

Killing two-tensors

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JOINT WORK WITH Thomas Leistner

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(arXiv:2604.17855)

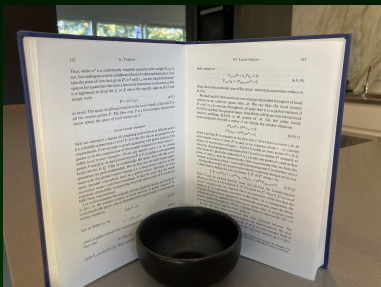
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Penrose at 95 and Twistors at 60

Local twistors



Spinors and Space-time
Roger Penrose and
Wolfgang Rindler
Cambridge University Press
1984 and 1986



$$\nabla_{A'}^{(A} \omega^{B)} = 0 \quad (6.9.8)$$

$$\left. \begin{aligned} \nabla_{AA'} \omega^B + i \epsilon_A^B \pi_{A'} &= 0 \\ \nabla_{AA'} \pi_{B'} + i P_{ABA'B'} \omega^B &= 0 \end{aligned} \right\} \quad (6.9.10)$$

local twistor transport

View as a connection

$$\nabla_{AA'} \begin{bmatrix} \omega^B \\ \pi_{B'} \end{bmatrix} = \begin{bmatrix} \nabla_{AA'} \omega^B + i \epsilon_A{}^B \pi_{A'} \\ \nabla_{AA'} \pi_{B'} + i P_{ABA'B'} \omega^B \end{bmatrix}$$

$$\underbrace{\qquad\qquad\qquad}_{\Psi} \mathcal{T} \longrightarrow \underbrace{\qquad\qquad\qquad}_{\Psi} \wedge^1 \otimes \mathcal{T}$$

By construction

$$\underbrace{\{\omega^A \mid \nabla_{A'}^{(A} \omega^{B)} = 0\}} \cong \underbrace{\left\{ \begin{bmatrix} \omega^B \\ \pi_{B'} \end{bmatrix} \mid \nabla_{AA'} \begin{bmatrix} \omega^B \\ \pi_{B'} \end{bmatrix} = 0 \right\}}$$

local twistors

conformal Killing spinors

twistor spinors

parallel sections

of the bundle $\mathcal{T}$,

a prolongation bundle

“But this equation leads to inconsistencies
on a conformally curved space-time”

Roger: 10/9/26

Killing fields

X^a a vector field on a Riemannian manifold with metric g_{ab}

$$\mathcal{L}_X g_{ab} = 2\nabla_{(a} X_{b)}$$

where $X_b \equiv X^a g_{ab}$ and $\nabla_a =$ Levi-Civita connection

Prolong the Killing equation $\nabla_{(a} \sigma_{b)} = 0$

$$\nabla_{(a} \sigma_{b)} = 0 \iff \boxed{\nabla_a \sigma_b = \mu_{ab} \in \wedge^2}$$

Compute

$$\nabla_{[a} \mu_{b]c} = \nabla_{[a} \nabla_{b]} \sigma_c = -\frac{1}{2} R_{ab}{}^d{}_c \sigma_d$$

But $\wedge^1 \otimes \wedge^2 \ni \Gamma_{abc} \mapsto \Gamma_{[ab]c} \in \wedge^2 \otimes \wedge^1$ an isomorphism

$$\implies \boxed{\nabla_a \mu_{bc} = R_{bc}{}^d{}_a \sigma_d}$$

closed system

Prolongation connection (Kostant 1955)

Tractor bundle on any Riemannian manifold

$$\mathbb{T} \equiv \frac{\wedge^1}{\wedge^2} \ni \begin{bmatrix} \sigma_b \\ \mu_{bc} \end{bmatrix} \xrightarrow{\nabla_a} \begin{bmatrix} \nabla_a \sigma_b - \mu_{ab} \\ \nabla_a \mu_{bc} - R_{bc}{}^d{}_a \sigma_d \end{bmatrix} \in \wedge^1 \otimes \mathbb{T}$$

is a **prolongation connection** for Killing fields:

$$\begin{bmatrix} \sigma_b \\ \mu_{bc} \end{bmatrix} \mapsto \sigma_b \quad \text{induces} \quad \left\{ \nabla_a \begin{bmatrix} \sigma_b \\ \mu_{bc} \end{bmatrix} = 0 \right\} \cong \{ \nabla_{(a} \sigma_{b)} = 0 \}$$

