Quasi—local mass, the Penrose property of Scri, and causality.

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Mass in General Relativity

- No local energy density.
- At space–like/null infinities: ADM/Trautman-Bondi.
- Quasi–local, for a closed space–like 2–surface $(\Sigma, h) \subset (M, g)$.
- Properties? (positivity, $\ldots$). Lots of possibilities.

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  2. Relation between mass, and global/causal properties of space–times.

     Cameron, P. and Dunajski M (2020) *On Schwarzschild causality in higher dimensions.* CQG **37**. `arXiv:2004.00086`.

     Cameron, P. (2023) *Positivity of mass in higher dimensions.* Annales Henri Poincare **24.** `arXiv:2010.05086`.

Quasi–local mass from isometric embeddings

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- $(\Sigma, h) \subset (M, g)$, closed, non–negative Gaussian curvature K, global isometric embedding $\iota : \Sigma \to \mathbb{R}^3$.
- Kijowski–Liu–Yau energy

$$E_{KLY} = \frac{1}{4\pi} \int_\Sigma (H - |\hat{H}|) \text{vol}_\Sigma, \quad \text{where}$$

H =mean curvature of $\Sigma \subset \mathbb{R}^3$, and $\hat{H}$ = mean curvature vector of $\Sigma \subset M$.

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- Pros: Amenable to computations (formula!), positive (Liu–Yau 2003). Cons: 'too positive' (Murchadha, Szabados, Tod 2004), excludes Σ with $K < 0$ (e.g. extreme Kerr horizon).

Kerr horizon

- Fix $(t, r = r_+)$ in the Boyer–Lindquist coordinates. Set $x = \cos\theta$

$$
\begin{aligned}
g &= \rho^2(B^{-1}dx^2 + Bd\phi^2), \quad \text{where} \quad B = \frac{(1+c^2)(1-x^2)}{1+c^2x^2}, \quad \text{and} \\
\rho^2 &= 2m(m + \sqrt{m^2 - J^2/m^2}), \quad c = \frac{2J}{\rho^2} \in [0,1], \quad x \in [-1,1].
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- Gaussian curvature

$$K = \frac{(c^2+1)^2(1-3c^2x^2)}{\rho^2(1+c^2x^2)^2} \geq \frac{(1-3c^2)}{\rho^2(c^2+1)} = K_{min}$$

Negative near antipodal points if $c \in (\sqrt{3}^{-1}, 1]$.

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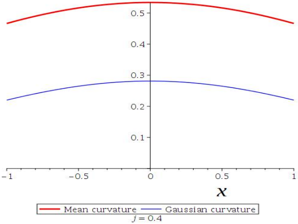
- Idea: Use an embedding in $\mathbb{R}^3$ for $c \in [0, \sqrt{3}^{-1}]$, and an embedding in $\mathbb{H}^3$ with maximal hyperbolic radius for $c \in (\sqrt{3}^{-1}, 1]$.

Slow rotating horizons

$$G = d\zeta^2 + dr^2 + r^2 d\phi^2, \quad \zeta = \pm\frac{\rho}{2}\int \frac{\sqrt{B(4-(B')^2)}}{B}dx, \quad r = \rho\sqrt{B}.$$

- Gaussian and mean curvatures, $c = \frac{1-\sqrt{1-j^2}}{j}, j \equiv J/m^2$.



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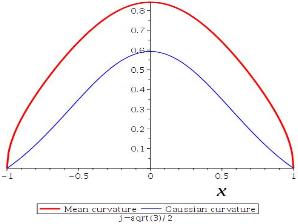

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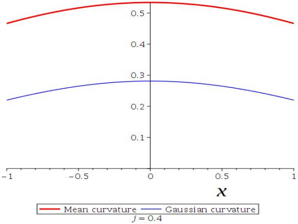

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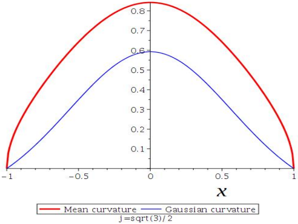

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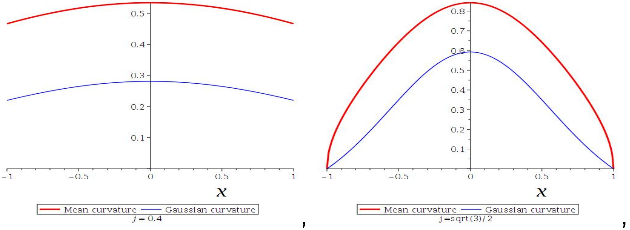

$$\begin{aligned} E(m,j) &= \frac{1}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi} H(x)\rho^2 d\phi dx \\ &= m\Big(2 - \frac{1}{4}j^2 - \frac{17}{320}j^4 - \frac{407}{17920}j^6 - \dots\Big). \end{aligned}$$

- **Theorem** (Pogorelov 1964). Any surface (Σ, h) with $K \geq -L^2$ can be globally isometrically embedded in the hyperbolic space $\mathbb{H}^3$ with Ricci scalar less or equal to $-6L^{-2}$.

