Handbook for the Undergraduate Mathematics Courses
Supplement to the Handbook
Honour School of Mathematics & Philosophy
Syllabus and Synopses for Part A 2010–11
for examination in 2011

Contents

1 Foreword

Notice of misprints or errors of any kind, and suggestions for improvements in this booklet should be addressed to the Academic Assistant in the Mathematical Institute.

[See the current edition of the *Examination Regulations* for the full regulations governing these examinations.]

In Part A each candidate shall be required to offer four written papers in Mathematics from the schedule of papers for Part A (given below). The two papers AC1(P) and AC2(P) will each be of 2 hours' duration and the two papers AO1(P) and AO2(P) will each be of $1\frac{1}{2}$ hours' duration. The four papers taken together will count as the equivalent of two 3 hour papers.

At the end of the Part A examination a candidate will be awarded a 'University Standardised Mark' (USM) for each Mathematics paper in Part A. The four USMs will be carried forward into the classification awarded at the end of the third year. In the calculation of any averages used to arrive at the final classification, USMs for AC1(P) and AC2(P) will be scaled by a factor of $\frac{2}{3}$ and those for AO1(P) and AO2(P) by a factor of $\frac{1}{3}$, so as to give the equivalent of two papers in total. [The differential weighting takes account of extra examination time allocated to AO1(P) and AO2(P) granted to assist candidates fully to demonstrate their knowledge of the optional material.]

The Schedule of Papers

Paper AC1(P) Algebra and Analysis

This paper will contain 6 short questions set on the CORE material in Algebra and Analysis for Part A of the FHS of Mathematics. Candidates are expected to answer all 6 questions. Each question is out of 10 marks.

Paper AC2(P) Algebra and Analysis

This paper will contain 6 longer questions set on the CORE material in Algebra and Analysis for Part A of the FHS of Mathematics. Each question is out of 25 marks. Candidates may submit answers to as many questions as they wish. The best answer in Algebra and the best answer in Analysis will count, together with the best of the remaining answers.

Papers AO1(P) and AO2(P) Options

These papers will contain questions on the OPTIONAL subjects listed below. In each paper there will be one question per 8 lectures, with Paper AO1(P) containing short questions each worth 10 marks and Paper AO2(P) containing longer questions each worth 25 marks. In AO1(P) a candidate may submit answers to as many questions as they wish, of which the best 3 will count. In AO2(P) a candidate may submit answers to as many questions as they wish, of which the best 2 will count.

Mark Schemes

Mark schemes for questions out of 10 will aim to ensure that the following qualitative criteria hold:

- 9-10 marks: a completely or almost completely correct answer, showing good understanding of the concepts and skill in carrying through arguments and calculations; minor slips or omissions only.

- 5-8 marks: a good though not complete answer, showing understanding of the concepts and competence in handling the arguments and calculations.

Mark schemes for questions out of 25 will aim to ensure that the following qualitative criteria hold:

- 20-25 marks: a completely or almost completely correct answer, showing excellent understanding of the concepts and skill in carrying through the arguments and/or calculations; minor slips or omissions only.

- 13-19 marks: a good though not complete answer, showing understanding of the concepts and competence in handling the arguments and/or calculations. In this range, an answer might consist of an excellent answer to a substantial part of the question, or a good answer to the whole question which nevertheless shows some flaws in calculation or in understanding or in both.

OPTIONAL SUBJECTS

From the FHS of Mathematics Part A

Introduction to Fields
Group Theory
Number Theory
Integration
Topology
Multivariable Calculus

From Honour Moderations in Mathematics Probability I and the first four lectures of Probability II

Candidates may also, with the support of their college tutors, apply to the Joint Committee for Mathematics and Philosophy for approval of other Optional Subjects as listed for Part A of the Honour School of Mathematics.

Syllabus and Synopses

The **syllabus** details in this booklet are those referred to in the *Examination Regulations* and have been approved by the Mathematics Teaching Committee for examination in Trinity Term 2011.

The **synopses** in this booklet give some additional detail, and show how the material is split between the different lecture courses. They also include details of recommended reading.

October 2010

2 CORE MATERIAL

2.1 Syllabus

2.1.1 Algebra

Vector spaces over an arbitrary field, subspaces, direct sums; quotient spaces; projection maps and their characterisation as idempotent operators.

Dual spaces of finite-dimensional spaces; annihilators; the natural isomorphism between a space and its second dual; dual transformations and their matrix representation with respect to dual bases.

Some theory of a single linear map on a finite-dimensional space: characteristic polynomial, minimal polynomial, Primary Decomposition Theorem, the Cayley–Hamilton Theorem; diagonalisability; triangular form. Statement of the Jordan normal form.

Real and complex inner product spaces. Orthogonal complements, orthonormal sets; the Gram–Schmidt process. Bessel's inequality; the Cauchy–Schwarz inequality.

The adjoint of a linear map on a finite-dimensional inner product space to itself. Eigenvalues and diagonalisability of self-adjoint linear maps.

Commutative rings with unity, integral domains, fields; units, irreducible elements, primes.

Ideals and quotient rings; isomorphism theorems. The Chinese Remainder Theorem [classical case of $\mathbb{Z}$ only].

