts in Mathematics, Readings in Mathematics 129 (Springer–Verlag, 1991). From Part I.

2. G. James, *The Representation Theory of the Symmetric Groups,* Lecture Notes in Mathematics 682 (Springer–Verlag, 1978).

3. G. James, A. Kerber, *The Representation Theory of the Symmetric Groups*, Encyclopaedia of Mathematics and its Applications 16, (Addison–Wesley, 1981). From Chapter 7.

4. R. Stanley, *Enumerative Combinatorics. Volume 2*, Cambridge Studies in Advanced Mathematics 62 (Cambridge University Press, 1999).

2.7 C2.2a: Commutative Algebra — Prof Segal — 16MT

Level: M-level **Method of Assessment:** Written examination.

Weight: Half-unit (OSS paper code tbc).

Recommended Prerequisites

A thorough knowledge of the second-year algebra courses, in particular rings, ideals and fields.

Overview

Amongst the most familiar objects in mathematics are the ring of integers and the polynomial rings over fields. These play a fundamental role in number theory and in algebraic geometry, respectively. The course explores the basic properties of such rings, and introduces the key concept of a module, which generalizes both abelian groups and the idea of a linear transformation on a vector space.

Synopsis

Introduction to modules. The structure of modules over a principal ideal ring. Prime ideals, maximal ideals, nilradical and Jacobson radical. Noetherian rings; Hilbert basis theorem. Minimal primes. Artin-Rees Lemma; Krull intersection theorem. Integral extensions. Prime ideals in integral extensions. Noether Normalization Lemma. Hilbert Nullstellensatz, maximal ideals. Krull dimension; Principal ideal theorem.; dimension of an affine algebra.

Reading

1. Atiyah, Macdonald, *Introduction to Commutative Algebra*, (Addison-Wesley, 1969).

2. Eisenbud *Commutative Algebra: with a View Toward Algebraic Geometry*, Grad. Texts Math. 150, (Springer-Verlag, 1995) Chapters 4, 5 and 13.

2.8 C2.2b: Homological Algebra — Dr Kremnizer — 16HT

Level: M-level **Method of Assessment:** Written examination.

Weight: Half-unit (OSS paper code tbc).

Synopsis

Chain complexes: complexes of R-modules, operations on chain complexes, long exact sequences, chain homotopies, mapping cones and cylinders (3 hours) Derived functors: delta functors, projective and injective resolutions, left and right derived functors (4 hours) Tor and Ext: Tor and flatness, Ext and extensions, universal coefficients theorems (3 hours) Group homology and cohomology: definition, interpretation of H^1 and H^2, universal central extensions, the Bar resolution (3 hours) Lie algebra homology and cohomology: Lie algebras and universal enveloping algebras, definition of homology and cohomology and relations to Tor and Ext, H^1 and H^2, universal central extensions (3 hours)

Reading

Weibel, Charles *An introduction to Homological algebra*
`http://books.google.co.uk/books/about/An_introduction_to_homological_algebra.html?id=flm-dBXfZ_gC\&redir_esc=y`

2.9 C2.3b: Infinite Groups — Dr Nikolov — 16HT

Level: M-level **Method of Assessment:** Written examination.

Weight: Half-unit (OSS paper code tbc).

Recommended Prerequisites

A thorough knowledge of the second-year algebra courses; in particular, familiarity with group actions, quotient rings and quotient groups, and isomorphism theorems will be assumed. Familiarity with the Commutative Algebra course will be helpful but not essential.

Overview

The concept of a group is so general that anything which is true of all groups tends to be rather trivial. In contrast, groups that arise in some specific context often have a rich and beautiful theory. The course introduces some natural families of groups, various questions that one can ask about them, and various methods used to answer these questions; these involve among other things rings and trees.

Synopsis

Free groups and their subgroups; finitely generated groups: counting finite-index subgroups; finite presentations and decision problems; Linear groups: residual finiteness; structure of soluble linear groups; Nilpotency and solubility: lower central series and derived series; structural and residual properties of finitely generated nilpotent groups and polycyclic groups; characterization of polycyclic groups as soluble Z-linear groups; Finitely generated groups acting on rooted trees: Gupta-Sidki groups and the General Burnside Problem.

