

Abstract

J. B. McLeod was a brilliant solver of problems in mathematical analysis, primarily differential equations. He received his FRS in 1992, and the citation reads in part: “Distinguished for many significant contributions to applied analysis, particularly to the theory of partial differential equations with applications to practical problems. ... By the exemplary precision and power of his publications and his lectures, he has become internationally recognized as the leading British authority on the useful applications of functional analysis.”

Also, in 2011 McLeod was awarded the Naylor Prize and Lectureship of the London Mathematical Society “in recognition of his important and versatile achievements in the analysis of nonlinear equations arising in applications to mechanics, physics, and biology.” He collaborated widely, and was a resource for many applied mathematicians who wanted to have a more rigorous foundation for their work. He leaves a hole which will be hard to fill.

J. Bryce McLeod

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May 23, 2016

1 Overview

When Bryce McLeod was ten years old, his home city of Aberdeen was under threat of German bombs. As a result, his schooling was partially interrupted, and so his parents sent him to his grandfather, former Head of Mathematics at Aberdeen Grammar School, for instruction. Apparently, this gentleman had lost track of what maths a ten-year-old would have been exposed to, and he began the first lesson with algebra, completing linear equations in around 15 minutes and then delving into the quadratic equation. Young Bryce, having seen nothing beyond arithmetic before, had no idea what these x 's and y 's were about, but was too in awe of his grandfather to admit this. He went home with an assignment, and agonized for hours trying to determine what was going on. But when he returned the next day he was able to solve every quadratic equation his grandfather gave him.^{1 2}

John Bryce McLeod was born on 23 December 1929, in Aberdeen. His parents were John McLeod, an engineer and Adeline Annie Bryce. His paternal grandfather was raised on a farm, but had been recognized by his teachers as bright and encouraged to attend Aberdeen University.

Bryce's father and one of his uncles were engineers and another uncle read mathematics at Cambridge, so he was following in a family tradition when he, at age 16, went off to study maths at Aberdeen University. Upon graduation he was awarded a stipend enabling him to continue his education at Oxford, where he received his second First Class undergraduate degree, again in mathematics. After an interlude for study abroad and National Service he completed his D. Phil. at Oxford in 1958, under the direction of E. C. Titchmarsh FRS, the leading analyst there at the time.

He took a position at Edinburgh University, but in 1960 accepted a Fellowship back in Oxford, at Wadham College. He became a University Lecturer, and his research in what is now called "applied analysis" flourished.

¹This story, and some other material in this Memoir, came from the interview McLeod did with Professor Sir John Ball of Oxford University in January, 2014. Both participants realized the seriousness of McLeod's illness at the time of the interview, a video of which is available online [30].

²Much of this section was taken from (Hastings 2014).

Applied analysis is largely the rigorous study of differential equations and optimization problems arising in the sciences and engineering. McLeod's research in this area was widely recognized in the UK and abroad, but not so much in Oxford, for reasons I will discuss later. Dissatisfaction with his situation within Oxford Mathematics, and also the mandatory retirement he saw looming, led to his departure for the University of Pittsburgh in 1987. Pitt had strong people in differential equations and neighboring Carnegie Mellon University also had a first class group in this area.

He stayed in Pittsburgh for 20 productive years. During this period, ironically, he received an inquiry from a senior mathematician at Cambridge inviting him to apply for a Chair there. He had to reply that he was beyond the mandatory retirement age.

McLeod's influence did much to resuscitate applied analysis in the UK. One indication of this was his FRS, awarded in 1992, while he was in Pittsburgh. Others around Britain, including John Ball, were encouraged in their interest in differential equations by his work. His Oxford graduate students gained Professorships at Exeter (later Canterbury), EPFL Lausanne, Heriot-Watt, Michigan, North Carolina State, and in Brazil.

McLeod collaborated widely in differential equations, where he was recognized as a problem solver of genius. These collaborations frequently developed when another mathematician had a problem from an applied area which he found intractable and brought it to McLeod's attention. Very often the result would be a new but simple way of looking at the problem which led to an ingenious solution.

It is symbolic of the revival of applied analysis in the UK in the last 30 years that the 2011 Naylor Prize and Lectureship of the London Mathematical Society was presented to J. B. McLeod, "in recognition of his important and versatile achievements in the analysis of nonlinear equations arising in applications to mechanics, physics, and biology."

In addition to the honors mentioned earlier, McLeod was awarded the Whittaker Prize of the Edinburgh Mathematical Society in 1965, the Keith Medal and Prize of the Royal Society of Edinburgh in 1987 and election as a FRSE in 1974. He died on August 20, 2014 and is survived by his wife, four children and three grandchildren.

2 Student years

Bryce, his father, and very likely his grandfather, were each the top student ("dux") in their turn at Aberdeen Grammar School, and hence had their pictures posted in a hallway. Many years later Bryce paid a return visit to the school, out of term. He was spotted by a custodian, who ordered this very casually dressed visitor to leave. Unfortunately, there had been a fire at the school in which the dux pictures were destroyed, and so Bryce was unable to point out his photo to the custodian as evidence that he should not be turned out.

It was assumed in those days that bright students from northern Scotland

would go to Aberdeen University. Bryce had known from his early teens that he wanted to study mathematics, and at Aberdeen he found a more than competent mathematical faculty. The best known mathematician there during Bryce's time was E. M. Wright, co-author with G. H. Hardy of a classic book on number theory and winner of the Senior Berwick Prize of the London Mathematical Society in 1978. It was Wright who encouraged Bryce to go further in mathematics and probably Wright who found the funds which enabled him to do so at Oxford. At that time he envisaged a life as a school teacher, following in the footsteps of his grandfather.

At Oxford Bryce fell under the influence of Theodore Chaundy, a mathematician remembered today especially for his work in hypergeometric functions. Bryce admired Chaundy greatly, and nearly twenty years later, having completed his D. Phil. with Titchmarsh and established himself in research, he wrote that when he came up to Christ Church as an undergraduate to read mathematics, he "became exposed to a mathematical mind which for sheer speed and restlessness, was quite unequalled in my experience then – or, for that matter, since." [11]

After completing his first class honours in mathematics at Oxford, Bryce took a somewhat unusual path. He was, under circumstances which he was unable to remember during the Ball interview, awarded a Rotary scholarship, apparently due to actions taken in Aberdeen. This enabled him to travel "anywhere in the world", presumably to study mathematics. Here, it appears to this outside observer, Chaundy's advice was a bit strange, for he recommended going to the University of British Columbia, not a powerhouse in pure mathematics at that time. Bryce duly followed this suggestion, and in [30] he expressed no regrets; on the contrary he was very positive about the contacts he made there, especially a long time friend who, as I will describe later, proved very helpful in his subsequent mathematical career.

The time at UBC apparently did cause him to shift his interest permanently from school teaching to mathematical research. When he returned, he was required to do national service, which involved low level teaching as an Education Officer in the RAF. After this he started his research with Titchmarsh.