Maximal ideals and their quotient rings.

Euclidean rings and their properties: polynomial rings as examples, theorem that their ideals are principal; theorem that their irreducible elements are prime; uniqueness of factorisation (proof non-examinable).

Gauss' Lemma; Eisenstein's criterion.

2.1.2 Analysis

The topology of Euclidean space and its subsets, particularly $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$: open sets, closed sets, subspace topology; continuous functions and their characterisation in terms of pre-images of open or closed sets; connected sets, path-connected sets; compact sets, Heine-Borel Theorem.

Complex differentiation. Holomorphic functions. Cauchy-Riemann equations (including z, $\bar{z}$ version). Real and imaginary parts of a holomorphic function are harmonic.

Path integration. Fundamental Theorem of Calculus in the path integral/holomorphic function setting. Power series and differentiation of power series. Exponential function, holomorphic branches of logarithm functions, fractional powers.

Cauchy's Theorem (proof excluded). Cauchy's Integral formulae. Taylor expansion. Liouville's Theorem. Identity Theorem. Morera's Theorem. Laurent's expansion. Classification of singularities. Calculation of principal parts and residues. Residue Theorem. Evaluation of integrals by the method of residues (straight forward examples including the use of simple estimates, and examples with simple poles on contour of integration).

Conformal mapping, Riemann mapping theorem (no proof): Möbius functions, exponential functions, fractional powers; mapping regions (not Christoffel transformations or Jowkowski's transformation).

2.2 Synopses of Lectures

2.2.1 Algebra — Dr Stewart & Dr Papazoglou — 24 lectures MT

Overview

Linear Algebra
The core of linear algebra comprises the theory of linear equations in many variables, the theory of matrices and determinants, and the theory of vector spaces and linear maps. All these topics were introduced in the Moderations course. Here they are developed further to provide the tools for applications in geometry, modern mechanics and theoretical physics, probability and statistics, functional analysis and, of course, algebra and number theory. Our aim is to provide a thorough treatment of some classical theory that describes the behaviour of linear maps on a finite-dimensional vector space to itself, both in the purely algebraic setting and in the situation where the vector space carries a metric derived from an inner product.

Rings
The rings part of the course introduces the student to some classic ring theory which is basic for other parts of abstract algebra, for linear algebra and for those parts of number theory that lead ultimately to applications in cryptography. The first-year algebra course contains a treatment of the Euclidean Algorithm in its classical forms for integers and for polynomial rings over a field; here the idea is developed *in abstracto*.

Learning Outcomes

Linear Algebra
Students will deepen their understanding of Linear Algebra. They will be able to define and obtain the minimal and characteristic polynomials of a linear map on a finite-dimensional vector space, and will understand and be able to prove the relationship between them; they will be able to prove and apply the Primary Decomposition Theorem, and the criterion for diagonalisability. They will have a good knowledge of inner product spaces, and be able to apply the Bessel and Cauchy–Schwarz inequalities; will be able to define and use the adjoint of a linear map on a finite-dimensional inner product space, and be able to prove and exploit the diagonalisability of a self-adjoint map.

Rings
By the end of the course students will have extended their knowledge of abstract algebra to include the key elements of classical ring theory. They will understand and be able to prove and use the Isomorphism Theorem. They will have a good knowledge of Euclidean rings, and be able to apply it.

Synopsis

1. **Linear Algebra**

   MT (17 lectures)
   Vector spaces over an arbitrary field, subspaces, direct sums; quotient vector spaces; induced linear map; projection maps and their characterisation as idempotent operators.

   [2 Lectures]

   Dual spaces of finite-dimensional spaces; annihilators; the natural isomorphism between a finite-dimensional space and its second dual; dual transformations and their matrix representation with respect to dual bases.

   [2-3 Lectures]

   Some theory of a single linear map on a finite-dimensional space: characteristic polynomial, minimal polynomial, Primary Decomposition Theorem, the Cayley-Hamilton Theorem (economically); diagonalisability; triangular form. Statement of the Jordan normal form.

   [4-5 Lectures]

   Real and complex inner product spaces: examples, including function spaces [but excluding completeness and L^2]. Orthogonal complements, orthonormal sets; the Gram–Schmidt process. Bessel's inequality; the Cauchy–Schwarz inequality.

   [4 Lectures]

   Some theory of a single linear map on a finite-dimensional inner product space: the adjoint; eigenvalues and diagonalisability of a self-adjoint linear map.

   [4 Lectures]

2. **Rings**

   MT (7 Lectures)

   Review of commutative rings with unity, integral domains, ideals, fields, polynomial rings and subrings of $\mathbb{R}$ and $\mathbb{C}$. The Chinese Remainder Theorem; the quotient ring by a maximal ideal is a field.

   [2 lectures]

   Euclidean rings and their properties : units, associates, irreducible elements, primes. The Euclidean Algorithm for a Euclidean ring; $\mathbb{Z}$ and $F[x]$ as prototypes; their ideals are principal; their irreducible elements are prime; factorisation is unique (proof not examinable).