Reading

1. D. J. S. Robinson, *A course in the theory of groups*, 2nd ed., Graduate texts in Mathematics, (Springer-Verlag, 1995). Chapters 2, 5, 6, 15.

2. D. Segal, *Polycyclic groups*, (CUP, 2005) Chapters 1 and 2.

2.10 C3.1a: Algebraic Topology — Prof. Tillmann — 16MT

Level: M-level. **Method of Assessment:** Written examination.

Weight: Half-unit (OSS paper codes tbc).

Recommended Prerequisites

Helpful but not essential: Part A Topology, B3.1a Topology and Groups.

Overview

Homology theory is a subject that pervades much of modern mathematics. Its basic ideas are used in nearly every branch, pure and applied. In this course, the homology groups of topological spaces are studied. These powerful invariants have many attractive applications. For example we will prove that the dimension of a vector space is a topological invariant and the fact that 'a hairy ball cannot be combed'.

Learning Outcomes

At the end of the course, students are expected to understand the basic algebraic and geometric ideas that underpin homology and cohomology theory. These include the cup product and Poincaré Duality for manifolds. They should be able to choose between the different homology theories and to use calculational tools such as the Mayer-Vietoris sequence to compute the homology and cohomology of simple examples, including projective spaces, surfaces, certain simplicial spaces and cell complexes. At the end of the course, students should also have developed a sense of how the ideas of homology and cohomology may be applied to problems from other branches of mathematics.

Synopsis

Chain complexes of free Abelian groups and their homology. Short exact sequences. Delta (and simplicial) complexes and their homology. Euler characteristic.

Singular homology of topological spaces. Relative homology and the Five Lemma. Homotopy invariance and excision (details of proofs not examinable). Mayer-Vietoris Sequence. Equivalence of simplicial and singular homology.

Degree of a self-map of a sphere. Cell complexes and cellular homology. Application: the hairy ball theorem.

Cohomology of spaces and the Universal Coefficient Theorem (proof not examinable). Cup products. Künneth Theorem (without proof). Topological manifolds and orientability. The fundamental class of an orientable, closed manifold and the degree of a map between manifolds of the same dimension. Poincaré Duality (without proof).

Reading

1. A. Hatcher, *Algebraic Topology* (Cambridge University Press, 2001). Chapters 3 and 4.
2. G. Bredon, *Topology and Geometry* (Springer, 1997). Chapters 4 and 5.
3. J. Vick, *Homology Theory,* Graduate Texts in Mathematics 145 (Springer, 1973).

2.11 C3.2b: Geometric Group Theory — Dr Papazoglou — 16HT

Level: M-level. **Method of Assessment:** Written examination.

Weight: Half-unit (OSS paper code tbc).

Recommended Prerequisites.

B3.1a Topology & Groups course is a helpful, though not essential, prerequisite.

Overview.

The aim of this course is to introduce the fundamental methods and problems of geometric group theory and discuss their relationship to topology and geometry.

The first part of the course begins with an introduction to presentations and the list of problems of M. Dehn. It continues with the theory of group actions on trees and the structural study of fundamental groups of graphs of groups.

The second part of the course focuses on modern geometric techniques and it provides an introduction to the theory of Gromov hyperbolic groups.

Synopsis.

Free groups. Group presentations. Dehn's problems. Residually finite groups.

Group actions on trees. Amalgams, HNN-extensions, graphs of groups, subgroup theorems for groups acting on trees.

Quasi-isometries. Hyperbolic groups. Solution of the word and conjugacy problem for hyperbolic groups.

If time allows: Small Cancellation Groups, Stallings Theorem, Boundaries.

Reading.

1. J.P. Serre, *Trees* (Springer Verlag 1978).

2. M. Bridson, A. Haefliger, *Metric Spaces of Non-positive Curvature, Part III* (Springer, 1999), Chapters I.8, III.H.1, III. *Gamma* 5.

3. H. Short *et al.*, 'Notes on word hyperbolic groups', *Group Theory from a Geometrical Viewpoint, Proc. ICTP Trieste* (eds E. Ghys, A. Haefliger, A. Verjovsky, World Scientific 1990)

   available online at: http://www.cmi.univ-mrs.fr/ hamish/

4. C.F. Miller, *Combinatorial Group Theory*, notes:
   http://www.ms.unimelb.edu.au/ cfm/notes/cgt-notes.pdf.

Additional Reading.

