Multivariate Quadratic Public-Key Cryptography In the NIST Competition

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Multivariate Cryptography

MPKC: Multivariate (Quadratic) Public Key Cryptosystem
Public Key: System of nonlinear multivariate equations

$$
\begin{aligned}
p^{(1)}(w_1,\ldots,w_n) &= \sum_{i=1}^{n}\sum_{j=i}^{n} p_{ij}^{(1)}\cdot w_i w_j \quad + \quad \sum_{i=1}^{n} p_i^{(1)}\cdot w_i \left(+p_0^{(1)}\right)\\
p^{(2)}(w_1,\ldots,w_n) &= \sum_{i=1}^{n}\sum_{j=i}^{n} p_{ij}^{(2)}\cdot w_i w_j \quad + \quad \sum_{i=1}^{n} p_i^{(2)}\cdot w_i \left(+p_0^{(2)}\right)\\
&\vdots\\
p^{(m)}(w_1,\ldots,w_n) &= \sum_{i=1}^{n}\sum_{j=i}^{n} p_{ij}^{(m)}\cdot w_i w_j \quad + \quad \sum_{i=1}^{n} p_i^{(m)}\cdot w_i \left(+p_0^{(m)}\right)
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$$

If degree d then Public Key size $= m\binom{n+d}{d}$, hence usually $d = 2$.

Security

The security of multivariate schemes is based on the

Problem MQ: Given m multivariate quadratic polynomials $p^{(1)}, \ldots, p^{(m)}$, find a vector $\mathbf{w} = (w_1, \ldots, w_n)$ such that $p^{(1)}(\mathbf{w}) = \ldots = p^{(m)}(\mathbf{w}) = 0$.

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- NP hard
- believed to be hard on average (even for quantum conputers): suppose we have a probabilistic Turing Machine T and a subexponential function η, T terminates with an answer to a random $MQ(n, m = an, \mathbb{F}_q)$ instance in time $\eta(n)$ with probability $\mathrm{negl}(n)$.
- higher order versions (MP for Multivariate Polynomials or PoSSo for Polynomial System Solving) clearly no less hard

However usually no direct reduction to MQ !! There are exceptions:

Identification Scheme of Sakumoto *et al* and MQDSS

An example 5-pass ID scheme depending only on MQ

- $\mathcal{P}$ be a set of random MQ polynomials
- Its "polar" form $D\mathcal{P}(\mathbf{x},\mathbf{y}) := \mathcal{P}(\mathbf{x}+\mathbf{y}) - \mathcal{P}(\mathbf{x}) - \mathcal{P}(\mathbf{y}) - \mathcal{P}(\mathbf{0})$
- $\mathcal{P}(\mathbf{s}) = \mathbf{p}$ is the public key, $\mathbf{s}$ is the secret.
- Peter picks and commits random $(\mathbf{r_0},\mathbf{t_0},\mathbf{e_0})$, sets $\mathbf{r_1} = \mathbf{s} - \mathbf{r_0}$ and commits $(\mathbf{r_1}, D\mathcal{P}(\mathbf{t_0},\mathbf{r_1}) + \mathbf{e_0})$.
- Vera sends random α,
- Peter sets and sends $\mathbf{t_1} := \alpha\mathbf{r_0} - \mathbf{t_0}$, $\mathbf{e_1} := \alpha\mathcal{P}(\mathbf{r_0}) - \mathbf{e_0}$.
- Vera sends challenge Ch, Peter sends $\mathbf{r}_{Ch}$.
- Vera checks the commit of either $(\mathbf{r_0}, \alpha\mathbf{r_0} - \mathbf{t_1}, \alpha\mathcal{P}(\mathbf{r_0}) - \mathbf{e_1})$ or $(\mathbf{r_1}, \alpha(\mathbf{p} - \mathcal{P}(\mathbf{r_1})) - D\mathcal{P}(\mathbf{t_1},\mathbf{r_1}) - \mathbf{e_1})$.

The Fiat-Shamir transform of this ID scheme is the MQDSS scheme.

Bipolar Construction

- Easily invertible quadratic map $\mathcal{Q} : \mathbb{F}^n \to \mathbb{F}^m$
- Two invertible linear maps $\mathcal{T}(: \mathbb{F}^m \to \mathbb{F}^m)$ and $\mathcal{S}(: \mathbb{F}^n \to \mathbb{F}^n)$
- *Public key*: $\mathcal{P} = \mathcal{T} \circ \mathcal{Q} \circ \mathcal{S}$ supposed to look random
- *Private key*: $\mathcal{S}, \ \mathcal{Q}, \ \mathcal{T}$ allows to invert the public key

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Encryption Schemes ($m \geq n$)

- Triangular schemes, ZHFE (broken)
- PMI+, IPHFE+
- Simple Matrix (not highly thought of)

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Signature Schemes ($m \leq n$)

- Unbalanced Oil and Vinegar
  - Rainbow (TTS)
- HFEv- (QUARTZ/Gui)
- pFLASH

NIST Candidates

Digital Signature Schemes (4 into second round)

- Transformed Zero-Knowledge: MQDSS
- HFEv-: GUI, GeMSS, DualModeMS
- Small Field: Rainbow, L(ifted)UOV, HiMQ3 (a version of TTS)

Encryption Schemes

- SRTPI (broken)
- DME (dubious)
- CFPKM (Polly Cracker)

Workflow

Decryption / Signature Generation

$$\mathbf{z} \in \mathbb{F}^m \xrightarrow{\mathcal{T}^{-1}} \mathbf{y} \in \mathbb{F}^m \xrightarrow{\mathcal{Q}^{-1}} \mathbf{x} \in \mathbb{F}^n \xrightarrow{\mathcal{S}^{-1}} \mathbf{w} \in \mathbb{F}^n$$

$$\mathbf{w} \in \mathbb{F}^n \xrightarrow{\mathcal{P}} \mathbf{z} \in \mathbb{F}^m$$