Bryce's first publication, joint with Chaundy, appeared in print in 1958, the year of his D. Phil. [1] The reviewer for Math Reviews said: "By an ingenious sequence of formal manipulations the authors prove that the form of the solution depends on a quadratic characteristic equation." His second paper [2] had the surprising title "On the commutator subring"; surprising because I doubt that many of his friends realize that he had a publication in algebra. The reviewer wrote "By a simple but ingenious computation, it is shown that the subring of S generated by all commutators $rs - sr$ is a two-sided ideal in S ." Again, "ingenious", a word not used lightly by mathematicians. But it can be said to apply to a large number of the proofs of J. B. McLeod (as he preferred to sign his papers).³

³Co-authors sometimes persuaded him to be less formal, using J. Bryce McLeod or even, in few instances, Bryce McLeod.

In [30] Bryce tells this story. Titchmarsh gave regular seminars on his work, often with Bryce as the only member of the audience, during one of which he mentioned a point which was still unresolved. Bryce thought he could say something about this, and after couple of days put some notes on the problem in Titchmarsh's "pigeonhole". In a week he found in his own pigeonhole the complete manuscript of a paper by McLeod and Titchmarsh. Apparently there was not one word exchanged between them in the preparation of this paper. In later years, and in the Ball video, Bryce made it clear that mathematical and social interchanges with other mathematicians were among his principle rewards for tackling such a hard topic as mathematics. It seems that there was none of this with his advisor.

3 Family

Bryce had a happy childhood and his parents, both intelligent people, accepted his inclination to disappear into his room to study maths. Like Bryce, his sister Morag obtained a first class degree at Aberdeen University, hers being in chemistry.

Bryce met his future wife, Eunice, while they were both at school. Eunice chose to skip University and become a nurse, continuing to work in the medical field after their marriage in 1956 and the concurrent move to Oxford. Their first two children, Kevin and Callum, were born in the UK and the last two, twins Bridget and Patrick, came on the scene in Madison Wisconsin, where Bryce took a year's leave in 1964-65.

Kevin was the only child who went into mathematics, obtaining his Ph. D. at the University of Minnesota under the direction of James Serrin and making his career at the University of Wisconsin, Milwaukee. He has one joint paper with his father. Callum became a musician, working in a mixture of classical music, as conductor and performer, and theatrical music, conducting the orchestra at the *Phantom of the Opera* in London for many years. Bridget went into school teaching, becoming a head teacher in Berkshire and organizing many school musical performances. Patrick took a degree in chemical engineering and became a Vice President at Dow Corning Corporation, working at various times in Belgium, France, and China. Bryce, with good reason, was proud of all his children. He was, however, much engrossed in his work, so we should give considerable credit to Eunice for their success. Beyond that, Eunice undoubtedly made Bryce's career possible with her so-called "supporting role" at home.

In 1964-65 Bryce took a leave at the Mathematics Research Center at the University of Wisconsin, Madison, accompanied by his family. He had made contact with the group there with the help of Tommy Hull, the friend from Vancouver whom I mentioned earlier [30]. This year was rewarding for Bryce mathematically and for the whole family socially. He found the MRC a revelation, particularly because he felt that applied analysis was properly appreciated there. From then until 1986 he spent most summers in Madison, with his family joining him for parts of these visits.

During those summers the family journeyed several times across the US in the Winnebago camper van that Bryce had bought to live in while in the US. At Bryce's funeral Patrick spoke of these journeys:

"I look back with great fondness on those road trips around the US – Mexico, the Canyon lands, Mesa Verde, the Badlands, California and the West Coast, New England during the bicentenary year, Banff, the Canadian National Parks. There were times, of course, when the experiences weren't fully appreciated by the rest of the family, and that would annoy Dad, but his intentions were always to provide us with opportunities to learn and to develop interests that many children of our ages would never have had."

Bryce was the family photographer and documented their travels. The photo below is one of the very few the family have of Bryce himself. It was taken on one of those cross country trips.



Although Bryce and his family travelled to many parts of the world, established temporary homes during sabbaticals in Wisconsin, Minnesota, and Sydney, and moved for 20 years to Pittsburgh, the house they bought in Abingdon in 1960 remained in many ways the center of their lives together. They kept this house throughout all of their sojourns abroad.

Nevertheless, from 1987 to 2007 Eunice and Bryce made a home in Pittsburgh, enjoying particularly the local classical music scene. Their Pittsburgh home, like the house in Abingdon, held a grand piano, for Bryce to play and accompany Eunice. It was with mixed feelings that they moved back to Abingdon after his retirement from Pitt.

Bryce was an excellent after dinner speaker. In my experience one of his best speeches was at the conference which was organized in Oxford for his 70th birthday. After Bryce died a Pittsburgh colleague, Carson Chow, wrote:

"One of the highlights of my career was being invited to a conference in his honour in Oxford in 2001. At the conference dinner, Bryce gave the most perfectly constructed speech I have ever heard. It was just like the way he did mathematics - elegantly and sublimely."

His family remembers particularly a speech he gave at his daughter Bridget's wedding, which brought her into a family with surname McGregor. His version

of the history of the McGregors, perhaps not entirely favorable to that ancient clan, had both families in stitches.

It appears, however, that mathematical audiences were not always so able to take his wit in stride. On at least two occasions speeches he gave at birthday celebrations for a mathematician were interpreted by some in the audience as veiled criticisms of the celebrant, couched in humor. But his family, who of course knew Bryce best, are sure that he was much too straight-forward for such a device. They maintain, for example, that when one honoree was described as being like Poo Bah in the *Mikado*, who held many important positions simultaneously, this was solely for its humorous effect.

4 His Mathematics

McLeod's mathematical specialty, applied analysis, is a bridge between pure and applied mathematics. In the 40's and 50's, major figures in this area in the UK included Titchmarsh, M. L. Cartwright FRS, and J. E. Littlewood FRS, but then the subject fell in stature as compared to such areas as abstract algebra and topology. After Titchmarsh's death in 1966 no specialist in differential equations held a Chair at Oxford (or Cambridge) until John Ball FRS was appointed Sedleian Professor thirty years later. I have included below a section about the reasons for the lack of interest in applied analysis at Oxford in the 70s and 80s, but first I will describe a selection of McLeod's most influential papers.⁴

4.1 Linear problems

Of McLeod's first 34 published papers 30 were in differential equations, and of these, most were on linear ordinary differential equations. This was to be expected for a student of Titchmarsh, whose mastery of the linear ODE domain was unsurpassed. Of his subsequent 128 papers, from 1968 to 2015, only 10 can be characterized as linear. And among his most cited papers, including two before 1968, all but one are on nonlinear problems. So my relative incompetence in linear ODEs is not the only reason that I will emphasize the nonlinear work here.

However some of McLeod's papers on linear problems do bear particular mention. A striking indication of the importance of his early work is that many of these papers continue to be cited in the 21st century. Using Google Scholar I have found that about half of McLeod's 34 papers published before 1968 had citations from the year 2000 or later. Among those on linear topics were papers on Schrödinger's equation [8], [3], and the number of L^2 solutions

⁴A suggestion to readers not trained in mathematics, taken from the excellent book (Yandell, 2002): If you come to material you don't understand, "skip a bit if you want –the biographical narrative will pick up again. Pretend you are reading *Moby Dick*, and have come to another chapter on whaling."

of a class of ODEs [6], [4]. Some of the early papers had citations 50 years or more after they were written.