On the **unit sphere** $R_{bc}{}^d{}_a = \delta_b{}^d g_{ac} - \delta_c{}^d g_{ab}$ so

$$\nabla_a \begin{bmatrix} \sigma_b \\ \mu_{bc} \end{bmatrix} = \begin{bmatrix} \nabla_a \sigma_b - \mu_{ab} \\ \nabla_a \mu_{bc} + g_{ab} \sigma_c - g_{ac} \sigma_b \end{bmatrix} \quad \text{and is flat!!}$$

$$\Rightarrow \dim\{\text{Killing fields on } S^n\} = n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$$

$$\dim \mathfrak{so}(n+1)$$

✓✓✓

Killing 2-tensors

Suppose that $\sigma_{bc} = \sigma_{(bc)}$

Killing 2-tensor $\nabla_{(a}\sigma_{bc)} = 0$ (cf. Killing field $\nabla_{(a}\sigma_{b)} = 0$)

Hamiltonian interpretation

$\sigma^{bc}p_bp_c: T^*M \rightarrow \mathbb{R}$ Poisson commutes with $g^{bc}p_bp_c$

FACT $K^2 \equiv \{\sigma_{bc} = \sigma_{(bc)} \mid \nabla_{(a}\sigma_{bc)} = 0\}$ is finite-dimensional
 $K^1 \equiv \{\sigma_b \mid \nabla_{(a}\sigma_{b)} = 0\}$ is finite-dimensional

$$\dim K^1 \leq \text{rank}(\wedge^1 \oplus \wedge^2) = n(n+1)/2$$

$$\dim K^2 \leq \text{rank}(\odot^2 \wedge^1 \oplus \dots) = n(n+1)^2(n+2)/12$$

Another prolongation bundle

Killing 2-tensors from Killing fields

If σ_b and $\tilde{\sigma}_b$ are Killing fields, then $\sigma_{(b}\tilde{\sigma}_{c)}$ is a Killing 2-tensor

Do they all arise like this??

$\odot^2 K^1 \longrightarrow K^2$ surjective? Maybe not: the Kerr metric
 injective? Maybe not: the round sphere

But sometimes both are okay: $\odot^2 K^1(\mathbb{CP}_m) \xrightarrow{\cong} K^2(\mathbb{CP}_m)$

Kerr metric: Carter constant (1968) is a hidden symmetry

Matveev & Nikolayevsky (2024) $\rightsquigarrow$ hidden symmetries on

$$\begin{array}{c} \mathbb{HP}_k \quad \text{for } k \geq 3 \\ \mathbb{OP}_2 \end{array}$$

E & Leistner (2026) $\rightsquigarrow$ hidden symmetries on $E_6/F_4, \dots$

Killing 2-tensors by prolongation

There is a prolongation connection

(EG: Tom Branson, Andi Čap, ME, & Rod Gover (2006))

$$\begin{array}{c} \square \\ \oplus \\ \square \\ \oplus \\ \square \end{array} \ni \begin{bmatrix} \sigma \\ \mu \\ \rho \end{bmatrix} \xrightarrow{\nabla} \begin{bmatrix} \nabla \sigma - \mu \\ \nabla \mu - R \bowtie \sigma - \rho \\ \nabla \rho - R \bowtie \mu - \nabla R \bowtie \sigma \end{bmatrix}$$

BUT, in general, there are some options, e.g.

$$\begin{array}{l} \square \square \\ \square \end{array} \begin{array}{l} = \{ \mu_{bcd} = \mu_{b(cd)} \mid \mu_{(bcd)} = 0 \} \quad \text{obvious choice?} \\ = \{ \mu_{bcd} = \mu_{b[cd]} \mid \mu_{[bcd]} = 0 \} \quad \text{better choice?} \end{array}$$

and, even then



it's a bit of a mess



The messy details (Thomas Leistner & ME)