Encryption / Signature Verification

Isomorphism of Polynomials

Due to the bipolar construction, the security of MPKCs is also based on the

Problem EIP (Extended Isomorphism of Polynomials): Given the public key $\mathcal{P}$ of a multivariate public key cryptosystem, find affine maps $\bar{\mathcal{S}}$ and $\bar{\mathcal{T}}$ as well as quadratic map $\bar{\mathcal{Q}}$ in class $\mathcal{C}$ such that $\mathcal{P} = \bar{\mathcal{T}} \circ \bar{\mathcal{Q}} \circ \bar{\mathcal{S}}$.

$\Rightarrow$ Hardness of problem depends much on the structure of the central map
$\Rightarrow$ Often EIP is really (a not so hard) MinRank
$\Rightarrow$ In general, not much is known about the complexity
$\Rightarrow$ Security analysis of multivariate schemes is a hard task

Generic (Direct) Attacks

Try to solve the public equation $\mathcal{P}(\mathbf{w}) = \mathbf{z}$ as an instance of the MQ-Problem, all algorithms have exponential running time (for $m \approx n$)

Known Best Generic Algorithms

- For larger q, FXL ("Hybridized XL" can Groverize)
- For $q = 2$, smart enumerative methods

Generic (Direct) Attacks

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Known Best Generic Algorithms

- For larger q, FXL ("Hybridized XL" can Groverize)
- For $q = 2$, Joux-Vitse's XL-with-enumeration Variant.

Complexity of Direct Attacks

How many equations are needed to meet given levels of security?

security level (bit)	number of equations			
	$\mathbb{F}_2$ *	$\mathbb{F}_{16}$	$\mathbb{F}_{31}$	$\mathbb{F}_{256}$
80	88	30	28	26
100	110	39	36	33
128	140	51	48	43
192	208	80	75	68
256	280	110	103	93

* depending on how we model the Joux-Vitse algorithm

XL Algorithm (Lazard, 1983; CKPS, 1999)

Given: nonlinear polynomials $f_1, \ldots, f_m$ of degree d

1. **eXtend** multiply each polynomial $f_1, \ldots, f_m$ by every monomial of degree $\leq D - d$
2. **Linearize**: Apply (sparse) linear algebra to solve the extended system

$$\text{Complexity} = 3 \cdot \binom{n + d_{\text{XL}}}{d_{\text{XL}}}^2 \cdot \binom{n}{d} \qquad \text{(for larger } q\text{)}$$

   or

2. or **Linearize and use an improved XL**: Many variants. . .

XL Variants

FXL – XL with k variables guessed or "hybridized"

if with k initial guesses / fixing / "hybridization":

$$\text{Complexity} = \min_k 3q^k \cdot \binom{n-k+d_{\mathrm{XL}}}{d_{\mathrm{XL}}}^2 \cdot \binom{n-k}{d}.$$

[generic method with the best asymptotic multiplicative complexity].

XL Variants

FXL – XL with k variables guessed or "hybridized"

Joux-Vitse ("Hybridized XL-related method")

1. **eXtend:** multiply each polynomial $f_1, \ldots, f_m$ by monomials, up to total degree $\leq D$
2. **Linearize**: Apply linear algebra to eliminate all monomials of total degree ≥ 2 in the first k variables (and get at least k such equations).
3. **Fix** $n - k$ variables, solve for the initial k in linear equations.

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XL2 – simplified F_4

1. **eXtend:** multiply each polynomial $f_1, \ldots, f_m$ by monomials, up to total degree $\leq D$
2. **Linearize**: Apply linear algebra to eliminate top level monomials
3. Multiply degree $D - 1$ equations by variables, **Eliminate Again**.

More Advanced Gröbner Bases Algorithms

- find a "nice" basis of the ideal $\langle f_1, \ldots, f_m \rangle$
- first studied by B. Buchberger
- later improved by Faugére et al. (F_4, F_5)
- With linear algebra constant $2 < \omega \leq 3$.

$$\mathrm{Complexity}(q, m, n) = O\left(\binom{n + d_{\mathrm{reg}} - 1}{d_{\mathrm{reg}}}^{\omega} \right) \qquad \text{(for larger } q\text{)}$$

- Can also be "Hybridized":

$$\mathrm{Complexity}(q, m, n) = \min_k \; q^k \cdot O\left(\binom{n - k + d_{\mathrm{reg}} - 1}{d_{\mathrm{reg}}}^{\omega} \right)$$

- Runs at the same degree as XL2.

Do not blithely set $\omega = 2$ here

Even if $\omega \to 2$, there is a huge constant factor which cannot be neglected.