The best known of McLeod's papers on linear equations is work he did around 1970 with Tosio Kato of the University of California at Berkeley [13]. Kato, one of the most eminent of McLeod's collaborators, visited Oxford in the early '70s. The problem he and McLeod worked on was about "wave motion in the overhead supply line to an electrified railway system". They obtained a number of results, but the work was unfinished when Kato returned to Berkeley. Later, when McLeod sent him a solution to the problem they had been stuck on, he wrote back: "How on earth did you think of that?" [30] Many of McLeod's collaborators over the years had the same question.

THE FUNCTIONAL-DIFFERENTIAL EQUATION

$$y'(x) = ay(\lambda x) + by(x) \text{ for } x > 0 \quad (1)$$

$$\lim_{x \rightarrow 0^+} y(x) = 1. \quad (2)$$

This equation, derived in Ockendon & Taylor 1971, is not an ordinary differential equation unless $\lambda = 1$. In [13] a may be complex, b is real, and λ is real and non-negative, but here I will only consider the physical case of real solutions with a real.

The cases $\lambda = 0$ and $\lambda = 1$ being trivial, we assume that $0 < \lambda < 1$ or $\lambda > 1$.⁵ The theory is more complete if $\lambda < 1$. In this case, setting $x = e^s$, $\lambda = e^c$, and $y(x) = z(s)$ gives a delay equation in standard form,

$$z'(s) = az(s+c) + bz(s)$$

where $c < 0$ since $\lambda < 1$. From this it is seen that the standard existence and uniqueness theory for linear delay-differential equations applies. The interest then is in the asymptotic behavior of solutions for large x .

In the case $b < 0$, for example, McLeod and Kato are able to show that if $\lambda < 1$ then every solution of (1) can be written in the form

$$y(x) = x^k [g(\log x) + o(1)] \text{ as } x \rightarrow \infty$$

where

$$k = \log \left(-\frac{b}{a} \right) / \log \lambda$$

and g is a C^∞ periodic function of period $|\log \lambda|$. There is such a solution for any such g .

Perhaps the most interesting twist in this paper is a relation which is revealed between the asymptotic behaviors for $\lambda < 1$ and $\lambda > 1$. This result is too technical to give here but I suspect that it was the cause of Kato's laudatory question mentioned above.

⁵Both cases are discussed in (Ockendon & Taylor 1971).

4.2 Non-linearity - before and after Edinburgh

4.2.1 Coagulation

But still, most of McLeod's influence eventually is likely to be from his papers on nonlinear differential equations, all but two published after a 1968 conference in Edinburgh which he helped organize and where he met James Serrin. It was in Serrin's Edinburgh lectures that McLeod first began to appreciate the interest and importance of nonlinearity in applied analysis. I start, though, with the paper [5] from 1962.

ON AN INFINITE SET OF NONLINEAR DIFFERENTIAL EQUATIONS

This paper was far ahead of its time. It received almost four times as many citations in the years 2011-2014 as it did up to 1980. In the video (see footnote 1), McLeod tells the story of meeting the theoretical chemist William Byers Brown, later Professor at the University of Manchester, while each was fulfilling his national service requirement by teaching at the Royal Air Force Technical College in Henlow, Bedfordshire. Byers Brown, a friend for many years thereafter, introduced McLeod to coagulation theory, and this was the subject of his first paper on nonlinear differential equations.

The title above is rather general but now it might be "On the discrete form of Smoluchowski's equation". This equation was developed in 1916 by a pioneer in statistical physics, Marian Smoluchowski, who worked at the University of Lwów in Poland. It is usually written as a single integral equation:

$$\frac{\partial n(x, t)}{\partial t} = \frac{1}{2} \int_0^x K(x-y) n(x-y, t) n(y, t) dy - \int_0^\infty K(x, y) n(x, t) n(y, t) dy.$$

The discrete version considered by McLeod in the first of his papers on the topic is, as his title indicates, an infinite system:

$$\begin{aligned} \frac{dn_1}{dt} &= -n_1 \sum_{i=1}^{\infty} K_{1i} n_i \\ \frac{dn_i}{dt} &= \frac{1}{2} \sum_{j=1}^{i-1} K_{j, i-j} n_j n_{i-j} - n_i \sum_{j=1}^{\infty} K_{ij} n_j, \quad i \geq 2. \end{aligned} \quad (3)$$

The standard initial conditions are

$$\begin{aligned} n_1(0) &= 1 \\ n_i(0) &= 0 \quad \text{if } i \geq 1 \end{aligned} \quad (4)$$

Smoluchowski's model has been studied extensively by mathematicians for the last 30 years, but in 1962 it was all but unknown in the mathematical community. McLeod states that the only previous pure mathematical work on the equation was for cases where the kernel K is bounded.

He first considered a distinctly unbounded case, $K(i, j) = ij$. What motivated him to do so is not made clear, but it is this part of the paper that has been most influential in the subsequent decades. His analysis of this case can be repeated in full:

For $i \geq 2$ multiply the equations by i and then sum over i , giving

$$\sum_{i=1}^{\infty} i \frac{dn_i}{dt} = \frac{1}{2} \sum_{i=1}^{\infty} i \sum_{j=1}^{i-1} j (i-j) n_j n_{i-j} - \sum_{i=1}^{\infty} i^2 n_i \sum_{j=1}^{\infty} j i n_j \quad (5)$$

Proceed “by noting that” if either term is multiplied out, then the total coefficient of $n_i n_j$ is $(i+j)ij$.” (for $i \neq j$).

This and (4) imply that

$$\begin{aligned} \sum_{i=1}^{\infty} i \frac{dn_i}{dt} &= 0 \\ \sum_{i=1}^{\infty} i n_i &= 1 \end{aligned}$$

and so

$$\begin{aligned} n_1' &= n_1 \\ n_1(t) &= e^t. \end{aligned}$$

This allows one to solve successively for n_2, n_3, \dots and there results the exact solution

$$n_j(t) = \frac{t^{j-1} j^{j-2}}{j! e^{j t}}, \quad 0 \leq t < 1.$$

It is remarkable that one can find an exact solution for such a complicated system. Perhaps the major step here was to hope that one could do so.

4.2.2 One of Serrin’s problems

THE EXISTENCE OF SIMILAR SOLUTIONS FOR SOME LAMINAR
BOUNDARY VALUE PROBLEMS
(1968, with James Serrin)

The equations are those of K. Stewartson (Stewartson 1949) for similarity solutions of the boundary layer equations for compressible flow over a surface. They are:

$$\begin{aligned} f''' + f f'' + \mu (h - f'^2) &= 0 \\ h'' + f h' &= 0 \end{aligned} \quad (6)$$

with boundary conditions

$$\begin{aligned} f(0) = f'(0) = 0, \quad h(0) = a \\ \lim_{x \rightarrow \infty} f'(x) = \lim_{x \rightarrow \infty} h(x) = 1 \end{aligned} \quad (7)$$

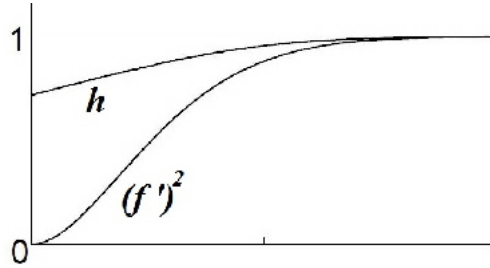
The problem addressed by McLeod and Serrin in [9] was the existence of a solution to this problem.