1. G. Baumslag, *Topics in Combinatorial Group Theory* (Birkhauser, 1993).

2. O. Bogopolski, *Introduction to Group Theory* (EMS Textbooks in Mathematics, 2008).

3. R. Lyndon, P. Schupp, *Combinatorial Group Theory* (Springer, 2001).

4. W. Magnus, A. Karass, D. Solitar,*Combinatorial Group Theory: Presentations of Groups in Terms of Generators and Relations* (Dover Publications, 2004).

5. P. de la Harpe, *Topics in Geometric Group Theory*, (University of Chicago Press, 2000).

2.12 C3.3b: Differentiable Manifolds — Prof. Hitchin — 16HT

Level: M-level. **Method of Assessment:** Written examination.

Weight: Half-unit (OSS paper code tbc).

Recommended Prerequisites

2nd year core algebra, topology, multivariate calculus. Useful but not essential: groups in action, geometry of surfaces.

Overview

A manifold is a space such that small pieces of it look like small pieces of Euclidean space. Thus a smooth surface, the topic of the B3 course, is an example of a (2-dimensional) manifold.

Manifolds are the natural setting for parts of classical applied mathematics such as mechanics, as well as general relativity. They are also central to areas of pure mathematics such as topology and certain aspects of analysis.

In this course we introduce the tools needed to do analysis on manifolds. We prove a very general form of Stokes' Theorem which includes as special cases the classical theorems of Gauss, Green and Stokes. We also introduce the theory of de Rham cohomology, which is central to many arguments in topology.

Learning Outcomes

The candidate will be able to manipulate with ease the basic operations on tangent vectors, differential forms and tensors both in a local coordinate description and a global coordinate-free one; have a knowledge of the basic theorems of de Rham cohomology and some simple examples of their use; know what a Riemannian manifold is and what geodesics are.

Synopsis

Smooth manifolds and smooth maps. Tangent vectors, the tangent bundle, induced maps. Vector fields and flows, the Lie bracket and Lie derivative.

Exterior algebra, differential forms, exterior derivative, Cartan formula in terms of Lie derivative. Orientability. Partitions of unity, integration on oriented manifolds.

Stokes' theorem. De Rham chomology. Applications of de Rham theory including degree.

Riemannian metrics. Isometries. Geodesics.

Reading

1. M. Spivak, *Calculus on Manifolds*, (W. A. Benjamin, 1965).

2. M. Spivak, *A Comprehensive Introduction to Differential Geometry*, Vol. 1, (1970).

3. W. Boothby, *An Introduction to Differentiable Manifolds and Riemannian Geometry*, 2nd edition, (Academic Press, 1986).

4. M. Berger and B. Gostiaux, *Differential Geometry: Manifolds, Curves and Surfaces.* Translated from the French by S. Levy, (Springer Graduate Texts in Mathematics, 115, Springer–Verlag (1988)) Chapters 0–3, 5–7.

5. F. Warner, *Foundations of Differentiable Manifolds and Lie Groups*, (Springer Graduate Texts in Mathematics, 1994).

6. D. Barden and C. Thomas, *An Introduction to Differential Manifolds.* (Imperial College Press, London, 2003.)

2.13 C3.4a: Algebraic Geometry — Dr Berczi — 16MT

Level: M-level. **Method of Assessment:** Written examination.

Weight: Half-unit (OSS paper codes tbc)

Recommended Prerequisites

Part A Group Theory and Introduction to Fields (B3 Algebraic Curves useful but no essential.

Overview

Algebraic geometry is the study of algebraic varieties: an algebraic variety is roughly speaking, a locus defined by polynomial equations. One of the advantages of algebraic geometry is that it is purely algebraically defined and applied to any field, including fields of finite characteristic. It is geometry based on algebra rather than calculus, but over the real or complex numbers it provides a rich source of examples and inspiration to other areas of geometry.

Synopsis

Affine algebraic varieties, the Zariski topology, morphisms of affine varieties. Irreducible varieties.