Remarks

Every cryptosystem can be represented as a set of nonlinear multivariate equations

- Direct attacks can be used in the cryptanalysis of other cryptographic schemes (in particular block and stream ciphers)
- The MQ (or PoSSo) Problem can be seen as one of the central problems in cryptography

Post-Quantum-ness of MQ

- A Grover attack against n-bit-input MQ takes $2^{\frac{n}{2}+1}n^3$ time.
- A Hybridized XL with Grover for enumeration on n boolean variables and as many equations still takes $2^{(0.471+o(1))n}$ in true (time-area) cost

Features of Multivariate Cryptosystems

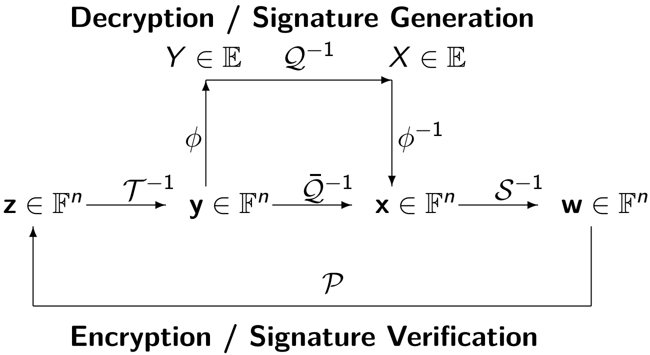
Advantages

- resistant against attacks with quantum computers
- reasonably fast
- only simple arithmetic operations required
  $\Rightarrow$ can be implemented on low cost devices
  $\Rightarrow$ suitable for security solutions for the IoT
- many practical signature schemes (UOV, Rainbow, HFEv-, ...)
- short signatures (e.g. 120 bit signatures for 80 bit security)

Disadvantages

- large key sizes (public key size $\sim 10 - 100$ kB)
- no security proofs
- mainly restricted to digital signatures

Big Field Schemes



Extension Fields

- $\mathbb{F}_q$: finite field with q elements
- $g(X)$ irreducible polynomial in $\mathbb{F}[X]$ of degree n
  $\Rightarrow \mathbb{F}_{q^n} \cong \mathbb{F}[X]/\langle g(X) \rangle$ finite field with q^n elements
- isomorphism $\phi : \mathbb{F}_q^n \to \mathbb{F}_{q^n}$, $(a_1, \ldots, a_n) \mapsto \sum_{i=1}^n a_i \cdot X^{i-1}$
- Addition in $\mathbb{F}_{q^n}$: Addition in $\mathbb{F}_q[X]$
- Multiplication in $\mathbb{F}_{q^n}$: Multiplication in $\mathbb{F}_q[X]$ modulo $g(X)$

The Matsumoto-Imai Cryptosystem (1988) or C^*

- $\mathbb{F}_q$: finite field of characteristic 2
- degree n extension field $\mathbb{E} = \mathbb{F}_{q^n}$
- isomorphism $\phi : \mathbb{F}_q^n \to \mathbb{E}$
- C^* parameter $\theta \in \mathbb{N}$ with

$$\gcd(q^\theta + 1, q^n - 1) = 1.$$

Key Generation

- *central map* $\mathcal{Q} : \mathbb{E} \to \mathbb{E}$, $X \mapsto X^{q^\theta+1} \Rightarrow \mathcal{Q}$ is bijective
- choose 2 invertible linear or affine maps $\mathcal{S}, \mathcal{T} : \mathbb{F}^n \to \mathbb{F}^n$
- *public key*: $\mathcal{P} = \mathcal{T} \circ \phi^{-1} \circ \mathcal{Q} \circ \phi \circ \mathcal{S} : \mathbb{F}^n \to \mathbb{F}^n$ quadratic multivariate map
- use the extended Euclidian algorithm to compute $h \in \mathbb{N}$ with

$$h \cdot \theta \equiv 1 \bmod q^n - 1$$

- *private key*: $\mathcal{S}, \mathcal{T}$

Linearization Attack against C^*

Given public key $\mathcal{P}$, $\mathbf{z}^\star \in \mathbb{F}^n$, find plaintext $\mathbf{w}^\star \in \mathbb{F}^n$, s.t. $\mathcal{P}(\mathbf{w}^\star) = \mathbf{z}^\star$

Proposed by J. Patarin in 1995

Taking the $q^\theta - 1$ st power of $Y = X^{q^\theta+1}$ and multiplying with XY yields

$$X \cdot Y^{q^\theta} = X^{q^{2\theta}} \cdot Y$$

$\Rightarrow$ bilinear equation in X and Y, hence, same in $\mathbf{w}$ and $\mathbf{z}$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{ij} w_i z_j + \sum_{i=1}^{n} \beta_i w_i + \sum_{j=1}^{n} \gamma_j z_j + \delta = 0. \quad (\star)$$

1. Compute $N \geq \frac{(n+1)\cdot(n+2)}{2}$ pairs $(\mathbf{z}^{(k)}/\mathbf{w}^{(k)})$ and substitute into $(\star)$.
2. Solve the resulting linear system for the coefficients α_{ij}, β_i, γ_j and δ.
   $\Rightarrow$ n bilinear equations in $w_1, \ldots, w_n, z_1, \ldots, z_n$
3. Substitute $\mathbf{z}^\star$ into these bilinear equations and solve for $\mathbf{w}^\star$.

pFLASH: Prefixed C^{*-}signature scheme

Natural restriction of $\mathcal{Q}$ to hyperplane = set coordinate to 0

Start from a C^* scheme with $\mathcal{Q}(x) = x^{1+q^\theta}$ with secret linear maps S and T. Let r and s be two integers between 0 and n. Let T^- be the projection of T on the last r coordinates and S^- be the restriction of S to the first $n-s$ coordinates. $\mathcal{P} = T^- \circ \mathcal{Q} \circ S^-$ is the public key and S^{-1} and T^{-1} are the secret key. This is pFLASH($\mathbb{F}_q$,$n-s$,$n-r$).

Inversion

To find $\mathcal{P}^{-1}(m)$ for $m \in \mathbb{F}_q^{n-r}$, the legitimate user first pads m randomly into vector $m' \in (\mathbb{F})^n$ and compute $T^{-1} \circ \mathcal{Q}^{-1} \circ S^{-1}(m')$. Repeat until this element has its last s coordinates to 0. Its $n-s$ first coordinates are a valid signature for m. When $r > s$, the process ends with probability 1 and costs on average q^s inversions of $\mathcal{Q}$.

pFLASH Parameters at NIST Cat. I-II

Suggested pFLASH($\mathbb{F}_{16}$,96-1,64) (146 kB pubkey, 6 kB prvkey).

