AN OPTIMAL TRANSPORT FORMULATION OF THE EINSTEIN EQUATIONS OF GENERAL RELATIVITY

A. MONDINO AND S. SUHR

Abstract. The goal of the paper is to give an optimal transport formulation of the full Einstein equations of general relativity, linking the (Ricci) curvature of a space-time with the cosmological constant and the energy-momentum tensor. Such an optimal transport formulation is in terms of convexity/concavity properties of the Shannon-Bolzmann entropy along curves of probability measures extremizing suitable optimal transport costs. The result gives a new connection between general relativity and optimal transport; moreover it gives a mathematical reinforcement of the strong link between general relativity and thermodynamics/information theory that emerged in the physics literature of the last years.

Contents

Key words and phrases. Ricci curvature, optimal transport, Lorentzian manifold, general relativity, Einstein equations, strong energy condition.

A. Mondino: University of Oxford, Mathematical Institut, email: Andrea.Mondino@maths.ox.ac.uk. Supported by the EPSRC First Grant EP/R004730/1 “Optimal transport and geometric analysis” and by the ERC Starting Grant 802689 “CURVATURE”.

S. Suhr: University of Bochum, email: Stefan.Suhr@ruhr-uni-bochum.de. Supported by the SFB/TRR 191 “Symplectic Structures in Geometry, Algebra and Dynamics”, funded by the Deutsche Forschungsgemeinschaft.

1. Introduction

In recent years, optimal transport revealed to be a very effective and innovative tool in several fields of mathematics and applications. By way of example, let us mention fluid mechanics (e.g. Brenier [14] and Benamou-Brenier [11]), partial differential equations (e.g. Jordan-Kinderleher-Otto [45] and Otto [54]), random matrices (e.g. Figalli-Guionnet [31]), optimization (e.g. Bouchitté-Buttazzo [17]), non-linear σ-models (e.g. Carfora [20]), geometric and functional inequalities (e.g. Cordero-Erausquin-Nazaret-Villani [23], Figalli-Maggi-Pratelli [32], Klartag [44], Cavalletti- Mondino [22]) Ricci curvature in Riemannian geometry (e.g. Otto-Villani [55], Cordero Erausquin-McCann-Schmuckenschläger [24], Sturm-VonRenesse [62]) and in metric measure spaces (e.g. Lott-Villani [46], Sturm [59, 60], Ambrosio-Gigli-Savaré [4]). For more details about optimal transport and its applications in both pure and applied mathematics, we refer the reader to the many books on the topic, e.g. [1, 3, 58, 66, 67].

Here let us just quote two of the many applications to partial differential equations. In the pioneering work of Jordan-Kinderleher-Otto [45] it was discovered a new optimal transport formulation of the Fokker-Planck equation (and in particular of the heat equation) as a gradient flow of a suitable functional (roughly, the Boltzmann-Shannon entropy defined below in (1.5) plus a potential) in the Wasserstein space (i.e. the space of probability measures with finite second moments endowed with the quadratic Kantorovich-Wasserstein distance); later, Otto [54] found a related optimal transport formulation of the porous medium equation. The impact of these works in the optimal transport community has been huge, and opened the way to a more general theory of gradient flows (see for instance the monograph by Ambrosio-Gigli-Savaré [3]).

The goal of the present work is to give a new optimal transport formulation of another fundamental class of partial differential equations: the Einstein equations of general relativity. First published by Einstein in 1915, the Einstein equations describe gravitation as a result of space-time being curved by mass and energy; more precisely, the space-time (Ricci) curvature is related to the local energy and momentum expressed by the energy-momentum tensor. Before entering into the topic, let us first recall that the Einstein equations are *hyperbolic* evolution equations (for a comprehensive treatment see the recent monograph by Klainerman-Nicoló [43]). Instead of a gradient flow/PDE approach, we will see the evolution from a geometric/thermodynamic/information point of view.

Next we briefly recall the formulation of the Einstein equations. Let M^n be an n-dimensional manifold ($n \geq 3$, the physical dimension being $n = 4$) endowed with a Lorentzian metric g, i.e. g is a nondegenerate symmetric bilinear form of signature $(-++\ldots+)$. Denote with Ric and Scal the Ricci and the scalar curvatures of (M^n, g). The Einstein equations read as

$$\mathrm{Ric} - \frac{1}{2}\mathrm{Scal}\, g + \Lambda g = 8\pi T, \tag{1.1}$$

where $\Lambda \in \mathbb{R}$ is the cosmological constant, and T is the energy-momentum tensor. Physically, the cosmological constant Λ corresponds to the energy density of the vacuum; the energy-momentum tensor is a symmetric bilinear form on M representing the density of energy and momentum, acting as the source of the gravitational field.