   [3 Lectures]

   Examples for applications: Gauss's Lemma and factorisation in $\mathbb{Z}[x]$; Eisenstein's criterion.

   [2 lectures]

Reading

Richard Kaye and Robert Wilson, *Linear Algebra* (OUP, 1998) ISBN 0-19-850237-0. Chapters 2–13. [Chapters 6, 7 are not entirely relevant to our syllabus, but are interesting.]

Peter J. Cameron, *Introduction to Algebra* (OUP, 1998) ISBN 0-19-850194-3. Chapter 2.

Alternative and further reading:

Joseph J. Rotman, *A First Course in Abstract Algebra* (Second edition, Prentice Hall, 2000), ISBN 0-13-011584-3. Chapters 1, 3.

I. N. Herstein, *Topics in Algebra* (Second edition, Wiley, 1975), ISBN 0-471-02371-X. Chapter 3. [Harder than some, but an excellent classic. Widely available in Oxford libraries; still in print.]

P. M. Cohn, *Classic Algebra* (Wiley, 2000), ISBN 0-471-87732-8. Various sections. [This is the third edition of his book previously called *Algebra I*.]

David Sharpe, *Rings and Factorization* (CUP, 1987), ISBN 0-521-33718-6. [An excellent little book, now sadly out of print; available in some libraries, though.]

Paul R. Halmos, *Finite-dimensional Vector Spaces*, (Springer Verlag, Reprint 1993 of the 1956 second edition), ISBN 3-540-90093-4. §§1–15, 18, 32–51, 54–56, 59–67, 73, 74, 79. [Now over 50 years old, this idiosyncratic book is somewhat dated but it is a great classic, and well worth reading.]

Seymour Lipschutz and Marc Lipson, *Schaum's Outline of Linear Algebra* (3rd edition, McGraw Hill, 2000), ISBN 0-07-136200-2. [Many worked examples.]

C. W. Curtis, *Linear Algebra—an Introductory Approach* (4th edition, Springer, reprinted 1994).

D. T. Finkbeiner, *Elements of Linear Algebra* (Freeman, 1972). [Out of print, but available in many libraries.]

J. A. Gallian, *Contemporary Abstract Algebra* (Houghton Mifflin Company, 2006).

There are very many other such books on abstract and linear algebra in Oxford libraries.

2.2.2 Analysis — Dr Qian — 24 lectures MT

Overview

The theory of functions of a complex variable is a rewarding branch of mathematics to study at the undergraduate level with a good balance between general theory and examples. It occupies a central position in mathematics with links to analysis, algebra, number theory, potential theory, geometry, topology, and generates a number of powerful techniques (for example, evaluation of integrals) with applications in many aspects of both pure and applied mathematics, and other disciplines, particularly the physical sciences.

In these lectures we begin by introducing students to the language of topology before using it in the exposition of the theory of (holomorphic) functions of a complex variable. The central aim of the lectures is to present Cauchy's Theorem and its consequences, particularly series expansions of holomorphic functions, the calculus of residues and its applications.

The course concludes with an account of the conformal properties of holomorphic functions and applications to mapping regions.

Learning Outcomes

Students will have been introduced to point-set topology and will know the central importance of complex variables in analysis. They will have grasped a deeper understanding of differentiation and integration in this setting and will know the tools and results of complex analysis including Cauchy's Theorem, Cauchy's integral formula, Lioville's Theorem, Laurent's expansion and the theory of residues.

Synopsis

(1-4) Topology of Euclidean space and its subsets, particularly $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$. Open sets, closed sets, subspace topology; continuous functions and their characterisation in terms of preimages of open or closed sets; connected sets, path-connected sets; compact sets, Heine-Borel Theorem (covered in Chapter 3 of Apostol).

(5-7)Complex differentiation. Holomorphic functions. Cauchy-Riemann equations. Real and imaginary parts of a holomorphic function are harmonic.

(8-11) Path integration. Power series and differentiation of power series. Exponential function and logarithm function. Fractional powers - examples of multifunctions.

(12-13) Cauchy's Theorem. (Sketch of proof only - students referred to various texts for proof.) Fundamental Theorem of Calculus in the path integral/holomorphic situation.

(14-16) Cauchy's Integral formulae. Taylor expansion. Liouville's Theorem. Identity Theorem. Morera's Theorem

(17-18) Laurent's expansion. Classification of singularities. Calculation of principal parts, particularly residues.

(19-21) Residue theorem. Evaluation of integrals by the method of residues (straight forward examples only but to include the use of Jordan's Lemma and simple poles on contour of integration).

(22-23) Conformal mapping Riemann mapping theorem (no proof): Möbius functions, exponential functions, fractional powers; mapping regions (not Christoffel transformations or Jowkowski's transformation).

(24) Summary and Outlook.

Reading

Main texts

H. A. Priestley, *Introduction to Complex Analysis* (second edition, Oxford Science Publications, 2003).

T. M. Apostol, *Mathematical Analysis* (Addison–Wesley, 1974)(Chapter 3 for the topology).

Reinhold Remmert, *Theory of Complex Functions* (Springer, 1989) (Graduate Texts in Mathematics 122).