The HFE Cryptosystem

- " Hidden Field Equations", proposed by Patarin in 1995
- BigField Scheme, can be used both for encryption and signatures
- finite field $\mathbb{F}$, extension field $\mathbb{E}$ of degree n, isomorphism $\phi : \mathbb{F}^n \to \mathbb{E}$

Original HFE

- central map $\mathcal{Q} : \mathbb{E} \to \mathbb{E}$ (not bijective, invert using Berlekamp Algorithm).

$$\mathcal{Q}(X) = \sum_{0 \le i \le j}^{q^i + q^j \le D} \alpha_{ij} X^{q^i + q^j} + \sum_{i=0}^{q^i \le D} \beta_i \cdot X^{q^i} + \gamma$$

$\Rightarrow \bar{\mathcal{Q}} = \phi^{-1} \circ \mathcal{Q} \circ \phi : \mathbb{F}^n \to \mathbb{F}^n$ quadratic

- degree bound D needed for efficient decryption / signature generation
- linear maps $\mathcal{S}, \mathcal{T} : \mathbb{F}^n \to \mathbb{F}^n$
- *public key*: $\mathcal{P} = \mathcal{T} \circ \bar{\mathcal{Q}} \circ \mathcal{S} : \mathbb{F}^n \to \mathbb{F}^n$
- *private key*: $\mathcal{S}$, $\mathcal{Q}$, $\mathcal{T}$

MinRank Attack against HFE

Look in extension field $\mathbb{E}$ (Kipnis and Shamir [KS99])

- the linear maps $\mathcal{S}$ and $\mathcal{T}$ relate to univariate maps $\mathcal{S}^\star(X) = \sum_{i=1}^{n-1} s_i \cdot X^{q^i}$ amd $\mathcal{T}^\star(X) = \sum_{i=1}^{n-1} t_i \cdot X^{q^i}$, with $s_i,\ t_i \in \mathbb{E}$.
- the public key $\mathcal{P}^\star$ can be expressed as $\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} p_{ij}^\star X^{q^i+q^j} = \underline{X} \cdot P^\star \cdot \underline{X}^T$,
- Components of $P^\star$ can be found by polynomial interpolation.
- Solve MinRank problem over $\mathbb{E}$.

No need to look in $\mathbb{E}$ (Bettale et al)

Perform the MinRank attack without recovering $\mathcal{P}^\star \Rightarrow$ HFE can be broken by using a MinRank problem over the base field $\mathbb{F}$.

$$\mathrm{Complexity}_{\mathrm{MinRank}} = \binom{n+r}{r}^{\omega}$$

with $2 < \omega \leq 3$ and $r = \lfloor \log_q(D-1) \rfloor + 1$.

Direct Attacks

- J-C Faugère solved HFE Challenge 1 (HFE over GF2, $d = 96$) in 2002
- Empirically HFE systems can be solved much faster than random
- Ding-Hodges Upper bound for d_{reg}

$$d_{reg} \leq \begin{cases} \frac{(q-1)\cdot(r-1)}{2} + 2 & q \text{ even and } r \text{ odd}, \\ \frac{(q-1)\cdot r}{2} + 2 & \text{otherwise}. \end{cases},$$

with $r = \lfloor \log_q(D-1) \rfloor + 1$.

$\Rightarrow$ Basic version of HFE is not secure

Variant Schemes

- Encryption Schemes IPHFE+ (inefficient), ZHFE (broken).
- Signature Schemes HFEv- (QUARTZ/GUI), MHFEv- (broken)

HFEv-

- finite field $\mathbb{F}$, extension field $\mathbb{E}$ of degree n, isomorphism $\phi : \mathbb{F}^n \to \mathbb{E}$
- central map $\mathcal{Q} : \mathbb{F}^v \times \mathbb{E} \to \mathbb{E}$, where the β_i and γ are affine.

$$\mathcal{Q}(X) = \sum_{0 \le i \le j}^{q^i + q^j \le D} \alpha_{ij} X^{q^i + q^j} + \sum_{i=0}^{q^i \le D} \beta_i(v_1, \ldots, v_v) \cdot X^{q^i} + \gamma(v_1, \ldots, v_v)$$

$\Rightarrow \bar{\mathcal{Q}} = \phi^{-1} \circ \mathcal{Q} \circ (\phi \times \mathrm{id}_v)$ quadratic map: $\mathbb{F}^{n+v} \to \mathbb{F}^n$

- linear maps $\mathcal{T} : \mathbb{F}^n \to \mathbb{F}^{n-a}$ and $\mathcal{S} : \mathbb{F}^{n+v} \to \mathbb{F}^{n+v}$ of maximal rank
- *public key*: $\mathcal{P} = \mathcal{T} \circ \bar{\mathcal{Q}} \circ \mathcal{S} : \mathbb{F}^{n+v} \to \mathbb{F}^{n-a}$
- *private key*: $\mathcal{S}$, $\mathcal{Q}$, $\mathcal{T}$

Signing Message digest $\mathbf{z}$

1. Compute $\mathbf{y} = \mathcal{T}^{-1}(\mathbf{z}) \in \mathbb{F}^n$ and $Y = \phi(\mathbf{y}) \in \mathbb{E}$
2. Choose random values for the vinegar variables $v_1, \ldots, v_v$
   Solve $\mathcal{Q}_{v_1,\ldots,v_v}(X) = Y$ over $\mathbb{E}$
   Can Repeat first step of Berlekamp until there is a unique solution.
3. Compute $\mathbf{x} = \phi^{-1}(X) \in \mathbb{F}^n$ and signature $\mathbf{w} = \mathcal{S}^{-1}(\mathbf{x}||v_1||\ldots||v_v)$.