$$\begin{array}{c} \square \\ \oplus \\ \square \\ \oplus \\ \square \end{array} \ni \begin{bmatrix} \sigma_{bc} \\ \mu_{bcd} \\ \rho_{bcde} \end{bmatrix} \xrightarrow{\nabla_a} \begin{bmatrix} \nabla_a \sigma_{bc} + 2\mu_{(bc)a} \\ \nabla_a \mu_{bcd} - (R \lhd \sigma)_{abcd} - \rho_{abcd} \\ \nabla_a \rho_{bcde} - (R \diamond \mu)_{abcde} + (\nabla R * \sigma)_{abcde} \end{bmatrix}$$

$$(R \lhd \sigma)_{abcd} \equiv \frac{2}{3} R_{cd}{}^e{}_a \sigma_{be} - \frac{1}{3} R_{bc}{}^e{}_a \sigma_{de} + \frac{1}{3} R_{bd}{}^e{}_a \sigma_{ce}$$

$$(R \diamond \mu)_{abcde} \equiv$$

$$\begin{aligned} & R_{bc}{}^f{}_a \mu_{fde} + R_{de}{}^f{}_a \mu_{fbc} - \frac{1}{2} R_{be}{}^f{}_a \mu_{fcd} - \frac{1}{2} R_{cd}{}^f{}_a \mu_{fbe} \\ & + \frac{1}{2} R_{bd}{}^f{}_a \mu_{fce} + \frac{1}{2} R_{ce}{}^f{}_a \mu_{fbd} \\ & + \frac{2}{3} (R_{a(b}{}^f{}_d) \mu_{(ce)f} - R_{a(c}{}^f{}_d) \mu_{(be)f} - R_{a(b}{}^f{}_e) \mu_{(cd)f} + R_{a(c}{}^f{}_e) \mu_{(bd)f}) \\ & - \frac{2}{3} (R_{bc}{}^f{}_d \mu_{(ea)f} - R_{bc}{}^f{}_e \mu_{(da)f} + R_{de}{}^f{}_b \mu_{(ca)f} - R_{de}{}^f{}_c \mu_{(ba)f}) \end{aligned}$$

$$(\nabla R * \sigma)_{abcde} \equiv$$

$$\begin{aligned} & \frac{2}{3} (\sigma_{f[b} \nabla_{c]} R_{de}{}^f{}_a + (\nabla_a R_{de}{}^f{}_{[b} \sigma_{c]f} + \sigma_{f[d} \nabla_{e]} R_{bc}{}^f{}_a + (\nabla_a R_{bc}{}^f{}_{[d} \sigma_{e]f}) \\ & + \frac{1}{3} \sigma_{af} (\nabla_c R_{de}{}^f{}_b - \nabla_b R_{de}{}^f{}_c) \end{aligned}$$

A better way

Recall $\odot^2 K^1 \rightarrow K^2$ induced by $\sigma_b \odot \tilde{\sigma}_c \mapsto \sigma(b\tilde{\sigma}_c)$

$$\odot^2 K^1 \rightarrow \{\text{parallel sections of } \odot^2 \mathbb{T}\} \quad \text{👁️👁️}$$

where

$$\odot^2 \mathbb{T} = \odot^2 \begin{pmatrix} \wedge^1 \\ \oplus \\ \wedge^2 \end{pmatrix} = \begin{matrix} \wedge^1 \odot \wedge^1 \\ \oplus \\ \wedge^1 \otimes \wedge^2 \\ \oplus \\ \wedge^2 \odot \wedge^2 \end{matrix} = \begin{matrix} \boxed{\begin{matrix} \square \\ \square \end{matrix}} \\ \oplus \\ \boxed{\begin{matrix} \square & \square \\ \square & \square \end{matrix}} \\ \oplus \\ \boxed{\begin{matrix} \square & \square & \square \\ \square & \square & \square \end{matrix}} \end{matrix} \oplus \begin{matrix} \wedge^3 \\ \oplus \\ \wedge^4 \end{matrix}$$

👁️👁️ $\left\{ \begin{array}{l} \text{an isomorphism} \\ \text{if locally symmetric} \\ \text{i.e. } \nabla_a R_{bc}{}^d{}_e = 0 \\ \text{e.g. } E_6/F_4 \end{array} \right.$

a familiar bundle

a nice subbundle

A fly in the ointment

So we have nice short exact sequence

$$0 \rightarrow \begin{matrix} \wedge^3 \\ \oplus \\ \wedge^4 \end{matrix} \rightarrow \odot^2 \mathbb{T} \rightarrow \begin{matrix} \square \\ \oplus \\ \square \\ \oplus \\ \square \end{matrix} \rightarrow 0$$

comes equipped
with a natural
connection

NOT a parallel subbundle

Fly swatting
and removal

?