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$$G = \frac{L^2}{z^2}\Big(dz^2 + dr^2 + r^2 d\phi^2\Big), \quad \text{where} \quad r(x) = \frac{\rho}{L}\sqrt{B(x)}z(x)$$

$$z(x) = \exp\Big(\int\Big(\frac{-\rho^2 BB' \pm \rho\sqrt{B(4\rho^2 B + 4L^2 - L^2(B')^2)}}{2B(\rho^2 B + L^2)}\Big)dx\Big)$$

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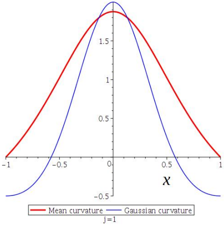
- Gaussian and mean curvatures



Quasi–local energy as a function of $j = J/m^2$.

$$H(x) = \frac{c\sqrt{1-x^2}(x^4(2c^4-5c^6)+x^2(-4c^6-12c^4+6c^2)+c^4+3c^2+9)}{\rho\sqrt{(1+c^2)(1+c^2x^2)^3}\sqrt{x^4(c^4-2c^6)+x^2(-c^6-4c^4+3c^2)+3}}.$$

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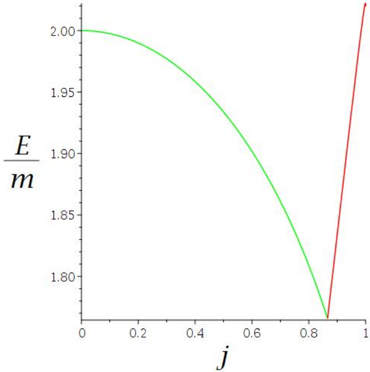

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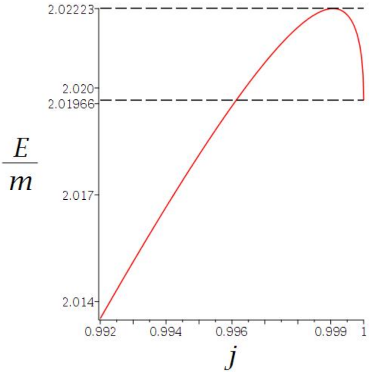

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- $j \sim 0.998$ for the equilibrium of a black hole absorbing matter and radiation from an accretion disk. (Kip Thorne 1974).
- Kerr boomerang at $j \sim 0.99434$ (Don Page 2021 `arXiv:2106.13262`).

Outlook of part I

Work in progress with Alex Colling

- For certain range of (Λ, m) the extreme Kerr-dS horizon can be embedded in both $\mathbb{R}^3$ and $\mathbb{H}^3$, and its Li–Yau mass can be computed.

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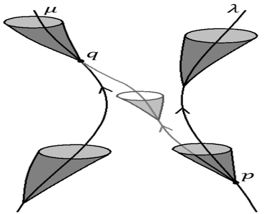
  - Compute a mass $M(X, \tau)$, and find an infimum of M over all τ: a variational problem for τ.
- Not applicable if the mean curvature of Σ in space–time vanishes (e.g. Kerr horizon) (but there exists a 'limiting procedures': Miller, Ray, Wang, Yau (2018) and Zhao, Andersson, Yau (2024)).

Mass and causality

- Asymptotic properties of space–time near space–like infinity depend on a mass.
- Shapiro effect: light–rays passing near a positive mass source are delayed.
- Anti–Shapiro effect: light–rays passing infinitely far from a positive mass source are advanced.
- Penrose, Sorkin, Woolgar (1993) *A Positive Mass Theorem Based on the Focusing and Retardation of Null Geodesics* `gr-qc/9301015v2`.

Roger Penrose (1980) *On Schwarzschild Causality - A Problem for 'Lorentz Covariant' General Relativity*. in Essays in General Relativity. 1-12.

- An asymptotically flat space–time admits a Penrose Property (PP) if any pair of endless timelike curves can be connected by a timelike curve.



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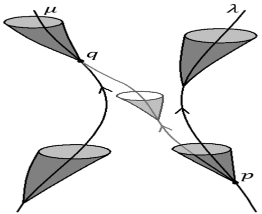
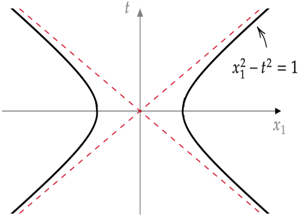


- Equivalent formulation: A weakly asymptotically simple Lorentzian manifold admits PP if and only if the timelike future of any point $p \in \mathcal{I}^-$ contains the whole of $\mathcal{I}^+$.