Security vs. Efficiency

Main Attacks

- MinRank Attack $\mathrm{Rank}(F) = r + a + v$

  $\Rightarrow \mathrm{Compl}_{\mathrm{MinRank}} = \binom{n+r+a+v}{r+a+v}^{\omega}$
- Direct attack [DY13]

$$d_{reg} \leq \begin{cases} \frac{(q-1)\cdot(r+a+v-1)}{2} + 2 & q \text{ even and } r+a \text{ odd,} \\ \frac{(q-1)\cdot(r+a+v)}{2} + 2 & \text{otherwise.} \end{cases},$$

with $r = \lfloor \log_q(D-1) \rfloor + 1$ and $2 < \omega \leq 3$.

Efficiency

Rate determining step: solving X from a univariate equation of degree D.

$$\mathrm{Complexity}_{\mathrm{Berlekamp}} = \mathcal{O}(D^3 + n \cdot D^2)$$

How to define a HFEv- like scheme over $\mathbb{F}_2$ [PCY+15]?

Collision Resistance of the hash function

To cover a hash value of k bit, the public key of a pure HFEv- scheme has to contain at least k equations over $\mathbb{F}_2$. $\Rightarrow$ public key $> k^3/2$ bits

security level	80	100	128	192	256
# equations	100	200	256	384	512
pubkey size (kB)	>250	> 500	> 1000	> 3000	> 8000

QUARTZ

- standardized by Courtois, Patarin in 2002
- HFEv$^-$ with $\mathbb{F} = \mathrm{GF}(2)$, $n = 103$, $D = 129$, $a = 3$ and $v = 4$
- public key: quadratic map $\mathcal{P} = \mathcal{T} \circ \mathcal{Q} \circ \mathcal{S} : \mathrm{GF}(2)^{107} \to \mathrm{GF}(2)^{100}$
- Prevent birthday attacks $\Rightarrow$ Generate four HFEv$^-$ signatures
  (for **w**, $\mathcal{H}(\mathbf{w}|00)$, $\mathcal{H}(\mathbf{w}|01)$ and $\mathcal{H}(\mathbf{w}|11)$)
- Combine them to a single signature of length
  $(n - a) + 4 \cdot (a + v) = 128$ bit

GeMSS, GUI (Generalized QUARTZ) Signature Generation

Input: HFEv- private key $(\mathcal{S}, \mathcal{Q}, \mathcal{T})$ message **d**, repetition factor k
Output: signature $\sigma \in \mathbb{F}_2^{(n-a)+k(a+v)}$

1: $\mathbf{h} \leftarrow \text{Hash}(\mathbf{d})$
2: $S_0 \leftarrow \mathbf{0} \in \mathrm{GF}(2)^{n-a}$
3: **for** $i = 1$ to k **do**
4: $\quad D_i \leftarrow$ first $n - a$ bits of $\mathbf{h}$
5: $\quad (S_i, X_i) \leftarrow \mathrm{HFEv}-^{-1}(D_i \oplus S_{i-1})$
6: $\quad \mathbf{h} \leftarrow \text{Hash}(\mathbf{h})$
7: **end for**
8: $\sigma \leftarrow (S_k || X_k || \ldots || X_1)$
9: **return** σ

Note that if any equation has zero (or more than 2 solutions for Gui), then we discard those vinegars and try again.

Signature Verification

Input: HFEv- public key $\mathcal{P}$, message $\mathbf{d}$, repetition factor k, signature $\sigma \in \mathbb{F}_2{}^{(n-a)+k(a+v)}$

Output: **TRUE** or **FALSE**

$\mathbf{h} \leftarrow \text{Hash}(\mathbf{d})$
$(S_k, X_k, \ldots, X_1) \leftarrow \sigma$
for $i = 1$ to k **do**
    $D_i \leftarrow$ first $n - a$ bits of $\mathbf{h}$
    $\mathbf{h} \leftarrow \text{Hash}(\mathbf{h})$
end for
for $i = k - 1$ to 0 **do**
    $S_i \leftarrow \mathcal{P}(S_{i+1} || X_{i+1}) \oplus D_{i+1}$
end for
if $S_0 = \mathbf{0}$ **then**
    return TRUE
else
    return FALSE
end if

Parameters for HFEv- (GeMSS,GUI) over $\mathbb{F}_2$?

Parameters are set by the complexity of MinRank and direct attacks

- For the complexity of the MinRank attack we have a concrete formula
- For the direct attack, we only have an upper bound on d_{reg}.

$$d_{reg} \leq \begin{cases} \frac{(q-1)\cdot(r+a+v-1)}{2} + 2 & q \text{ even and } r+a \text{ odd,} \\ \frac{(q-1)\cdot(r+a+v)}{2} + 2 & \text{otherwise.} \end{cases} \quad (\star)$$

Experiments show that these estimate for d_{reg} is reasonably tight.

Parameter Choice of HFEv- over $\mathbb{F}_2$

Aggressive $\Rightarrow$ Choose D as small as possible (GUI)

- $D = 5 \Rightarrow r = \lfloor \log_2(D-1) \rfloor + 1 = 3$
- $D = 9 \Rightarrow r = \lfloor \log_2(D-1) \rfloor + 1 = 4$
- $D = 17 \Rightarrow r = \lfloor \log_2(D-1) \rfloor + 1 = 5$

Increase a and v ($0 \leq v - a \leq 1$) to reach the required security level.
Conservate choice: choose $D = 513$ and n as needed (GeMSS).