Mark J. Ablowitz, Athanassios S. Focas, *Complex Variables, Introduction and Applications*(2nd edition, Cambridge Texts in Applied Mathematics, 2003).

Further Reading

L. Ahlfors, *Complex Analysis* (McGraw-Hill, 1979).

Theodore Gamelin, *Complex Analysis* (Springer, 2000).

E. C. Titchmarsh, *The Theory of Functions* (2nd edition, Oxford University Press).

I. Stewart and D. Tall, *Complex Analysis*, (CUP, 1983).

J.P Gilman, I Krn and R.E Rogriguez: *Complex Analysis*, (Springer 2007) (Graduate Texts in Mathematics 122.) This book is included for its extra material showing where the subject can lead.

3 OPTIONS

3.1 Syllabus

3.1.1 Introduction to Fields

Fields, subfields, finite extensions; examples. Degree of an extension, the Tower Theorem. Simple algebraic extensions; splitting fields, uniqueness (proof not to be examined); examples. Characteristic of a field. Finite fields: existence; uniqueness (proof not to be examined). Subfields. The multiplicative group of a finite field. The Frobenius automorphism.

3.1.2 Group Theory

Groups: subgroups, normal subgroups and quotient groups; elementary results concerning symmetric and alternating groups; important examples of groups, including the general Linear groups. Isomorphism theorems for groups. Simplicity; composition series. Finite soluble groups. Actions of groups on sets; examples, including coset spaces, groups acting on themselves by translation and conjugation, the Möbius groups. Orbits, transitivity, stabilisers, equivalence of a transitive space with a coset space, kernels of such actions, examples. Symmetry groups of geometric objects including regular polyhedra.

3.1.3 Number Theory

The ring of integers; congruences; rings of integers modulo n; the Chinese Remainder Theorem. Wilson's Theorem; Fermat's Little Theorem for prime modulus. Euler's phi-function; Euler's generalisation of Fermat's Little Theorem to arbitrary modulus. Quadratic residues modulo primes. Quadratic reciprocity. Factorisation of large integers; basic version of the RSA encryption method.

3.1.4 Integration

Measure spaces. Outer measure, null set, measurable set. The Cantor set. Lebesgue measure on the real line. Counting measure. Probability measures. Construction of a non-measurable set (non-examinable). Measurable function, simple function, integrable function. Reconciliation with the integral introduced in Moderations.

A simple comparison theorem. Integrability of polynomial and exponential functions over suitable intervals. Changes of variable. Fatou's Lemma (proof not examinable). Monotone Convergence Theorem (proof not examinable). Dominated Convergence Theorem. Corollaries and applications of the Convergence Theorems (including term-by-term integration of series).

Theorems of Fubini and Tonelli (proofs not examinable). Differentiation under the integral sign. Change of variables.

Brief introduction to L^p spaces. Hölder and Minkowski inequalities (proof not examinable).

3.1.5 Topology

Metric spaces. Examples to include metrics derived from a norm on a real vector space, particularly l^1, l^2, l^∞ norms on $\mathbb{R}^n$, the *sup* norm on the bounded real-valued functions on a set, and on the bounded continuous real-valued functions on a metric space. Continuous functions (ϵ, δ definition). Uniformly continuous functions; examples include Lipschitz functions and contractions. Open balls, open sets, accumulation points of a set. Completeness (but not completion). Contraction Mapping Theorem. Completeness of the space of bounded real-valued functions on a set, equipped with the *sup* norm, and the completeness of the space of bounded continuous real-valued functions on a metric space, equipped with the *sup* metric.

Axiomatic definition of an abstract topological space in terms of open sets. Continuous functions, homeomorphisms. Closed sets. Accumulation points of sets. Closure of a set ($\bar{A} = A$ together with its accumulation points). Interior of a set. Continuity if $f(\bar{A}) \subseteq \overline{f(A)}$. Examples to include metric spaces (definition of topological equivalence of metric spaces), discrete and indiscrete topologies, subspace topology, cofinite topology, quotient topology. Base of a topology. Product topology on a product of two spaces and continuity of projections. Hausdorff topology.

Connected spaces: closure of a connected space is connected, union of connected sets is connected if there is a non-empty intersection, continuous image of a connected space is connected. Path-connectedness implies connectedness. Connected open subset of a normed vector space is path-connected.

Compact sets, closed subset of a compact set is compact, compact subset of a Hausdorff space is closed. Heine-Borel Theorem in $\mathbb{R}^n$. Product of two compact spaces is compact. A continuous bijection from a compact space to a Hausdorff space is a homeomorphism. Equivalence of sequential compactness and abstract compactness in metric spaces.

Further discussion of quotient spaces: simple classical geometric spaces such as the torus and Klein bottle.

3.1.6 Multivariable Calculus

Definition of a derivative of a function from $\mathbb{R}^m$ to $\mathbb{R}^n$; examples; elementary properties; partial derivatives; the chain rule; the gradient of a function from $\mathbb{R}^m$ to $\mathbb{R}$; Jacobian. Continuous partial derivatives imply differentiability, Mean Value Theorems. Higher order derivatives.

The Implicit Function Theorem (proof for special case, non-examinable), the Inverse Function Theorem (proof non-examinable).