Modify the connection on $\odot^2 \mathbb{T}$

- without changing its parallel sections
- so that $\wedge^3 \oplus \wedge^4$ is now parallel !

Can do this using its own curvature !!

The modified connection on $\odot^2 \mathbb{T}$

Curvature of the induced connection on $\odot^2 \mathbb{T}$

$$\odot^2 \mathbb{T} = \odot^2 \left(\begin{array}{c} \wedge^1 \\ \oplus \\ \wedge^2 \end{array} \right) \ni \begin{bmatrix} \sigma \\ \mu \\ \rho \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ \boxed{R \bowtie \mu + (\nabla R) \bowtie \sigma} \\ *** \end{bmatrix}$$

$$\boxed{} \in \wedge^1 \otimes \boxplus \xrightarrow[\text{automorphism}]{\text{carefully chosen}} \wedge^1 \otimes \boxplus \ni \boxed{R \diamond (\mu, \sigma)}$$

$$\tilde{\nabla} \begin{bmatrix} \sigma \\ \mu \\ \rho \end{bmatrix} \equiv \nabla \begin{bmatrix} \sigma \\ \mu \\ \rho \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ R \diamond (\mu, \sigma) \end{bmatrix} \Rightarrow \begin{array}{c} \wedge^3 \\ \oplus \\ \wedge^4 \end{array}$$

is a parallel
subbundle
of $\odot^2 \mathbb{T}$

The prolongation connection from thin air

So NOW we have nice short exact sequence

$$0 \rightarrow \begin{matrix} \wedge^3 \\ \oplus \\ \wedge^4 \end{matrix} \rightarrow \odot^2 \mathbb{T} \rightarrow \begin{matrix} \square \\ \oplus \\ \square \\ \oplus \\ \square \end{matrix} \rightarrow 0$$

NOW equipped
with a modified
connection

NOW a parallel subbundle

Thm

The induced
connection on

is the prolongation
connection !!



Consequences

Locally symmetric case: the kernel of $\odot^2 K^1(M) \rightarrow K^2(M)$ is controlled by a connection on

$$\begin{array}{c} \wedge^3 \\ \oplus \\ \wedge^4 \end{array} \leftrightarrow \frac{\text{Killing-Yano 3-forms}}{\phi_{bcd} \mid \nabla_a \phi_{bcd} \text{ is skew}}$$

But F. Belgun, A. Moroianu, & U. Semmelmann (DGA 2006)
or P.-A. Nagy (JoLT 2013) $\Rightarrow$

There are no Killing-Yano 3-forms on an irreducible locally symmetric space M of compact type unless

- $M \subseteq S^n$ or
- $M \subseteq G$, a compact Lie group

Group case: $\ker \odot^2 K^1(G) \rightarrow K^2(G) = \text{span of } \underline{\text{Cartan 3-form}}$

What about E_6/F_4 ?

As homogeneous bundles (use LiE (Joseph Wolf & ME 2011))

$$\mathfrak{e}_6 = \begin{array}{c} 0 & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ & & | & & \\ & & 1 & \bullet & \end{array} = \underbrace{\begin{array}{c} 0 & 0 & 0 & 1 \\ \bullet & \bullet & \bullet & \bullet \end{array}}_{\wedge^1} \oplus \underbrace{\begin{array}{c} 1 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet \end{array}}_K$$

Killing two-tensors on E_6/F_4 are

$$\odot^2 \left(\begin{array}{c} 0 & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ & & | & & \\ & & 1 & \bullet & \end{array} \right) \oplus \boxed{\begin{array}{c} 0 & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ & & | & & \\ & & 1 & \bullet & \end{array}}_{\text{hidden symmetries}}$$

Dimension is $\frac{78 \times 79}{2} + 78 = 3081 + 78 = \underline{\underline{3159}}$

The End

Thanks for listening

HAPPY BIRTHDAY ROGER