Projective space and general position points. Projective varieties, affine cones over projective varieties. The Zariski topology on projective varieties. The projective closure of affine variety. Morphisms of projective varieties. Projective equivalence.

Veronese morphism: definition, examples. Veronese morphisms are isomorphisms onto their image; statement, and proof in simple cases. Subvarieties of Veronese varieties. Segre maps and products of varieties, Categorical products: the image of Segre map gives the categorical product.

Coordinate rings. Hilbert's Nullstellensatz. Correspondence between affine varieties (and morphisms between them) and finitely generate reduced k-algebras (and morphisms between them). Graded rings and homogeneous ideals. Homogeneous coordinate rings.

Categorical quotients of affine varieties by certain group actions. The maximal spectrum.

Discrete invariants projective varieties: degree dimension, Hilbert function. Statement of theorem defining Hilbert polynomial.

Quasi-projective varieties, and morphisms of them. The Zariski topology has a basis of affine open subsets. Rings of regular functions on open subsets and points of quasi-projective varieties. The ring of regular functions on an affine variety in the coordinate ring. Localisation and relationship with rings of regular functions.

Tangent space and smooth points. The singular locus is a closed subvariety. Algebraic re-formulation of the tangent space. Differentiable maps between tangent spaces.

Function fields of irreducible quasi-projective varieties. Rational maps between irreducible varieties, and composition of rational maps. Birational equivalence. Correspondence between dominant rational maps and homomorphisms of function fields. Blow-ups: of affine space at appoint, of subvarieties of affine space, and general quasi-projective varieties along general subvarieties. Statement of Hironaka's Desingularisation Theorem. Every irreducible variety is birational to hypersurface. Re-formulation of dimension. Smooth points are a dense open subset.

Reading

KE Smith et al, *An Invitation to Algebraic Geometry*, (Springer 2000), Chapters 1–8.

Further Reading

1. M Reid, *Undergraduate Algebraic Geometry*, LMS Student Texts 12, (Cambridge 1988).
2. K Hulek, *Elementary Algebraic Geometry*, Student Mathematical Library 20. (American Mathematical Society, 2003).

2.14 C3.4b: Lie Groups — Prof. Joyce — 16MT

Level: M-level **Method of Assessment:** Written examination.

Weight: Half-unit (OSS paper code tbc).

Recommended Prerequisites

2nd year Groups in Action, Topology, Multivariable Calculus.

Overview

The theory of Lie Groups is one of the most beautiful developments of pure mathematics in the twentieth century, with many applications to geometry, theoretical physics and mechanics, and links to both algebra and analysis. Lie groups are groups which are simultaneously manifolds, so that the notion of differentiability makes sense, and the group multiplication and inverse maps are differentiable. However this course introduces the theory in a more concrete way via groups of matrices, in order to minimise the prerequisites.

Learning Outcomes

Students will have learnt the basic theory of topological matrix groups and their representations. This will include a firm understanding of root systems and their role for representations.

Synopsis

The exponential map for matrices, Ad and ad, the Campbell–Baker–Hausdorff series.

Linear Groups, their Lie algebras and the Lie correspondence. Homomorphisms and coverings of linear groups. Examples including $SU(2)$, $SO(3)$ and $SL(2;\mathbb{R}) \cong SU(1,1)$.

The compact and complex classical Lie groups. Cartan subgroups, Weyl groups, weights, roots, reflections.

Informal discussion of Lie groups as manifolds with differentiable group structures; quotients of Lie groups by closed subgroups.

Bi-invariant integration on a compact group (statement of existence and basic properties only). Representations of compact Lie groups. Tensor products of representations. Complete reducibility, Schur's lemma. Characters, orthogonality relations.

Statements of Weyl's character formula, the theorem of the highest weight and the Borel–Weil theorem, with proofs for $SU(2)$ only.

Reading

1. W. Rossmann, *Lie Groups: An Introduction through Linear Groups*, (Oxford, 2002), Chapters 1–3 and 6.

2. A. Baker, *Matrix Groups: An Introduction to Lie Group Theory*, (Springer Undergraduate Mathematics Series).

Further Reading

1. J. F. Adams, *Lectures on Lie Groups* (University of Chicago Press, 1982).

2. R. Carter, G. Segal and I. MacDonald, *Lectures on Lie Groups and Lie Algebras* (LMS Student Texts, Cambridge, 1995).

3. J. F. Price, *Lie Groups and Compact Groups* (LMS Lecture Notes 25, Cambridge, 1977).

2.15 C4.1a: Functional Analysis — Prof. Kristensen — 16MT

Level: M-level **Method of Assessment:** Written examination.