The definition of a submanifold of $\mathbb{R}^m$, its tangent space at a point. Examples, defined parametrically and implicitly, including curves and surfaces in $\mathbb{R}^3$.

Lagrange multipliers.

Option from Honour Moderations in Mathematics

3.1.7 Probability

Sample space as the set of all possible outcomes, events and probability function. Permutations and combinations, examples using counting methods, sampling with or without replacement. Algebra of events. Conditional probability, partitions of sample space, theorem of total probability, Bayes's Theorem, independence. Examples with statistical implications.

Random variables. Probability mass function. Discrete distributions: Bernoulli, binomial, Poisson, geometric; situations in which these distributions arise. Expectation: mean and variance. Probability generating functions, use in calculating expectations. Bivariate discrete distribution, conditional and marginal distributions. Extensions to many random variables. Independence for discrete random variables. Random walks (finite state space only). Solution of linear and quadratic difference equations with applications to random walks.

Expectations of functions of more than one random variable. Random samples. Conditional expectation, application of theorem of total probability to expectation of a random variable. Sums of independent random variables. Examples from well-known distributions.

Continuous random variables. Cumulative distribution function for both discrete and continuous random variables. Probability density functions. Examples: uniform, exponential, gamma, normal. Practical examples. Expectation. Cumulative distribution function and probability density function for a function of a single continuous random variable. Simple examples of joint distributions of two or more continuous random variables, independence, expectation (mean and variance of sums of independent, identically distributed random variables).

3.2 Synopses of Lectures

3.2.1 Introduction to Fields — Dr Kremnizer — 8 lectures HT

Weeks 1 to 4 in Hilary Term.

Overview

Informally, finite fields are generalisations of systems of real numbers such as the rational or the real numbers— systems in which the usual rules of arithmetic (including those for division) apply. Formally, fields are commutative rings with unity in which division by non-zero elements is always possible. It is a remarkable fact that the finite fields may be completely classified. Furthermore, they have classical applications in number theory, algebra, geometry, combinatorics, and coding theory, and they have newer applications in other areas. The aim of this course is to show how their structure may be elucidated, and to present the main theorems about them that lead to their various applications.

Learning Outcomes

Students will have a sound knowledge of field theory including the classification of finite fields. They will have an appreciation of the applications of this theory.

Synopsis

Fields, subfields and their intersections. Statement of the Fundamental Theorem of Algebra; the splitting field for a rational polynomial as the minimal subfield of $\mathbb{C}$ that contains all its roots, its Galois group over $\mathbb{Q}$ (basic concept only). The link between the structure of the Galois group and the solubility of equations (not examinable). [$1\frac{1}{2}$ lectures]

The characteristic of a field, prime subfields. [$\frac{1}{2}$ lecture]

Extensions of fields; examples. Degree of a finite extension, the Tower Theorem. [1 lecture]

Simple algebraic extensions; splitting fields, uniqueness (proof sketched but not examinable); examples. [2 lectures]

Finite fields: existence and uniqueness (proof sketched but not examinable), subfields. The multiplicative group of a finite field, the Frobenius automorphism. [3 lectures]

Reading

P.J. Cameron *Introduction to Algebra* (2nd. ed., OUP, 2008) pp. 99-103, 220-223, 268-276.

Joseph J. Rotman, *A First Course in Abstract Algebra* (Second edition, Prentice Hall, 2000), ISBN 0-13-011584-3. Chapters 1,3.

Further Reading

I. N. Herstein, *Topics in Algebra* (Wiley, 1975). ISBN 0-471-02371-X 5.1, 5.3, 7.1. [Harder than some, but an excellent classic. Widely available in Oxford libraries; still in print.]

P. M. Cohn, *Classic Algebra* (Wiley, 2000), ISBN 0-471-87732-8, parts of Chapter 6. [This is the third edition of his book on abstract algebra, in Oxford libraries.]

There are many other such books on abstract algebra in Oxford libraries.

3.2.2 Group Theory — Dr Szendroi — 8 lectures HT

Weeks 5 to 8 in Hilary Term.

Overview

This group theory course develops the theory of finite groups begun in Mods. In this course we will present an introduction to general "structural" theory via the Jordan-Hölder Theorem for finite groups and a basic study of finite soluble groups. This will be followed by a discussion of the concept of a "group acting on a set" which lies at the heart of the application to solving quadratic, cubic and quartic equations over the rationals but which appears wherever groups are studied throughout mathematics.

Learning Outcomes

Students will begin to have a deeper knowledge of group structure and theory, particularly finite groups. They will have an appreciation of some of the important properties of groups including simplicity, solubility and actions of a group on a set. Examples include S_n, A_n, the Möbius group, and the symmetry group of Platonic solids.