This was the first of many uses by McLeod of the so-called “shooting method” for proving the existence of a solution to an ODE boundary value problem. Although the method had been used earlier, for example by Wazewski, I suspect that McLeod was not aware of this and came up with it himself, developing a more straight-forward version.

In the shooting method, one assumes enough additional initial conditions to specify a unique solution, and then tries to find values for these additional initial conditions such that the boundary conditions at infinity are satisfied. Thus, suppose that

$$f''(0) = \alpha, \quad h'(0) = \beta. \quad (8)$$

The goal is to choose α and β so that (7) is satisfied. The idea is to sort out various ways in which the solution can go wrong, and fail to satisfy the boundary conditions at infinity. Based on numerical computations one can expect graphs of $(f')^2$ and h for a solution satisfying (7) to look like this:



In particular, we look for a solution with f' increasing. My presentation below is slightly simplified from that of McLeod and Serrin because they treated a more general class of systems, including one other important application. For the problem (6)-(7), one can find two specific ways that the solution (f, h) to (6)-(8) can go wrong and not solve (7), or even the conditions $f'' \geq 0$ and $f'(\infty)^2 = h(\infty)$.

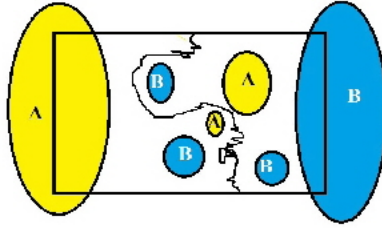
- (i) $f'' > 0$ and $(f')^2$ increases above h . This happens if, for a given $\beta = h'(0)$, α is very large
- (ii) f'' becomes negative. This happens if α is zero, because $f'''(0) < 0$.

Suppose that \mathcal{A} is the set of initial conditions $(\alpha, \beta) \in R^2$ such that (i) occurs, and \mathcal{B} is the set of (α, β) such that (ii) occurs.

McLeod and Serrin show, essentially, that there is a rectangle R in the (α, β) plane such that the left side of R is contained in \mathcal{A} and the right side of R is contained in \mathcal{B} . Also, \mathcal{A} and \mathcal{B} are disjoint and open. Then they use the following result from point set topology.

Lemma: *Under the given conditions on \mathcal{A} and \mathcal{B} there is a continuum $\Gamma \subset R$ which intersects neither \mathcal{A} nor \mathcal{B} and which connects the top and the bottom of R .*⁶

⁶Topologists today would prove this using some form of degree theory or algebraic topology. McLeod and Serrin give an elementary proof “from scratch”.



It is easy to show that if $(\alpha, \beta) \in \Gamma$ then $f'(\infty)^2$ and $h'(\infty)$ exist and are equal.

Their final step, relatively straight forward, is to show that the limits above are continuous functions of (α, β) in Γ and that R can be chose so that at every point in the top of $R \cap \Gamma$ the two limits are above 1 and at every point in the bottom of $R \cap \Gamma$ these limits are below 1. The result follows.

4.3 Swirling flow

In the period 1969-1975 McLeod wrote seven papers on the general problem of the symmetric flow above an infinite rotating disk, or between two infinite rotating disks. These problems were introduced by von Karman and are called problems in “swirling flow”. Here I will briefly describe the results in two of these papers.

4.3.1 Flow between two disks

ON THE FLOW BETWEEN TWO COUNTER-ROTATING INFINITE
PLANE DISKS
(1969, with S. Parter)

One of the contacts McLeod made at the Math Research Center in Madison was S. V. Parter, with whom he wrote two papers on swirling flow. One of these, [14], resolved a dispute between two distinguished applied mathematicians about the boundary layer behavior of the flow.

In the situation where the angular velocities of the two rotating disks were of equal magnitude but opposite in signs, K. Stewartson had maintained that the main body of the flow was at rest, with boundary layers at each plate (Stewartson 1953), while G. K. Batchelor argued that the transition between the angular velocities of the two plates occurred in a narrow region in the middle (Batchelor 1951). By making the asymptotic analysis of Stewartson rigorous, McLeod and Parter disproved the conjecture of Batchelor.

To me this paper beautifully illustrates one role that applied analysis can play for applied mathematicians. Other papers of McLeod in which rigorous mathematics was used to disprove “results” which had been obtained by formal asymptotic arguments include [20] and [26]. Among papers where he confirmed the results of more applied researchers I can mention [10], [27], and, I presume,

most of the papers which he co-authored with applied mathematicians, such as [28], [22], and [19].

4.3.2 Flow above a single disk – use of continuation

THE EXISTENCE OF AXIALLY SYMMETRIC FLOW ABOVE A ROTATING DISK (1971)

The paper [12] solved perhaps the most basic problem in the area, one which had eluded some excellent mathematicians since von Karman developed the model in 1921. I chose it to discuss for two reasons: it is an impressive piece of analysis, and it marks McLeod’s first use of a technique which he was to use again in some important work in areas unrelated to fluid mechanics, that is, the technique of continuation.

The idealized physical problem studied here is that of single infinite disk rotating with angular velocity Ω_0 , and fluid occupying the half-space $x > 0$ above this disk. The fluid is assumed to have an imposed angular velocity of Ω_∞ at $x = \infty$. Von Karman showed that the problem reduces to the study of two ODEs with boundary conditions:

$$f''' + ff'' + \frac{1}{2}(g^2 - f'^2) = \frac{1}{2}\Omega_\infty^2 \quad (9)$$

$$g'' + fg' = f'g \quad (10)$$

$$f(0) = a, f'(0) = 0 \quad (11)$$

$$g(0) = \Omega_0, f'(\infty) = 0, g(\infty) = \Omega_\infty \quad (12)$$

Here a is a parameter measuring possible suction at the plate. The problem with this term in it was brought to McLeod’s attention by his colleague Hilary Ockendon.

The relation of x , f , and g to physical quantities is via a similarity substitution in the original partial differential equations and we omit those details.

McLeod had previously proved existence of a solution to this problem when $\Omega_\infty = 0$. In this paper he proves existence if Ω_0 and Ω_∞ have the same sign, for any value of a . In assessing this proof we must remember that this was done in the years before “bifurcation” theory became a prominent topic in applied analysis. What he does here anticipates this theory by analyzing the bifurcation curve of solutions in, for example, the $(\Omega_0, f''(0))$ plane. I point out the following remark in the paper:

“There is evidently a close affinity between this pattern of proof and the general existence theorem of Leray & Schauder (1934). There the existence of a non-zero index (which is closely allied to the idea of an odd number of solutions) for one value of a parameter is used to prove the existence for other values of the parameter. It must be possible, if not probable, that the existence theorem of the present paper can be treated as an application (although a highly non-trivial one) of the Leray-Schauder result, but it does seem that the work

involved in approaching the problem from this angle leads to a more complicated presentation rather than a simpler one, and it is not attempted here.”