Quantum Attacks and Impact

A determined multivariate system of m equations over $\mathbb{F}_2$ can be solved using $2^{m/2} \cdot 2 \cdot m^3$ operations using a quantum computer.

- This does not affect signatures in general because the hashes are typically twice as wide as the design security.
- Alas, this wipes out much of GUI's gains.

$\Rightarrow$ very large public key size

security level	80	100	128	192	256
min # equations	117	155	208	332	457

Proposed Parameters (Signature includes 128-bit salt)

NIST Category level (bit)	Parameters $\mathbb{F}_q, n, D, a, v, k$	public key size (kB)	private key size (kB)	signature size (bit)
I	Gui ($\mathbb{F}_2$,184,33,16,16,2)	416.3	19.1	360
III	Gui ($\mathbb{F}_2$,312,129,24,20,2)	1,955.1	59.3	504
V	Gui ($\mathbb{F}_2$,448,513,32,28,2)	5,789.2	155.9	664

Quantum Attacks and Impact

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Proposed Parameters (Signature includes 128-bit salt)

NIST Category level (bit)	Parameters $\mathbb{F}_q, n, D, \Delta, v, nb\_ite$	public key size (kB)	private key size (kB)	signature size (bit)
I	GeMSS ($\mathbb{F}_2$,174,513,12,12,4)	417	14.5	384
III	GeMSS ($\mathbb{F}_2$,265,513,22,20,4)	1,304	40.3	704
V	GeMSS ($\mathbb{F}_2$,354,513,30,33,4)	3,604	83.7	832

HFEv- - Summary

- short signatures
- security well respected
- conflict between security and efficiency
- restricted to very small fields, hence very large keys
- 109M cycles keygen, 676M cycles signing, about 107k cycles verifying at NIST Cat. 1.

Oil-Vinegar Polynomials [Patarin 1997]

Let $\mathbb{F}$ be a (finite) field. For $o, v \in \mathbb{N}$ set $n = o + v$ and define

$$p(x_1, \ldots, x_n) = \underbrace{\sum_{i=1}^{v} \sum_{j=i}^{v} \alpha_{ij} \cdot x_i \cdot x_j}_{v \times v \text{ terms}} + \underbrace{\sum_{i=1}^{v} \sum_{j=v+1}^{n} \beta_{ij} \cdot x_i \cdot x_j}_{v \times o \text{ terms}} + \underbrace{\sum_{i=1}^{n} \gamma_i \cdot x_i}_{\text{linear terms}} + \delta$$

$x_1, \ldots, x_v$: Vinegar variables $x_{v+1}, \ldots, x_n$: Oil variables, no $o \times o$ terms.
If we randomly set $x_1, \ldots, x_v$, result is linear in $x_{v+1}, \ldots, x_n$

(Unbalanced) Oil-Vinegar matrix

$\tilde{p}$ the homogeneous quadratic part of $p(x_1, \ldots, x_n)$ can be written as quadratic form $\tilde{p}(\mathbf{x}) = \mathbf{x}^T \cdot M \cdot \mathbf{x}$ with

$$M = \left(\begin{array}{c|c} *_{v \times v} & *_{o \times v} \\ \hline *_{v \times o} & 0_{o \times o} \end{array} \right)$$

where $*$ denotes arbitrary entries subject to symmetry.

Kipnis-Shamir OV attack when $o = v$

$$\mathcal{O} := \{\mathbf{x} \in \mathbb{F}^n : x_1 = \ldots = x_v = 0\} \quad \text{"Oilspace"}$$
$$\mathcal{V} := \{\mathbf{x} \in \mathbb{F}^n : x_{v+1} = \ldots = x_n = 0\} \quad \text{"Vinegarspace"}$$

Let E, F be invertible "OV-matrices", i.e. $E, F = \begin{pmatrix} \star & \star \\ \star & 0 \end{pmatrix}$ Then

$E \cdot \mathcal{O} \subset \mathcal{V}$. Since the two has the same rank, equality holds, so $(F^{-1} \cdot E) \cdot \mathcal{O} = \mathcal{O}$, i.e. $\mathcal{O}$ is an invariant subspace of $F^{-1} \cdot E$.

Common Subspaces

Let H_i be the matrix representing the homogeneous quadratic part of the i-th public polynomial. Then we have $H_i = S^T \cdot E_i \cdot S$, i.e. $S^{-1}(\mathcal{O})$ is an invariant subspace of the matrix $(H_j^{-1} \cdot H_i)$, and we find S^{-1}.

tl;dr Summary of the Standard UOV Attack

- for $v \leq o$, breaks the balanced OV scheme in polynomial time.
- For $v > o$ the complexity of the attack is about $q^{v-o} \cdot o^4$.

$\Rightarrow$ Choose $v \approx 2 \cdot o$ (unbalanced Oil and Vinegar (UOV)) [KP99]

Other Attacks

- **Collision Attack**: $o \geq \frac{2^{2\ell}}{\log_2(q)}$ for ℓ-bit security.
- **Direct Attack**: Try to solve the public equation $\mathcal{P}(\mathbf{w}) = \mathbf{z}$ as an instance of the MQ-Problem. The public systems of UOV behave much like random systems, but they are highly underdetermined ($n = 3 \cdot m$)

  Result [Thomae]: A multivariate system of m equations in $n = \omega \cdot m$ variables can be solved in the same time as a determined system of $m - \lfloor \omega \rfloor + 1$ equations.

  $\Rightarrow$ m has to be increased by 2.