Weight: Half-unit (OSS paper code 2A64)

Recommended Prerequisites

Part A Topology, B4 Analysis

Overview

This course builds on B4, by extending the theory of Banach spaces and operators. As well as developing general methods that are useful in Operator Theory, we shall look in more detail at the structure and special properties of "classical" sequence-spaces and function-spaces.

Synopsis

Normed spaces and Banach spaces; dual spaces, subspaces, direct sums and completions; quotient spaces and quotient operators.

Baire's Category Theorem and its consequences (review).

Classical Banach spaces and their duals; smoothness and uniform convexity of norms.

Compact sets and compact operators. Ascoli's theorem.

Hahn–Banach extension and separation theorems; the bidual space and reflexivity.

Weak and weak* topologies. The Banach–Alaoglu theorem and Goldstine's theorem. Weak compactness.

Schauder bases; examples in classical spaces. Gliding-hump arguments.

Fredholm operators.

Reading

1. M. Fabian et al., *Functional Analysis and Infinite-Dimensional Geometry* (Canadian Math. Soc, Springer 2001), Chapters 1,2,3,6,7.

Alternative Reading

1. N. L. Carothers, *A Short Course on Banach Space Theory*, (LMS Student Text, Cambridge University Press 2004).

2.16 C9.1a: Modular Forms — tbc — 16MT

Level: M-Level. **Method of Assessment:** Written examination.

Weight: Half-unit (OSS paper code tbc)

Prerequisites

Part A Analysis and Algebra (core material) and Part A Group Theory. Part A Number Theory is useful but not essential. B3b Algebraic Curves is recom- mended but some background reading on the notions of a Riemann surface and its genus will suffice.

Overview

The course aims to introduce students to the beautiful theory of modular forms, one of the cornerstones of modern number theory. This theory is a rich and challenging blend of methods from complex analysis and linear algebra, and an explicit application of group actions and the theory of Riemann surfaces.

Learning Outcomes

The student will learn about modular curves and spaces of modular forms, and understand in special cases how to compute their genus and dimension, respectively. They will see how modular curves parameterise families of elliptic curves, and that modular forms can be described explicitly via their q-expansions, and they will be familiar with explicit examples of modular forms. They will learn about the rich algebraic structure on spaces of modular forms, given by Hecke operators and the Petersson inner product, how spaces of modular forms of different level are related, and how modular forms may be used in number theory.

Synopsis

The modular group and the upper half-plane. Lattices and elliptic curves. Modular forms of level 1. Examples of modular forms: Eisenstein series, Ramanujan's function Δ. Congruence subgroups and fundamental domains. Modular forms of higher level. Hecke operators. The Petersson inner product. Old and new forms. Applications in number theory.

Reading

1. F. Diamond and J. Shurman, *A First Course in Modular Forms,* Graduate Texts in Mathematics 228, (Parts of Chapters 1-5), Springer-Verlag, 2005.

2. J.-P. Serre, *A Course in Arithmetic,* (Chapter VII), Graduate Texts in Mathematics 7, Springer-Verlag, 1973.

2.17 C9.1b Elliptic Curves — Prof. Kim — 16HT

Level: M-Level. **Method of Assessment:** Written examination.

Weight: Half-unit (OSS paper code 2B82).

Recommended Prerequisites

It is helpful, but not essential, if students have already taken a standard introduction to algebraic curves and algebraic number theory. For those students who may have gaps in their background, I have placed the file "Preliminary Reading" permanently on the Elliptic Curves webpage, which gives in detail (about 30 pages) the main prerequisite knowledge for the course. Go first to: http://www.maths.ox.ac.uk/courses/material then click on "C9.1b Elliptic Curves" and then click on the pdf file "Preliminary Reading".