In this sentence we can see the essence of McLeod’s approach to problems of this sort. He believes, and often shows, that in many cases getting to the heart of a particular problem with standard analysis yields more insights and easier proofs than application of wide-ranging theories.

In McLeod’s proof Ω_∞ is considered fixed, and positive for definiteness, and Ω_0 is a parameter. There is a trivial (constant) solution when $\Omega_0 = \Omega_\infty$. He shows, importantly, that this solution is unique, and by methods akin to the implicit function theorem, but in infinite dimensions, he then shows that there is a locally unique solution for $\Omega_0 - \Omega_\infty$ small. Assuming that $\Omega_0 > \Omega_\infty$ and a solution exists for Ω_0 in some interval $(\Omega_\infty, \Omega^*)$, he concludes that a solution exists if $\Omega_0 = \Omega^*$. Further, he proves that these solutions can be chosen so that $g > 0$, meaning that the whole body of fluid is rotating in the same direction.

He wishes to show that solutions exist for $\Omega_0 > \Omega^*$ and close to Ω^* . Using series expansions and some deep but classical results from analysis, he finds that if this is not the case then there must be a second branch of solutions which goes back from Ω^* and merges with the first branch at $\Omega_0 = \Omega_\infty$. But this contradicts the local uniqueness near Ω_∞ proved earlier, and this is the basic step in proving existence for all $\Omega_0 > \Omega_\infty$. A similar proof gives existence for $0 < \Omega_0 < \Omega_\infty$.

5 Two important papers on PDEs.

The term differential equation is usually taken to refer to one of two types, ordinary differential equations, in which there is only one independent variable, often time, and partial differential equations, where there are several independent variables, such as the three spatial coordinates. The majority of McLeod’s work in differential equations involved ODEs, though often these are of a type derived from a PDE. However of his six most cited papers, only one can be considered purely a problem in ODEs. The three with the greatest number of citations are all on partial differential equations, which is an indication of the greater importance attached by much of the modern applied analysis community to work in multivariable problems. I will now give some details on two of these three.

5.1

THE APPROACH OF SOLUTIONS OF NONLINEAR DIFFUSION
EQUATIONS TO TRAVELLING FRONT SOLUTIONS
(with Paul Fife, 1977)

Many authors refer to this article, [15], as the “classic paper of Fife and McLeod”. The general topic is nonlinear diffusion in a homogeneous medium where there can also occur chemical reactions. Fife and McLeod considered

so-called “excitable media”, in which an initial stimulus at one point develops into a wave traveling out from the initial point at a steady speed. With one spatial variable the relevant pde is of the form used by R. FitzHugh and by J. Nagumo:

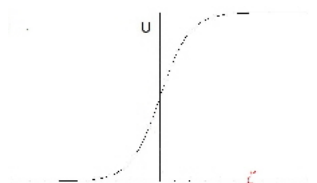
$$u_t = u_{xx} + u(1-u)(u-a) \tag{13}$$

where $0 < a < \frac{1}{2}$ and initial conditions are assumed:

$$u(x, 0) = \phi(x), \quad -\infty < x < \infty \quad .$$

First one looks for travelling wave solutions, by which is meant a solution of the form $u(x, t) = U(x + ct)$. A physically relevant solution must be bounded, and it is not hard to show that if $0 < a < \frac{1}{2}$ and $c > 0$ then a non-constant bounded travelling wave solution must satisfy

$$\lim_{\zeta \rightarrow -\infty} U(\zeta) = 0, \quad \lim_{\zeta \rightarrow \infty} U(\zeta) = 1$$



front moves to the left

The following result is straightforward using the “shooting method” mentioned above:

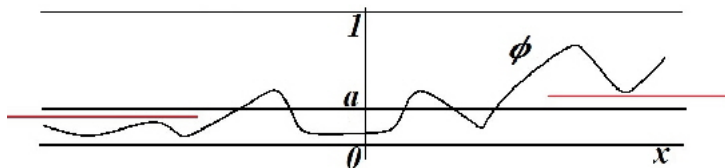
Theorem: *There is a unique value of c for which such a solution exists.*

We can then ask: Is this wave front stable?

Stability theorems for nonlinear partial differential equations are almost all local. If the initial condition is “sufficiently close” in some sense to the exact wave form, then solutions tend to a translation of this wave as $t \rightarrow \infty$. The result of Fife and McLeod is very different.

Theorem: *Suppose that ϕ is continuous and $0 < \phi(x) < 1$ for all x . Suppose also that*

$$\limsup_{x \rightarrow -\infty} \phi(x) < a, \quad \liminf_{x \rightarrow \infty} \phi(x) > a$$



Then for some x_0 and positive constants K and ω ,

$$|u(x, t) - U(x + ct - x_0)| < Ke^{-\omega t}$$

for all x and all $t > 0$.

This is a very strong result because the initial condition has only the restriction that it be physically reasonable ($0 < \phi(x) < 1$) and lie at least some small amount below the “threshold” a for large negative x and at least some small amount above a for large positive x . As shown in the figure, this allows a wide variety of initial conditions. The principle tools used in the proof are a priori estimates and comparison theorems for parabolic equations.

5.2

BLOW-UP OF POSITIVE SOLUTIONS OF SEMILINEAR HEAT EQUATIONS (with Avner Friedman, 1985)

McLeod wrote two papers with Friedman on this topic. Both have been widely cited. Here we will discuss one of their results for the following class of problems.

$$\begin{aligned} u_t &= \Delta u + f(u) && \text{in } \Omega, \\ u(x, 0) &= \phi(x) && \text{if } x \in \Omega \\ u(x, t) &= 0 && \text{if } x \in \partial\Omega, \quad 0 < t < T \end{aligned}$$

$\Omega \subset R^n$ bounded, C^2 boundary

Denote the closure of Ω by $\bar{\Omega}$ and assume that

$$\lim_{t \rightarrow T^-} \max_{x \in \bar{\Omega}} u(x, t) = \infty$$

Then the problem is to identify the blow-up set:

$$\left\{ x \mid \lim_{t \rightarrow T^-} u(x, t) = \infty \right\}.$$

A special case is when the domain and initial data are radially symmetric:

$$\begin{aligned} \Omega &= B_R \\ \phi &= \phi(r) \end{aligned}$$

and with

$$f(u) = u^p$$

for some $p > 1$.