Synopsis

Brief revision of group theory: homomorphisms, normal subgroups and quotient groups, First Isomorphism Theorem. The groups S_4 and S_5 and their normal subgroups. Simplicity of A_5. [1 lecture]

Second and Third Isomorphism Theorems. Automorphisms. Semidirect products. [1 lecture]

Simplicity, composition series and Jordan-Hölder Theorem (finite groups only); examples. [See Peter J. Cameron, *Introduction to Algebra* (Oxford University Press, 1998), section 7.1.3 and 7.1.4, pages 185-187.] [1 lecture]

Finite soluble groups; subgroups, quotients and extensions. Insolubility of S_n for $n > 4$. [1 lecture]

Actions of groups on sets, equivalence of actions, examples including coset spaces and conjugation actions.[$1\frac{1}{2}$ lectures]

Orbits, transitivity, stabilisers, kernels of actions, equivalence of a transitive action with a coset space. Examples, including Möbius groups and symmetry groups of Platonic solids. [2 $\frac{1}{2}$ lectures]

Reading

P.J. Cameron *Introduction to Algebra* (2nd. ed., OUP, 2008) pp. 124-146, 237-250.

Further Reading

Peter M. Neumann, G. A. Stoy, E. C. Thompson, *Groups and Geometry* (OUP, 1994, reprinted 2002), ISBN 0-19-853451-5. Chapters 1-9, 15.

Geoff Smith, Olga Tabachnikova, *Topics in Groups Theory* (Springer Undergraduate Mathematics Series, 2002) ISBN 1-85233-2. Chapter 3.

M. A. Armstrong, *Groups and Symmetry* (Springer, 1988), ISBN 0-387-96675-7. Chapters 1-19.

Joseph J. Rotman, *A First Course in Algebra* (Second Edition, Prentice Hall, 2000). Chapter 2.

3.2.3 Number Theory — Prof. Tillmann — 8 lectures TT

Overview

Number theory is one of the oldest parts of mathematics. For well over two thousand years it has attracted professional and amateur mathematicians alike. Although notoriously ‘pure’ it has turned out to have more and more applications as new subjects and new technologies have developed. Our aim in this course is to introduce students to some classical and important basic ideas of the subject.

Synopsis

The ring of integers; congruences; ring of integers modulo n; the Chinese Remainder Theorem. [2 lectures]

Wilson’s Theorem; Fermat’s Little Theorem for prime modulus; Euler’s generalisation of Fermat’s Little Theorem to arbitrary modulus; primitive roots. [2 lectures]

Quadratic residues modulo primes. Quadratic reciprocity. [2 lectures]

Factorisation of large integers; basic version of the RSA encryption method. [2 lectures]

Reading

Alan Baker, *A Concise Introduction to the Theory of Numbers* (Cambridge University Press, 1984) ISBN: 0521286549 Chapters 1,3,4.

David Burton, *Elementary Number Theory* (McGraw-Hill, 2001).

Dominic Welsh, *Codes and Cryptography*, (Oxford University Press, 1988), ISBN 0-19853-287-3. Chapter 11.

3.2.4 Integration — Prof. Etheridge — 16 lectures HT

Overview

The course will exhibit Lebesgue's theory of integration in which integrals can be assigned to a huge range of functions on the real line, thereby greatly extending the notion of integration presented in Mods. The theory will be developed in such a way that it can be easily extended to a wider framework including summation of series and probability theory (although no knowledge of probability will be required), but measures other than Lebesgue's will only be lightly touched.

Operations such as passing limits, infinite sums, or derivatives, through integral signs, or reversing the order of double integrals, are often taken for granted in courses in applied mathematics. Actually, they can occasionally fail. Fortunately, there are powerful convergence and other theorems allowing such operations to be justified under conditions which are widely applicable. The course will display these theorems and a wide range of their applications.

This is a course in rigorous applications. Its principal aim is to develop understanding of the statements of the theorems and how to apply them carefully. Knowledge of technical proofs concerning the construction of Lebesgue measure and the integral will not be an essential part of the course, and such proofs will usually be omitted from the lectures.

Synopsis

Motivation: Why do we need a more general theory of integration?
The notion of measure.
Key examples: Lebesgue measure, probability measure, counting measure.
Measurable functions, integrable functions (via simple functions). Reconciliation with Mods Analysis III. Changes of variable.
Comparison Theorem.
Fatou's Lemma.
Monotone Convergence Theorem.
Dominated Convergence Theorem.
Corollaries and applications of the Convergence Theorems (term-by-term integration of series etc). Differentiation under the integral sign.
Double integrals, theorems of Fubini and Tonelli, changes of variable.
A very brief introduction to L^p spaces. Hölder and Minkowski inequalities.

Reading

A. Etheridge, *Integration,* Mathematical Institute Lecture Notes

M. Capinski & E. Kopp, *Measure, Integral and Probability* (Second Edition, Springer, 2004).

F. Jones, *Lebesgue Integration on Euclidean Space* (Second Edition, Jones & Bartlett, 2000).

Further Reading

R. G. Bartle, *The Elements of Integration* (Wiley, 1966).

D. S. Kurtz & C. W. Swartz, *Theories of Integration* (Series in Real Analysis Vol.9, World Scientific, 2004).

H. A. Priestley, *Introduction to Integration* (OUP 1997).

[Useful for worked examples, although adopts a different approach to construction of the integral].

H. L. Royden, *Real Analysis* (Third Edition, Macmillan, 1988).

E. M. Stein & R. Shakarchi, *Real Analysis: Measure Theory, Integration and Hilbert Spaces* (Princeton Lectures in Analysis III, Princeton University Press,2005).