Overview

Elliptic curves give the simplest examples of many of the most interesting phenomena which can occur in algebraic curves; they have an incredibly rich structure and have been the testing ground for many developments in algebraic geometry whilst the theory is still full of deep unsolved conjectures, some of which are amongst the oldest unsolved problems in mathematics. The course will concentrate on arithmetic aspects of elliptic curves defined over the rationals, with the study of the group of rational points, and explicit determination of the rank, being the primary focus. Using elliptic curves over the rationals as an example, we will be able to introduce many of the basic tools for studying arithmetic properties of algebraic varieties.

Learning Outcomes

On completing the course, students should be able to understand and use properties of elliptic curves, such as the group law, the torsion group of rational points, and 2-isogenies between elliptic curves. They should be able to understand and apply the theory of fields with valuations, emphasising the p-adic numbers, and be able to prove and apply Hensel's Lemma in problem solving. They should be able to understand the proof of the Mordell–Weil Theorem for the case when an elliptic curve has a rational point of order 2, and compute ranks in such cases, for examples where all homogeneous spaces for descent-via-2-isogeny satisfy the Hasse principle. They should also be able to apply the elliptic curve method for the factorisation of integers.

Synopsis

Non-singular cubics and the group law; Weierstrass equations.
Elliptic curves over finite fields; Hasse estimate (stated without proof).
p-adic fields (basic definitions and properties).
1-dimensional formal groups (basic definitions and properties).

Curves over p-adic fields and reduction mod p.
Computation of torsion groups over $\mathbb{Q}$; the Nagell–Lutz theorem.
2-isogenies on elliptic curves defined over $\mathbb{Q}$, with a $\mathbb{Q}$-rational point of order 2.
Weak Mordell–Weil Theorem for elliptic curves defined over $\mathbb{Q}$, with a $\mathbb{Q}$-rational point of order 2.
Height functions on Abelian groups and basic properties.
Heights of points on elliptic curves defined over $\mathbb{Q}$; statement (without proof) that this gives a height function on the Mordell–Weil group.
Mordell–Weil Theorem for elliptic curves defined over $\mathbb{Q}$, with a $\mathbb{Q}$-rational point of order 2.
Explicit computation of rank using descent via 2-isogeny.
Public keys in cryptography; Pollard's $(p-1)$ method and the elliptic curve method of factorisation.

Reading

1. J.W.S. Cassels, *Lectures on Elliptic Curves,* LMS Student Texts 24 (Cambridge University Press, 1991).

2. N. Koblitz, *A Course in Number Theory and Cryptography,* Graduate Texts in Mathematics 114 (Springer, 1987).

3. J.H. Silverman and J. Tate, *Rational Points on Elliptic Curves,* Undergraduate Texts in Mathematics (Springer, 1992).

4. J.H. Silverman, *The Arithmetic of Elliptic Curves,* Graduate Texts in Mathematics 106 (Springer, 1986).

Further Reading

1. A. Knapp, *Elliptic Curves, Mathematical Notes 40* (Princeton University Press, 1992).

2. G, Cornell, J.H. Silverman and G. Stevans (editors), *Modular Forms and Fermat's Last Theorem* (Springer, 1997).

3. J.H. Silverman, *Advanced Topics in the Arithmetic of Elliptic Curves,* Graduate Texts in Mathematics 151 (Springer, 1994).

2.18 C10.1a: Stochastic Differential Equations —Prof. Hambly—16MT

Level: M-level **Method of Assessment:** Written examination.

Weight: Half-unit (OSS paper code 2A83)

Recommended Prerequisites Part A integration and B10a Martingales Through Measure Theory, is expected.

Overview

Stochastic differential equations have been used extensively in many areas of application, including finance and social science as well as chemistry. This course develops the basic theory of Itô's calculus and stochastic differential equations.

Learning Outcomes

The student will have developed an appreciation of stochastic calculus as a tool that can be used for defining and understanding diffusive systems.

Synopsis

Brownian motion: basic properties, reflection principle, quadratic variation. Itô's calculus: stochastic integrals with respect to martingales, Itô's lemma, Levy's theorem on characteristic of Brownian motion, exponential martingales, exponential inequality, Girsanov's theorem, The Martingale Representation Theorem. Stochastic differential equations: strong solutions, questions of existence and uniqueness, diffusion processes, Cameron–Martin formula, weak solution.