Theorem: *If $\phi'(r) < 0$ for $r > 0$ then blowup occurs only at $r = 0$.*

The proof uses the maximal principle four times. Friedman and McLeod introduce the function $w(r) = r^{n-1}u_r$, showing by the maximal principle that it is negative for $r > 0$ and $0 < t < T$. They then consider a function

$$J = w + c(r)F(u)$$

where c and F are to be determined. They impose several conditions on the functions c and F , including for example that

$$f'F - fF' - \frac{2c'}{r^{n-1}}F'F + \frac{2(n-1)}{r^n}cF'F + \left(c^n - \frac{n-1}{r}c'\right)\frac{F}{c} \geq 0.$$

They eventually find that if $c(r) = \varepsilon r^n$ and $F(u) = u^\gamma$ with $1 < \gamma < p$ then their conditions are satisfied. Further use of the maximal principle allows them to conclude that for any $\gamma \in (1, p)$ there is an $\varepsilon > 0$ such that if $r > 0$ then

$$\int_{u(r,t)}^{\infty} \frac{1}{s^\gamma} ds \geq \frac{1}{2}\varepsilon r^2$$

for $0 < t < T$. This beautiful inequality implies that for $r > 0$, $u(r, t)$ is bounded on $(0, T)$, proving the Theorem.

5.3 A partial success

I suspect that the problem McLeod most wanted to solve among those where he was unsuccessful was the Stokes conjecture on the “wave of greatest height”. This conjecture was made by Lord Stokes (Stokes 1880) and involves waves in deep water. Stokes’ conjecture is the existence of a wave of greatest height and the validity of a clever formal argument he gave showing that the angle formed at the crest of this wave is $\frac{2\pi}{3}$.

An important step in proving this conjecture was taken in 1921 by A. Nekrasov (Nekrasov 1921). By what is known in fluid mechanics as a hodograph transformation, the region under one period, with wavelength λ , can be mapped onto the unit disk in the complex plane. Nekrasov showed that if $\phi(s)$ is the slope of the wave profile at the point corresponding to the point e^{is} on the unit circle, then ϕ satisfies the integral equation

$$\phi(s) = \frac{1}{3} \int_{-\pi}^{\pi} \frac{1}{\pi} \log \frac{\sin \frac{1}{2}(s+t)}{\sin \frac{1}{2}|s-t|} \frac{\sin \phi(t)}{\frac{1}{\mu} + \int_0^t \sin \phi(w) dw} dt. \quad (14)$$

Here μ is defined in terms the wave speed c and other physical parameters.

In 1961 Y. Krasovski (Krasovski 1961) showed that for each β with $0 < \beta < \frac{\pi}{6}$ there is a μ and a corresponding solution of (14) such that $\phi \geq 0$ and

$$\sup_{0 \leq s \leq \pi} \phi(s) = \beta. \quad (15)$$

However this does not give the range of μ for which there are solutions to (14). In 1978 it was shown by G. Keady and J. Norbury (Keady & Norbury 1978) that there is a solution for every $\mu > 3$. Each of the resulting waves is smooth at the crest.

The case $\mu = \infty$ corresponds to a stagnation point at the wave crest, and it is the case where, for a given c , the wave reaches the greatest height. Note that the equation (14) makes sense without the term $\frac{1}{\mu}$. The Stokes conjecture is that (14) has a solution ϕ_∞ in this case, the wave slope $\phi_\infty(s)$ is discontinuous at the crest ($s = 0$), and $\lim_{|s| \rightarrow 0} \phi_\infty(s) = \pm \frac{\pi}{6}$. Krasovski conjectured that $|\phi_\infty(s)| \leq \frac{\pi}{6}$ for all s in $[-\pi, \pi]$, but in [16] McLeod showed that ϕ_∞ takes on values above $\frac{\pi}{6}$ for large μ . In that paper as well he greatly improved on a proof the year before by Toland (Toland, 1978) that the solution ϕ_∞ exists. The validity of Stokes' limiting argument remained a challenge, however.

The full Stokes conjecture was finally proved in a paper by C. Amick, E. Fraenkel, and J. Toland (Amick et al, 1982), in which the authors made use of the following approximation to (14):

$$\theta(x) = \frac{1}{3} \int_0^\infty k(x, y) \frac{\sin \theta(y)}{\int_0^y \sin \theta(t) dt} dy, 0 < x < \infty \quad (16)$$

where $k(x, y) = \frac{1}{\pi} \log \frac{x+y}{|x-y|}$. (McLeod had used a similar approximation in [16].)

The advantage of (16) is that it admits the exact solution $\Theta(x) = \frac{\pi}{6}$. A key step in (Amick et al, 1982) is their

Theorem 2.1: If $\phi(x) = \frac{\pi}{6}$ is the only solution of (16) taking values in $(0, \frac{\pi}{3}]$, then any solution satisfying

$$\liminf_{s \rightarrow 0} \theta(s) > 0, \quad \sup_{s \in (0, \infty)} \theta(s) \leq \frac{\pi}{3} \quad (17)$$

has the property $\theta(s) \mapsto \frac{\pi}{6}$ as $s \mapsto 0$.

Of this result the authors of (Amick et al, 1982) wrote in a prominently displayed Acknowledgement:

“We are heavily indebted to J. B. McLeod for an emphatic statement of Theorem 2.1 (for $\lambda < \infty$) to one of us, during a conversation in January 1980. Although we were aware already of the usefulness of the approximate kernel k , it was McLeod's remark that ultimately led us to concentrate attention on the approximate integral equation.”

6 The Painlevé equations

McLeod wrote seven papers on a now famous set of six nonlinear ordinary differential equations due to Painlevé. I was fortunate to be involved in the first of these [17].

This is not the place to explain the background or theory of these equations, which have become an important part of mathematical physics. See for example

the article “Painlevé equations – nonlinear special functions” by Peter Clarkson (Clarkson 2003), a student of Bryce who has become the leading authority on Painlevé equations since writing his D. Phil. thesis on one of them in 1984. Here I will describe how I came upon the problem we wrote about, and how Bryce solved it.

In 1978 I spent some time at Cornell, where I talked with G. S. S. Ludford, a leading applied mathematical authority on combustion theory. He showed me the equation

$$y'' - xy = 2y^{2\tau+1}$$

where τ is a positive constant, which he had encountered in work with C. deBoer (Ludford & deBoer 1975). The simplest version of the problem they posed “for a mathematician” was to prove that this equation has exactly one non-constant solution which exists on $(-\infty, \infty)$. Ludford and deBoer also pointed out that when $\tau = 1$ this is the so-called “second Painlevé transcendent”.

I had been using the “shooting method”, mentioned earlier, for about as long as Bryce, having learned a rudimentary form of the technique from an off-hand remark of my Ph. D. advisor, N. Levinson. I thought the problem could be done that way, and perhaps, though my memory is unclear, had an outline of a proof before I talked about it with Bryce when I visited him in Madison later that year. But during that Madison visit I learned of some numerical work being done by students of Whitham at Cal Tech which raised the problem to a different level.

This work was for the Painlevé case of $\tau = 1$:

$$y'' - xy = 2y^3 \tag{18}$$

It is not hard to show that the solution being sought, call it y_* , tends to zero at infinity, and it is reasonable to conclude that its asymptotics for large x resemble those for the well known Airy equation

$$y'' - xy = 0 \tag{19}$$

One expects that

$$\lim_{k \rightarrow \infty} \frac{y(x)}{Ai(x)} = k,$$

Ai being the so-called Airy function from mathematical physics and k some positive constant. What the Cal Tech group had shown numerically was that, to many decimal places, the solution of (18) which exists on $(-\infty, \infty)$ is characterized by having $k = 1$. The obvious question is: why?