Other Attacks

- **Collision Attack**: $o \geq \frac{2^{2\ell}}{\log_2(q)}$ for ℓ-bit security.
- **Direct Attack**: Try to solve the public equation $\mathcal{P}(\mathbf{w}) = \mathbf{z}$ as an instance of the MQ-Problem. The public systems of UOV behave much like random systems, but they are highly underdetermined $(n = 3 \cdot m) \;\Rightarrow\; m$ has to be increased by 2.
- **UOV-Reconciliation attack**: Try to find a linear transformation $\mathcal{S}$ ("good keys") which transforms the public matrices H_i into the form of UOV matrices

$$(S^T)^{-1} \cdot H_i \cdot S^{-1} = \begin{pmatrix} \star & \star \\ \star & 0 \end{pmatrix}, \qquad S = \begin{pmatrix} 1 & \star \\ 0 & 1 \end{pmatrix}$$

$\Rightarrow$ Each Zero-term yields a quadratic equation in the elements of S.
$\Rightarrow$ S can be recovered by solving several MQ systems (the hardest with v variables, m equations if $v < m$).

Summary of UOV

Safe Parameters for UOV($\mathbb{F}, o, v$)

security level (bit)	scheme	public key size (kB)	private key size (kB)	hash size (bit)	signature (bit)
80	UOV($\mathbb{F}_{16}$,40,80)	144.2	135.2	160	480
	UOV($\mathbb{F}_{256}$,27,54)	89.8	86.2	216	648
100	UOV($\mathbb{F}_{16}$,50,100)	280.2	260.1	200	600
	UOV($\mathbb{F}_{256}$, 34,68)	177.8	168.3	272	816
128	UOV($\mathbb{F}_{16}$,64,128)	585.1	538.1	256	768
	UOV($\mathbb{F}_{256}$,45,90)	409.4	381.8	360	1,080
192	UOV($\mathbb{F}_{16}$,96,192)	1,964.3	1,786.7	384	1,152
	UOV($\mathbb{F}_{256}$,69,138)	1,464.6	1,344.0	552	1,656
256	UOV($\mathbb{F}_{16}$,128,256)	4,644.1	4,200.3	512	1,536
	UOV($\mathbb{F}_{256}$,93,186)	3,572.9	3,252.2	744	2,232

What we know today about UOV

- unbroken since 1999 $\Rightarrow$ high confidence in security
- not the fastest multivariate scheme
- very large keys, (comparably) large signatures

Rainbow Digital Signature

Ding and Schmidt, 2004

- Patented by Ding (May have had patent by T.-T. Moh, expired)
- TTS is its variant with sparse central map

Rainbow Digital Signature

Ding and Schmidt, 2004

- Finite field $\mathbb{F}$, integers $0 < v_1 < \cdots < v_u < v_{u+1} = n$.
- Set $V_i = \{1, \ldots, v_i\}$, $O_i = \{v_i + 1, \ldots, v_{i+1}\}$, $o_i = v_{i+1} - v_i$.
- Central map $\mathcal{Q}$ consists of $m = n - v_1$ polynomials $f^{v_1+1}, \ldots, f^{(n)}$ of the form

$$f^{(k)} = \sum_{i,j \in V_\ell} \alpha_{ij}^{(k)} x_i x_j + \sum_{i \in V_\ell, j \in O_\ell} \beta_{ij}^{(k)} x_i x_j + \sum_{i \in V_\ell \cup O_\ell} \gamma_i^{(k)} x_i + \delta^{(k)},$$

  with coefficients $\alpha_{ij}^{(k)}$, $\beta_{ij}^{(k)}$, $\gamma_i^{(k)}$ and $\delta^{(k)}$ randomly chosen from $\mathbb{F}$ and ℓ being the only integer such that $k \in O_\ell$.
- Choose randomly two affine (or linear) transformations $\mathcal{T} : \mathbb{F}^m \to \mathbb{F}^m$ and $\mathcal{S} : \mathbb{F}^n \to \mathbb{F}^n$.
- *public key*: $\mathcal{P} = \mathcal{T} \circ \mathcal{Q} \circ \mathcal{S} : \mathbb{F}^n \to \mathbb{F}^m$
- *private key*: $\mathcal{T}$, $\mathcal{Q}$, $\mathcal{S}$

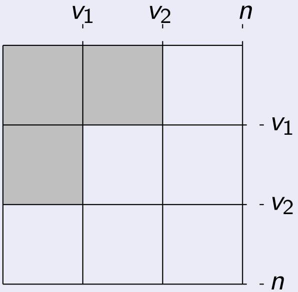
Idea of Rainbow

Inversion of the central map

- Invert the single UOV layers recursively.
- Use the variables of the i-th layer as Vinegars of the $i+1$-th layer.

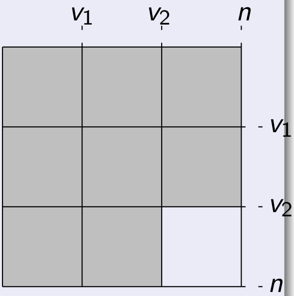
Illustration: Rainbow with two layers

$F^{(k)} =$



$v_1 + 1 \leq k \leq v_2$

$F^{(k)} =$



$v_2 + 1 \leq k \leq n$

Idea of Rainbow

Inversion of the central map

- Invert the single UOV layers recursively.
- Use the variables of the i-th layer as Vinegars of the $i+1$-th layer.

Input: Rainbow central map $\mathcal{Q} = (f^{(v_1+1)}, \ldots, f^{(n)})$, vector $\mathbf{y} \in \mathbb{F}^m$.
Output: vector $\mathbf{x} \in \mathbb{F}^n$ with $\mathcal{Q}(\mathbf{x}) = \mathbf{y}$.