Reading — Main Texts

1. Dr Qian's online notes:

   www.maths.ox.ac.uk/courses/course/15721

2. B. Oksendal, *Stochastic Differential Equations: An introduction with applications* (Universitext, Springer, 6th edition). Chapters II, III, IV, V, part of VI, Chapter VIII (F).

3. F. C. Klebaner, *Introduction to Stochastic Calculus with Applications* (Imperial College Press, 1998, second edition 2005). Sections 3.1 – 3.5, 3.9, 3.12. Chapters 4, 5, 11.

Alternative Reading

1. H. P. McKean, *Stochastic Integrals* (Academic Press, New York and London, 1969).

Further Reading

1. N. Ikeda & S. Watanabe, *Stochastic Differential Equations and Diffusion Processes* (North–Holland Publishing Company, 1989).

2. I. Karatzas and S. E. Shreve, *Brownian Motion and Stochastic Calculus,* Graduate Texts in Mathematics 113 (Springer-Verlag, 1988).

3. L. C. G. Rogers & D. Williams, *Diffusions, Markov Processes and Martingales Vol 1 (Foundations) and Vol 2 (Ito Calculus)* (Cambridge University Press, 1987 and 1994).

2.19 C11.1a: Graph Theory — Prof. McDiarmid — 16MT

Level: M-level **Method of Assessment:** Written examination.

Weight: Half-unit (OSS paper code 2A85).

Recommended Prerequisites:

None.

Overview

Graphs are among the simplest mathematical structures, but nevertheless have a very rich and well-developed structural theory. Graph Theory is an important area of mathematics, and also has many applications in other fields such as computer science.

The main aim of the course is to introduce the analysis of discrete structures, and particularly the use of extremal methods.

Learning Outcomes

The student will have developed an appreciation of extremal methods in the analysis and understanding of graphical structures.

Synopsis

Introduction. Trees. Euler circuits. Planar graphs.
Matchings and Hall's Theorem. Connectivity and Menger's Theorem.
Extremal problems. Long paths and cycles. Turán's Theorem. Erdős–Stone Theorem.
Graph colouring. The Theorem of Brooks. The chromatic polynomial.
Ramsey's Theorem.
Szemerédi's Regularity Lemma.

Reading

1. B. Bollobás, *Modern Graph Theory,* Graduate Texts in Mathematics 184 (Springer-Verlag, 1998)

Further Reading

1. J. A. Bondy and U. S. R. Murty, *Graph Theory: An Advanced Course*, Graduate Texts in Mathematics 244 (Springer–Verlag, 2007).
2. R. Diestel, *Graph Theory,* Graduate Texts in Mathematics 173 (third edition, Springer-Verlag, 2005).
3. D. West, *Introduction to Graph Theory* (second edition, Prentice–Hall, 2001).

2.20 CD : Dissertations on a Mathematical Topic

Level : M-level

Weight : Half-unit (5,000 words) or whole-unit (10,000).

Students may offer either a whole-unit or a half-unit dissertation on a Mathematical topic for examination at Part C. A whole-unit is equivalent to a 32-hour lecture course and a half-unit is equivalent to a 16-hour lecture course. Students will have approximately 8 hours of supervision for a whole-unit dissertation or 4 hours for a half-unit distributed over Michaelmas and Hilary terms. In addtion there are lectures on writing mathematics and using Latex in Michaelmas and Hilary terms. See the lecture list for details.

Students considering offering a dissertation should read the *Guidance Notes on Extended Essays and Dissertations in Mathematics* available at:

http://www.maths.ox.ac.uk/current-students/undergraduates/projects/.

Application

Students must apply to the Mathematics Projects Committee for approval of their proposed topic in advance of beginning work on their dissertation. Proposals should be addressed to the Chairman of the Projects Committee, c/o Mrs Helen Lowe, Room DH61, Dartingtion House and are accepted from the end of Trinity Term. All proposals must be received before 12noon on Friday of Week 0 of Michaelmas Full Term. For CD dissertations candidates should take particular care to remember that the project must have substantial mathematical content. The application form is available at http://www.maths.ox.ac.uk/current-students/undergraduates/projects/. Once a title has been approved, it may only be changed by approval of the Chairman of the Projects Committee.