Bryce, as a student of Titchmarsh, knew all about Airy functions. I will never forget the question he asked me when I arrived at his office one morning:

“Did you notice that by using the Airy transform you can get an exact solution of the Painlevé?”

No, I had not noticed. But this insight, plus Bryce’s unparalleled technical competence in handling asymptotic analysis of this sort, led to a proof that $k = 1$, a remarkable result on what can be called the “connection” problem of

relating the behavior of the solution as $x \rightarrow \infty$ to its continued existence as $x \rightarrow -\infty$. The importance of this result in the area of solitons only seems to grow over time.

7 Oxford

Clearly McLeod was stimulated by the mathematical atmosphere at Oxford. Initially he interacted mostly with Chaundy and Titchmarsh, but after the pivotal 1968 conference in Edinburgh mentioned above his horizons broadened tremendously. We described earlier one of the ODE boundary value problems from fluid mechanics presented in Edinburgh by James Serrin, and McLeod's work on these marked the beginning of his almost complete shift to nonlinear differential equations.

During much of McLeod's time at Oxford he was well supported by outside funding from the SRC and its successors, enabling him to bring important visitors to the Mathematical Institute, such as Serrin, Tosio Kato, Paul Fife, and Avner Friedman. He also benefited from the lively Oxford applied maths group spearheaded by Alan Tayler, John Ockendon FRS, Hilary Ockendon, Sam Howison, and later Jon Chapman. In particular, the important collaboration between McLeod and Kato, which I discussed above, originated from a Study Group on mathematics in industry, one of a long and on-going series pioneered by Leslie Fox and Alan Tayler. The particular model studied by McLeod and Kato was formulated by Tayler and John Ockendon in (Ockendon & Tayler 1971). A number of other papers, such as [22] and [21], were either in collaboration with Oxford applied mathematicians or inspired by their work, and McLeod had some joint grant support with members of this group. In addition he co-advised some applied maths doctoral students.

He had some outstanding D. Phil. students at Oxford, including Jack Carr, Peter Clarkson, Joe Conlon, Michael Shearer, Charles Stuart, and others, some of whom brought their skills to industry and the City. According to his wife Eunice, the thing Bryce most enjoyed about Oxford was the teaching, both undergraduate and graduate, because the students were so good. The undergraduate teaching was through Wadham College, and he said in [30] that life in Wadham was one of the things that made it difficult to leave when they went to Pittsburgh.

There were two conferences organized in his honor between 2001 and 2007. The first, mentioned earlier, was for his 70th birthday, and was held in Oxford, and the second was on the occasion of his retirement from Pittsburgh. At each one a number of his Oxford students gave talks, and without exception they began by expressing their gratitude for the positive attitude McLeod brought to their collaboration. As one said, "Bryce encouraged (his students) ... to get on with it."

Oxford, like other major mathematical centers, has an amazing number of visitors, and these provide constant cross fertilization of mathematical ideas from around the world. This alone must have made it difficult to leave. So why

did he leave Oxford for Pittsburgh?

As he said in [30],

“The (Oxford University) faculty board in mathematics, I think it has to be said, just wasn’t interested in applied analysis. There was always the feeling that Oxford wasn’t supporting the subject as it should.”

Research mathematicians, particularly so-called “pure” mathematicians, are sometimes categorized as either “problem solvers” or “theory builders”. McLeod was definitely a problem solver, and described himself as such in the introduction to [29].

An illuminating essay on this subject was written by Timothy Gowers FRS (Gowers 2000). In this essay Gowers, a Fields Medalist and theory building mathematician, argues for the importance of both theory building and problem solving maths. Gowers refers to writings of Sir Michael Atiyah FRS, also a Fields medalist and perhaps the dominant voice in pure mathematics at Oxford from 1972 to 1989, and he includes the following quote from an interview for The Mathematical Intelligencer in 1984:

INTERVIEWER: How do you select a problem to study?

ATIYAH: I think that presupposes an answer. I don’t think that’s the way I work at all. Some people may sit back and say, “I want to solve this problem” and they sit down and say, “How do I solve this problem?” I don’t. I’m interested in mathematics; I talk, I learn, I discuss and then interesting questions simply emerge. I have never started off with a particular goal, except the goal of understanding mathematics.”

Gowers says:

“.. the subjects that appeal to theory-builders are, at the moment, much more fashionable than the ones that appeal to problem-solvers. Moreover, mathematicians in the theory-building areas often regard what they are doing as the central core (Atiyah uses this exact phrase⁷) of mathematics, with (problem oriented subjects) thought of as peripheral”

By contrast, the problems that McLeod worked on could often be described as quite special, and not part of a general theory. He wrote

“ ... differential equations .. are very individual things and have to be tackled most often in a very individual way.” [11]

Another factor at Oxford was the rather sharp line that was drawn between pure and applied mathematics. McLeod was the Tutor in Pure Mathematics at Wadham college, and the senior pure mathematicians Oxford in the 70’s and 80’s did not encourage work of his sort. For example the quotes above show that Atiyah had a very different view of what constituted important mathematical research from McLeod.

It is not clear, however, that McLeod would have been happier if he had been officially in the Oxford applied math group. He was a pure mathematician in the

⁷in (Atiyah 1988)

sense that to him, it was crucial to give rigorous proofs of results. This is usually not the principle concern of applied mathematicians, particularly of the British school, who mostly look for what are sometimes called “formal” arguments, based on sophisticated, sometimes ingenious, manipulations and calculations, without worrying too much about the analytical details.

Today also there is a very sharp line between pure and applied mathematics in Oxford. That line, or maybe it is a half-plane, is the one that separates the “pure maths” wing of the magnificent new Mathematical Institute building from the “applied maths” wing. What changed from the 1980’s was that for the short time that McLeod had an office in the new building, it was in the applied wing, though his mathematics was of the same type as before. Indeed, during all his years in Pittsburgh he maintained an office for use in the summers in the quarters occupied by the Oxford Centre for Industrial and Applied Mathematics, led for much of that time by John Ockendon. Also the Oxford Center for Nonlinear Partial Differential Equations, headed by John Ball, is housed in the applied wing of the new building. Thus for several reasons it was natural that Bryce would have a retirement office in that wing.

Fortunately, the physical separation of pure and applied mathematicians stops in the basement, where the classrooms and cafeteria are located. From experience I know that it is possible to descend in one elevator to this level and then inadvertently take the wrong elevator back up, and find yourself in the opposite wing from where you started. Let us hope that mathematical ideas can take the same path. Indeed, McLeod’s work shows that they can.

I end this section with a quote from Gilbert Strang of MIT, a distinguished mathematician with wide ranging interests and much influence through his writings.

“Bryce was a true analyst – he solved problems! His papers gave answers rather than abstractions and they led the subject onward.”

8 Pittsburgh

Bryce published at a greater rate in Pittsburgh than he had in Oxford, probably because his teaching duties were less. The best known of the Pittsburgh papers was written with his colleague Bard Ermentrout, a leading mathematical biologist.