- Minkowski space–time in $(d+1)$ dim: PP doesn't hold.

$$ds^2 = dt^2 - dx_1{}^2 - \cdots - dx_n{}^2, \quad x_1{}^2 - t^2 = 1, x_2 = \cdots = x_n = 0$$



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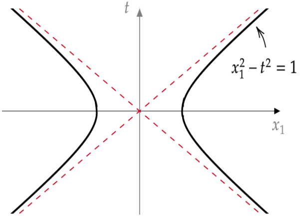

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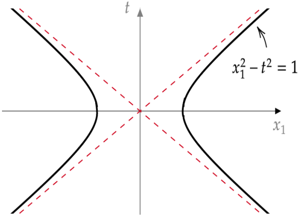

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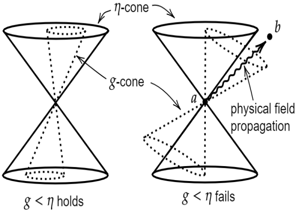
- **Theorem** (Penrose 1980) Schwarzschild in (3+1) dim: PP holds.
- **Theorem** (Cameron-D. 2020) Kerr in (3+1) dim: PP holds.

A problem for perturbative quantum gravity

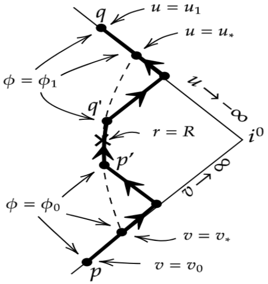
- Lorentz covariant quantum gravity:

$$g \sim \eta + \epsilon h + \ldots, \hat{O}_g(x) \sim \hat{O}^{(0)}_\eta(x) + \epsilon \hat{O}^{(1)}_\eta(x) + \ldots$$

- Causality and QFT: $[\hat{O}(a), \hat{O}(b)]$ must be zero if (a, b) space separated, and may be non–zero otherwise.
- Consistency: lightcones of g should be inside lightcones of η: Time like curves w.r.t g must be timelike w.r.t η. Write $g < \eta$.
- This fails for Schwarzschild and Kerr.



- Schwarzchild
  1. $\Delta\phi \geq (1 - r_s/r_{min})^{-1/2}\pi$ along a null geodesic γ between $r = r_{min}$ and $r = \infty$.
  2. Connect $p \in \mathcal{I}^-$ with (v_0, ϕ_0) to $q \in \mathcal{I}^+$ with (u_1, ϕ_1) by 4 segments of radial null geodesics, and γ (used to adjust ϕ). Smooth the corners to obtain a time–like curve.



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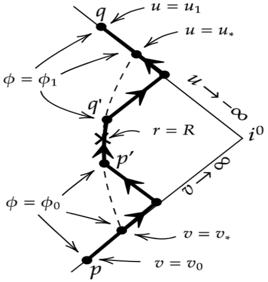

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- Kerr
  1. Use the Pretorius-Israel compactification `gr-qc/9803080`.
  2. Enclose Kerr light–cones by those of a *quasi–Schwarzschild* metric.

Penrose property in other dimensions

- $(2+1)$–dimensional Schwarzschild and deficit angle

$$ds^2 = dt^2 - dr^2 - r^2 d\phi^2, \quad 0 \leq \phi \leq 2\pi(1-4m).$$

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- **Theorem.** (Cameron, D. 2020) The Penrose property is satisfied by Schwarzschild spacetime of mass m and varying spacetime dimension according to the following table

Spacetime dimension	$m > 0$	$m \leq 0$
3	Yes	No
4	Yes	No
≥ 5	No	No

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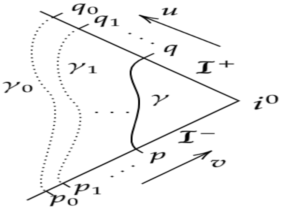

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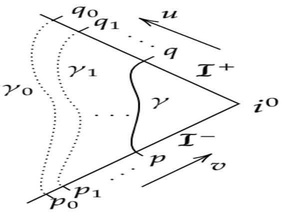

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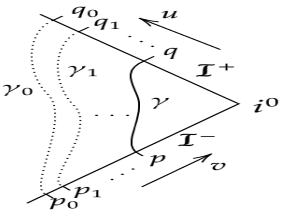

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- ... But then it has to be a null geodesic with no–conjugate points. Contradiction.

Conclusions

- Two approaches to mass:
- Quasi–local, based on isometric embeddings in $\mathbb{R}^3$ and $\mathbb{H}^3$.
- Global, based on causality and properties of space–times near i^0.
- Both applicable to Kerr: quasi–local mass of Kerr outer horizon, and Penrose property for Kerr space–time.
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Thank you!