Choose random values for the variables $x_1, \ldots, x_{v_1}$ and substitute these values into the polynomials $f^{(i)}$ $(i = v_1 + 1, \ldots n)$.
for $\ell = 1$ to u **do**
  Perform Gaussian Elimination on the polynomials $f^{(i)}$ $(i \in O_\ell)$ to get the values of the variables x_i $(i \in O_\ell)$.
  Substitute the values of x_i $(i \in O_\ell)$ into the polynomials $f^{(i)}$ $(i = v_{\ell+1} + 1, \ldots, n)$.
end for

Idea of Rainbow

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- Use the variables of the i-th layer as Vinegars of the $i+1$-th layer.

Signature Generation from message d

1. Use a hash function $\mathcal{H} : \{0,1\} \to \mathbb{F}^m$ to compute $\mathbf{z} = \mathcal{H}(d) \in \mathbb{F}^m$
2. Compute $\mathbf{y} = \mathcal{T}^{-1}(\mathbf{z}) \in \mathbb{F}^m$.
3. Compute a pre-image $\mathbf{x} \in \mathbb{F}^n$ of $\mathbf{y}$ under the central map $\mathcal{Q}$
4. Compute the signature $\mathbf{w} \in \mathbb{F}^n$ by $\mathbf{w} = \mathcal{S}^{-1}(\mathbf{x})$.

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Inversion of the central map

- Invert the single UOV layers recursively.
- Use the variables of the i-th layer as Vinegars of the $i+1$-th layer.

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4. Compute the signature $\mathbf{w} \in \mathbb{F}^n$ by $\mathbf{w} = \mathcal{S}^{-1}(\mathbf{x})$.

Signature Verification from message d, signature $\mathbf{z} \in \mathbb{F}^n$

1. Compute $\mathbf{z} = \mathcal{H}(d)$.
2. Compute $\mathbf{z}' = \mathcal{P}(\mathbf{w})$.

Accept the signature $\mathbf{z} \Leftrightarrow \mathbf{w}' = \mathbf{w}$.

Security

Rainbow is an extension of UOV
$\Rightarrow$ All attacks against UOV can be used against Rainbow, too.

Additional structure of the central map allows several new attacks

- **MinRank Attack**: Look for linear combinations of the matrices H_i of low rank (complexity $q^{v_1} o_1(m^3/3 + mn^2)$).
- **HighRank Attack**: Look for the linear representation of the variables appearing the lowest number of times in the central polynomials. (Complexity $q^{o_u} o_u(n^3/3 + o_u n^2)$, can Groverize)
- **Rainbow-Band-Separation Attack**: Variant of the UOV-Reconciliation Attack using the additional Rainbow structure

Choosing Parameter Selection for Rainbow is interesting

MinRank Attack

Minors Version

Set all rank $r+1$ minors of $\sum_i \alpha_i H_i$ to 0.

Kernel Vector Guessing Version

- Guess a vector $\mathbf{v}$, let $\sum_i \alpha_i H_i \mathbf{v} = 0$, hope to find a non-trivial solution.
- (If $m > n$, guess $\lceil \frac{m}{n} \rceil$ vectors.)
- Takes $q^r(m^3/3 + mn^2)$ time to find a rank r kernel.

Accumulation of Kernels and Effective Rank

In the first stage of Rainbow, there are $o_1 = v_2 - v_1$ equations and v_2 variables. The rank should be v_2. But if your guess corresponds to $x_1 = x_2 = \cdots = x_{v_1} = 0$, then about $1/q$ of the time we find a kernel. The easy way to see this is that there are $q^{o_1 - 1}$ different kernels. We say that "effectively the rank is $v_1 + 1$".

Rainbow Band Separation

Extension to UOV reconciliation to use the special Rainbow form.

n variables, $n+m-1$ quadratic equations

1. Let $w_i := w'_i - \lambda_i w'_n$ for $i \le v$, $w_i = w'_i$ for $i > v$. Evaluate $\mathbf{z}$ in $\mathbf{w}'$.
2. Find m equations by letting all $(w'_n)^2$ terms vanish; there are v of λ_i's.
3. Set all cross-terms involving w'_n in $z_1 - \sigma_1^{(1)} z_{v+1} - \sigma_2^{(1)} z_{v+2} - \cdots - \sigma_o^{(1)} z_m$ to be zero and find $n-1$ more equations.
4. Solve $m+n-1$ quadratic equations in $o+v=n$ unknowns.
5. Repeat, e.g. next set $w'_i := w''_i - \lambda_i w''_{n-1}$ for $i < v$, and let every $(w''_{n-1})^2$ and $w''_n w''_{n-1}$ term be 0. Also set $z_2 - \sigma_1^{(2)} z_{v+1} - \sigma_2^{(2)} z_{v+2} - \cdots - \sigma_o^{(2)} z_m$ to have a zero second-to-last column. [$2m+n-2$ equations in n unknowns.]

Rainbow - Summary

- no weaknesses found since 2007
- efficient (25.5kcycles verifying, 75.5kcycles signing at NIST Cat. 1)
- suitable for low cost devices
- shorter signatures and smaller key sizes than UOV

Parameters for Rainbow

NIST Security Category	parameters $\mathbb{F}, v_1, o_1, o_2$	public key size (kB)	private key size (kB)	hash size (bit)	signature (bit)
I	$\mathbb{F}_{16}$,32,32,32	148.5	97.9	256	512
III	$\mathbb{F}_{256}$,68,36,36	703.9	525.2	576	1,248
V	$\mathbb{F}_{256}$,92,48,48	1,683.3	1,244.4	768	1,632

Thank you for Listening

That's it Folks!