Assessment

Each project is blind double marked. The marks are reconciled through discussion between the two assessors, overseen by the examiners. Please see the *Guidance Notes on Extended Essays and Dissertations in Mathematics* for detailed marking criteria and class descriptors.

Submission

THREE copies of your dissertation, identified by your candidate number only, should be sent to the Chairman of Examiners, FHS of Mathematics Part C, Examination Schools, Oxford, to arrive no later than **12noon on Friday of week 9, Hilary Term 2013**. An electronic copy of your dissertation should also be submitted via the Mathematical Institute website. Further details may be found in the *Guidance Notes on Extended Essays and Dissertations in Mathematics*.

2.21 Computer Science: Half Units

Students in Part C may take half units drawn from Part C of the Honour School of Mathematics and Computing. For full details of these half units see the Department of Computer Science's website (http://www.cs.ox.ac.uk/teaching/courses/)

Please note that these three courses will be examined by mini-project (as for MSc students). Mini-projects will be handed out to candidates on the last Friday of the term in which the subject is being taught, and you will have to hand it in to the Exam Schools by noon on Monday of Week 1 of the following term. The mini-project will be designed to be completed in about three days. It will include some questions that are more open-ended than those on a standard sit-down exam. The work you submit should be your own work, and include suitable references.

Please note that the Computer Science courses in Part C are 50% bigger than those in earlier years, i.e. for each Computer Science course in the 3rd year undergraduates are expected to undertake about 10 hours of study per week, but 4th year courses will each require about 15 hours a week of study. Lecturers are providing this extra work in a variety of ways, e.g. some will give 16 lectures with extra reading, classes and/or practicals, whereas others will be giving 24 lectures, and others still will be doing something in between. Students will need to look at each synopsis for details on this.

The Computer Science half units available are as follows:

- CCS1a Categories, Proofs and Processes
- CCS3b Quantum Computer Science
- CCS4b Automata, Logics and Games

2.22 OD : Dissertations on a Mathematically related Topic

Level : M-level

Weight : Half-unit (5,000 words) or whole-unit (10,000 words).

Students may offer either a whole-unit or a half-unit dissertation on a Mathematically related topic for examination at Part C. For example, applications of mathematics to another field (eg Maths in Music), historical topics, topics concentrating on the analysis of statistical data, or topics concentrating on the production of computer-generated data are acceptable as topics for an OD dissertation. (Topics in mathematical education are not allowed.)

A whole-unit is equivalent to a 32-hour lecture course and a half-unit is equivalent to a 16-hour lecture course. Students will have approximately 8 hours of supervision for a whole-unit dissertation or 4 hours for a half-unit distributed over Michaelmas and Hilary terms. In addtion there are lectures on writing mathematics and using Latex in Michaelmas and Hilary terms. See the lecture list for details.

Candidates considering offering a dissertation should read the *Guidance Notes on Extended Essays and Dissertations in Mathematics* available at:

http://www.maths.ox.ac.uk/current-students/undergraduates/projects/.

Application

Students must apply to the Mathematics Projects Committee for approval of their proposed topic in advance of beginning work on their dissertation. Proposals should be addressed to the Chairman of the Projects Committee, c/o Mrs Helen Lowe, Room DH61, Dartingtion House and are accepted from the end of Trinity Term. All proposals must be received before 12noon on Friday of Week 0 of Michaelmas Full Term. The application form is available at http://www.maths.ox.ac.uk/current-students/undergraduates/projects/.

Once a title has been approved, it may only be changed by approval of the Chairman of the Projects Committee.

Assessment

Each project is blind double marked. The marks are reconciled through discussion between the two assessors, overseen by the examiners. Please see the *Guidance Notes on Extended Essays and Dissertations in Mathematics* for detailed marking criteria and class descriptors.

Submission

THREE copies of your dissertation,identified by your candidate number only, should be sent to the Chairman of Examiners, FHS of Mathematics Part C, Examination Schools, Oxford, to arrive no later than **12noon on Friday of week 9, Hilary Term 2013**. An electronic copy of your dissertation should also be submitted via the Mathematical Institute website. Further details may be found in the *Guidance Notes on Extended Essays and Dissertations in Mathematics.*