8.1 A problem from neurobiology

EXISTENCE OF TRAVELLING WAVES FOR A NEURAL NETWORK
(with Bard Ermentrout, 1991)

Ermentrout brought to McLeod the equation

$$u_t = -u + \int_{-\infty}^{\infty} k(x-y) S(u(y,t)) dy, \quad (20)$$

an equation derived from a limiting process in which a large number of neurons are connected in a one dimensional array, with excitatory “all-to-all” synaptic interaction. The goal is to obtain a traveling front, as in the Fife-McLeod paper discussed earlier. But in the generality which they considered, the equation does not reduce to a system of pde’s, and so the mathematics is even more difficult.

In [23] Ermentrout and McLeod obtained existence of fronts for a wide range of symmetric positive kernels k with $\int_{-\infty}^{\infty} k(s) ds = 1$, and firing functions S which are smooth, increasing, and bounded. The basic method was continuation, which we saw above in McLeod’s work on swirling flow. But here the setting is infinite dimensional.

This paper has often been cited by subsequent users of continuation in similar settings. As Ermentrout and McLeod described it, the idea is to move continuously “from the general problem to one where everything is known.” (Actually, they started at the known end and moved to the general problem.)

The starting point is to use the kernel

$$k(s) = \frac{1}{2}e^{-|s|}.$$

In this case, the equation does reduce to a pde, by use of the Fourier Transform in the convolution. The resulting equation

$$u_t = u_{xx} + S(u) - u.$$

turns out to be similar to the problem of Fife and McLeod. Making the traveling wave substitution $u = u(x - ct)$ gives

$$-cu' = u'' + S(u) - u,$$

and choosing S so that $\int_0^1 (S(u) - u) = 0$, it is easily shown by a phase plane argument that there is a standing wave ($c = 0$) connecting $u = 0$ to $u = 1$. (In dynamical system terms, this is a heteroclinic orbit in the (u, u') phase plane.)

Then they vary S and k gradually, finding a wave speed c at each point, until one reaches the given S and k of the problem. The linearized operator in the continuation is

$$L\phi = \int_{-\infty}^{\infty} k(x + cs - y) S'(u(y)) \phi(y) dy,$$

considered on $C_0(-\infty, \infty)$. It is shown that this operator is Fredholm with index 0 and has 1 as a simple eigenvalue, which is what enables continuation to proceed. The paper illustrates McLeod’s mastery of modern functional analysis techniques when required.

9 Later work

While the paper with Ermentrout is probably the best known of McLeod’s later papers, a number of others have achieved significant recognition. In particular

the papers “On the uniqueness of flow of a Navier-Stokes fluid due to a stretching boundary”, with K. R. Rajagopal (1989) and “Smooth static solutions of the Einstein/Yang-Mills equations” with J. Smoller, A. G. Wasserman, and S. T. Yau (1991) have been particularly influential in their respective areas. We mention also [20], [25], and [24].

The last of these was written with C-K Law, one of the most active researchers among McLeod’s Pittsburgh doctoral students and a leading applied analyst in Taiwan. Another active Pitt student, C-B Wang, wrote an excellent dissertation on Painlevé III and continued to work with McLeod through the 2000’s. His recent well-reviewed monograph (Wang 2013) was published by Springer.

McLeod collaborated with at least five other Pitt mathematics faculty, most often W. C. Troy and me, and K. R. Rajagopal, mentioned above, was in the Pitt Engineering school. He also had many discussions with mathematicians from neighboring Carnegie Mellon University, and in particular wrote an interesting paper with D. Kinderlehrer.

In the period after his retirement Bryce and I wrote the book [29]. With a focus on existence theory for ODE boundary value problems, it was designed as a text for students with a background which included the basic existence theorem for initial value problems due to Picard and analysis of phase planes. It contains some new proofs of known results and also some previously unpublished theorems. In the course of writing this book Bryce came up with a beautiful new proof of an important result due to A. C. Lazer and D. E. Leach (Lazer & Leach 1969), one of several on ODEs and pdes which Lazer wrote with two collaborators in around 1970 and which have been widely cited in the fifty years since. I will end this memoir by giving the crux of Bryce’s proof, since to me it encapsulates his ability to look at a problem from a totally new angle and thereby obtain deep new insights.

The problem, in simplest form, is about periodic solutions for equations of the form

$$y'' + n^2y + g(y) = p(t) \tag{21}$$

where n is a non-zero integer, p is continuous and periodic, say with period 2π , g is smooth,

$$\lim_{y \rightarrow \infty} g(y) \text{ and } \lim_{y \rightarrow -\infty} g(y) \text{ exist,}$$

and for all y ,

$$g(-\infty) < g(y) < g(\infty).$$

The Lazer - Leach result is a surprising blend of simple harmonic analysis and nonlinear ODE theory. It states that a necessary and sufficient condition for (21) to have a periodic solution is that

$$\sqrt{A^2 + B^2} < 2(g(\infty) - g(-\infty)),$$

where

$$A = \int_0^{2\pi} p(s) \sin ns \, ds, \quad B = \int_0^{2\pi} p(s) \cos ns \, ds.$$

The original proof was an elegant application of Schauder's fixed point theorem, and we include it in our book.⁸ Bryce had not known of the work until I called it to his attention, sometime around 2010 when he was in England and I in the US. I wrote to him suggesting that we include it in the book for its elegance and importance, and attached a copy of the paper. A few weeks later Bryce wrote back that he had not read the Lazer-Leach proof, for fear of prejudicing his view of the problem, but that he had his own proof, which began with the following result:

Lemma: *Suppose that in addition to the hypotheses above g satisfies a local Lipschitz condition. For any $r > 0$ consider (21) with the following initial conditions:*

$$y(0) = r \cos \beta, \quad y'(0) = r \sin \beta.$$

If r is sufficiently large, say $r > r_0$, then for every β this solution satisfies

$$(y(2\pi) - y(0)) \cos \beta - (y'(2\pi) - y'(0)) \sin \beta > 0.$$

This lemma is not hard to prove, and the local Lipschitz condition is easily removed at the end of the proof of the theorem. Now assume that there is no periodic solution. For every r and β there are R and γ , both depending continuously on r and β , such that

$$y(2\pi) - y(0) = R \sin \gamma, \quad y'(2\pi) - y'(0) = R \cos \gamma.$$

The lemma implies that if $r > r_0$ then $\sin(\gamma - \beta) > 0$. As β goes from 0 to 2π , γ must increase by 2π because the initial conditions at $\beta = 0$ and $\beta = 2\pi$ are the same.

If there is no periodic solution then as r decreases from above r_0 , γ and R continue to be well-defined, for all $\beta > 0$. Also, since γ is continuous in r and β , it must continue to increase by 2π as β goes from 0 to 2π . As $r \rightarrow 0$ however, R is bounded away from 0; say $R \geq \delta > 0$, because we are assuming that at $R = 0$ the solution is not periodic. For very small r the solution varies only a little for initial conditions on the circle of radius r , so with $R \geq \delta$, γ cannot increase by 2π , a contradiction.

Bryce found this proof when he was over 80.

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⁸One goal of our book was to show different ways of attacking the same problem.

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Thanks to: Hilary and John Ockendon, Edward Fraenkel, Eunice McLeod, Trevor Stuart, and two anonymous referees for advice and comments.