3.2.5 Topology — Prof. Lackenby — 16 lectures HT

Overview

The ideas, concepts and constructions in general topology arose from extending the notions of continuity and convergence on the real line to more general spaces. The first class of general spaces to be studied in this way were metric spaces, a class of spaces which includes many of the spaces used in analysis and geometry. Metric spaces have a distance function which allows the use of geometric intuition and gives them a concrete feel. They allow us to introduce much of the vocabulary used later and to understand the formulation of continuity which motivates the axioms in the definition of an abstract topological space.

The axiomatic formulation of a topology leads to topological proofs of simplicity and clarity often improving on those given for metric spaces using the metric and sequences. There are many examples of topological spaces which do not admit metrics and it is an indication of the naturality of the axioms that the theory has found so many applications in other branches of mathematics and spheres in which mathematical language is used.

Learning Outcomes

The outcome of the course is that a student should understand and appreciate the central results of general topology and metric spaces, sufficient for the main applications in geometry, number theory, analysis and mathematical physics, for example.

Synopsis

Metric spaces. Examples to include metrics derived from a norm on a real vector space, particularly l^1, l^2, l^∞ norms on $\mathbb{R}^n$, the *sup* norm on the bounded real-valued functions on a set, and on the bounded continuous real-valued functions on a metric space. Continuous functions (ϵ, δ definition). Uniformly continuous functions; examples include Lipschitz functions and contractions. Open balls, open sets, accumulation points of a set. Completeness (but not completion). Contraction Mapping Theorem. Completeness of the space of bounded real-valued functions on a set, equipped with the *sup* norm, and the completeness of the space of bounded continuous real-valued functions on a metric space, equipped with the *sup* metric. Example: completion of a metric space. [3 lectures].

Axiomatic definition of an abstract topological space in terms of open sets. Continuous functions, homeomorphisms. Closed sets. Accumulation points of sets. Closure of a set ($\bar{A} = A$ together with its accumulation points). Interior of a set. Continuity if $f(\bar{A}) \subseteq \overline{f(A)}$. Examples to include metric spaces (definition of topological equivalence of metric spaces), discrete and indiscrete topologies, subspace topology, cofinite topology, quotient topology. Base of a topology. Product topology on a product of two spaces and continuity of projections. Hausdorff topology. [5 lectures]

Connected spaces: closure of a connected space is connected, union of connected sets is connected if there is a non-empty intersection, continuous image of a connected space is connected. Path-connectedness implies connectedness. Connected open subset of a normed vector space is path-connected. [2 lectures]

Compact sets, closed subset of a compact set is compact, compact subset of a Hausdorff space is closed. Heine-Borel Theorem in $\mathbb{R}^n$. Product of two compact spaces is compact. A continuous bijection from a compact space to a Hausdorff space is a homeomorphism. Equivalence of sequential compactness and abstract compactness in metric spaces. [4 lectures]

Further discussion of quotient spaces explaining some simple classical geometric spaces such as the torus and Klein bottle. [2 lectures]

Reading

W. A. Sutherland, *Introduction to Metric and Topological Spaces* (Oxford University Press, 1975). Chapters 2-6, 8, 9.1-9.4.
(New edition to appear shortly.)

J. R. Munkres, *Topology, A First Course* (Prentice Hall, 1974), chapters 2, 3, 7.

Further Reading

B. Mendelson, *Introduction to Topology* (Allyn and Bacon, 1975). (cheap paperback edition available).

G. Buskes, A. Van Rooij, *Topological Spaces* (Springer, 1997).

N. Bourbaki, *General Topology* (Springer, 1998).

J. Dugundji, *Topology* (Allyn and Bacon, 1966), chapters 3, 4, 5, 6, 7, 9, 11. [Although out of print, available in some libraries.]

3.2.6 Multivariable Calculus — Prof. Drutu — 8 lectures TT

Overview

In this course, the notion of the total derivative for a function $f\colon \mathbb{R}^m \to \mathbb{R}^n$ is introduced. Roughly speaking, this is an approximation of the function near each point in $\mathbb{R}^n$ by a linear transformation. This is a key concept which pervades much of mathematics, both pure and applied. It allows us to transfer results from linear theory locally to nonlinear functions. For example, the Inverse Function Theorem tells us that if the derivative is an invertible linear mapping at a point then the function is invertible in a neighbourhood of this point. Another example is the tangent space at a point of a surface in $\mathbb{R}^3$, which is the plane that locally approximates the surface best.

Synopsis

Definition of a derivative of a function from $\mathbb{R}^m$ to $\mathbb{R}^n$; examples; elementary properties; partial derivatives; the chain rule; the gradient of a function from $\mathbb{R}^m$ to $\mathbb{R}$; Jacobian. Continuous partial derivatives imply differentiability, Mean Value Theorems. Higher order derivatives. [3 lectures]

The Inverse Function Theorem and the Implicit Function Theorem (proofs non-examinable). [2 lectures]

The definition of a submanifold of $\mathbb{R}^m$. Its tangent and normal space at a point, examples, including two-dimensional surfaces in $\mathbb{R}^3$. [2 lectures]

Lagrange multipliers. [1 lecture]

Reading

Theodore Shifrin, *Multivariable Mathematics* (Wiley, 2005). Chapters 3-6.

T. M. Apostol, *Mathematical Analysis: Modern Approach to Advanced Calculus (World Students)* (Addison Wesley, 1975). Chapters 12 and 13.

S. Dineen, *Multivariate Calculus and Geometry* (Springer, 2001). Chapters 1-4.

J. J. Duistermaat and J A C Kolk, *Multidimensional Real Analysis I, Differentiation* (Cambridge University Press, 2004).

Further Reading

William R. Wade, *An Introduction to Analysis* (Second Edition, Prentice Hall, 2000). Chapter 11.

M. P. Do Carmo, *Differential Geometry of Curves and Surfaces* (Prentice Hall, 1976).

Stephen G. Krantz and Harold R. Parks, *The Implicit Function Theorem: History, Theory and Applications* (Birkhaeuser, 2002).

Option from Honour Moderations in Mathematics

3.2.7 Probability I — Dr Laws — 8 MT

Overview

An understanding of random phenomena is becoming increasingly important in today's world within social and political sciences, finance, life sciences and many other fields. The aim of this introduction to probability is to develop the concept of chance in a mathematical framework. Discrete random variables are introduced, with examples involving most of the common distributions.

Learning Outcomes

Students should have a knowledge and understanding of basic probability concepts, including conditional probability. They should know what is meant by a random variable, and have met the common distributions and their probability mass functions. They should understand the concepts of expectation and variance of a random variable. A key concept is that of independence which will be introduced for events and random variables.

Synopsis

Motivation, relative frequency, chance. (What do we mean by a 1 in 4 chance?) Sample space as the set of all possible outcomes—examples. Events and the probability function. Permutations and combinations, examples using counting methods, sampling with or without replacement. Algebra of events. Conditional probability, partitions of sample space, theorem of total probability, Bayes's Theorem, independence.

Random variable. Probability mass function. Discrete distributions: Bernoulli, binomial, Poisson, geometric; situations in which these distributions arise. Expectation: mean and variance. Probability generating functions, use in calculating expectations. Bivariate discrete distribution, conditional and marginal distributions. Extensions to many random variables. Independence for discrete random variables. Conditional expectation. Solution of linear and quadratic difference equations with applications to random walks.

Main Reading

1. D. Stirzaker, *Elementary Probability* (Cambridge University Press, 1994), Chapters 1–4, 5.1–5.6, 6.1–6.3, 7.1, 7.2, 7.4, 8.1, 8.3, 8.5 (excluding the joint generating function).

2. D. Stirzaker, *Probability and Random Variables: A Beginner's Guide* (Cambridge University Press, 1999).

Further Reading

1. J. Pitman, *Probability* (Springer-Verlag, 1993).

2. S. Ross, *A First Course In Probability* (Prentice-Hall, 1994).

3. G. R. Grimmett and D. J. A. Welsh, *Probability: An Introduction* (Oxford University Press, 1986), Chapters 1–4, 5.1–5.4, 5.6, 6.1, 6.2, 6.3 (parts of), 7.1–7.3, 10.4.

3.3 Probability II — Dr Marchini — 8 HT

Learning Outcomes

Students should have a knowledge and understanding of basic probability concepts, including conditional probability. They should know what is meant by a random variable, and have met the common distributions, and their probability density functions. They should understand the concepts of expectation and variance of a random variable. A key concept is that of independence which will be introduced for events and random variables. The emphasis in this course is a continuation of discrete variables studied in Probability I, followed by continuous random variables, with examples involving the common distributions.

Synopsis

Random walks (finite state space only). Expectations of functions of more than one random variable. Random sample. Conditional expectation, application of theorem of total probability to expectation of a random variable. Sums of independent random variables. Examples from well-known distributions.

Continuous random variables, motivation. Cumulative distribution functions for both discrete and continuous random variables. Probability density function - analogy with mass and density of matter. Examples: uniform, exponential, gamma, normal. Practical examples. Expectation. Continuous distribution functions and probability density functions for functions of a single continuous random variable. Simple examples of joint distributions of two or more continuous random variables; independence, expectation (mean and variance of sums of independent, identically distributed random variables).

Main Reading

1. D. Stirzaker, *Elementary Probability* (Cambridge University Press, 1994), Chapters 1–4, 5.1–5.6, 6.1–6.3, 7.1, 7.2, 7.4, 8.1, 8.3, 8.5 (excluding the joint generating function).

2. D. Stirzaker, *Probability and Random Variables: A Beginner's Guide* (Cambridge University Press, 1999).

Further Reading

1. J. Pitman, *Probability* (Springer-Verlag, 1993).

2. S. Ross, *A First Course In Probability* (Prentice-Hall, 1994).

3. G. R. Grimmett and D. J. A. Welsh, *Probability: An Introduction* (Oxford University Press, 1986), Chapters 1–4, 5.1–5.4, 5.6, 6.1, 6.2, 6.3 (parts of), 7.1–7.3, 10.4.