1.1. **Statement of the main results.** Recall that in a Lorentzian manifold (M^n, g), a non-zero tangent vector $v \in T_xM$ is said *time-like* if $g(v, v) < 0$. If M admits a continuous no-where vanishing time-like vector field X, then (M, g) is said to be *time-oriented* and it is called a *space-time*. The vector field X induces a partition on the set of time-like vectors, into two equivalence classes: the *future pointing* tangent vectors v for which $g(X, v) < 0$ and the *past pointing* tangent vectors v for which $g(X, v) > 0$. The closure of the set of future pointing time-like vectors is denoted

$$\mathcal{C} = \mathrm{Cl}(\{v \in TM : \; g(v, v) < 0 \text{ and } g(X, v) < 0\}) \subset TM.$$

A physical particle moving in the space-time $(M, g, \mathcal{C})$ is represented by a *causal curve* which is an absolutely continuous curve, γ, satisfying

$$\dot\gamma_t \in \mathcal{C} \text{ a.e. } t \in [0, 1].$$

If the particle cannot reach the speed of light (e.g. massive particle), then it is represented by a *chronological* curve which is an absolutely continuous curve, γ, satisfying

$$\dot\gamma_t \in \mathrm{Int}(\mathcal{C}) \text{ a.e. } t \in [0, 1],$$

where $\mathrm{Int}(\mathcal{C})$ is the interior of the cone $\mathcal{C}$ made by future pointing time-like vectors. The Lorentz length of a causal curve is

$$L_g(\gamma) = \int_0^1 \sqrt{-g(\dot\gamma_t, \dot\gamma_t)}\, dt.$$

A point y is in the future of x, denoted $y >> x$, if there is a future oriented chronological curve from x to y; in this case, the *Lorentz distance* or *proper time* between x and y is defined by

$$\sup\{L_g(\gamma) : \; \gamma_0 = x \text{ and } \gamma_1 = y, \; \gamma \text{ chronological}\} > 0,$$

which is achieved by a geodesic which is called a *maximal geodesic*. See for example [28, 53, 68]

In this paper, we consider the following Lorentzian Lagrangian on TM for $p \in (0, 1)$:

$$\mathcal{L}_p(v) := \begin{cases} -\frac{1}{p}(-g(v, v))^{\frac{p}{2}} & \text{if } v \in \mathcal{C} \\ +\infty & \text{otherwise.} \end{cases} \tag{1.2}$$

Note that if p were 1 this would be the negative of the integrand for the Lorentz length given above. Here we study $p \in (0, 1)$ because this Lorentzian Lagrangian $\mathcal{L}_p$ has good convexity properties for such p [see Lemma 2.1].

Let $\mathrm{AC}([0,1],M)$ denote the space of absolutely continuous curves from $[0,1]$ to M. The Lagrangian action $\mathcal{A}_p$, corresponding to the Lagrangian $\mathcal{L}_p$ and defined for any $\gamma \in \mathrm{AC}([0,1],M)$, is given by

$$\mathcal{A}_p(\gamma) := \int_0^1 \mathcal{L}_p(\dot\gamma_t)dt \in (-\infty,0] \cup \{+\infty\}. \tag{1.3}$$

Observe that $\mathcal{A}_p(\gamma) \in (-\infty,0]$ if and only if γ is a causal curve. Note that, if p were 1, this would be the negative of the Lorentz length of γ or the proper time along γ. Thus $-\mathcal{A}_p(\dot\gamma)$ can be seen as a kind of non-linear p-proper time along γ, enjoying better convexity properties. The reader may note the parallel with the theory of Riemannian geodesics, where one often studies the energy functional $\int|\dot\gamma|^2$ in place of the length functional $\int|\dot\gamma|$, due to the analogous advantages.

The choice of the minus sign in (1.3) is motivated by optimal transport theory, in order to have a minimization problem instead of a maximization one (as in the sup defining the Lorentz distance between points above). It is readily checked that the critical points of $\mathcal{A}_p$ with negative action are time-like geodesics [see Lemma 2.2]. The advantage of $\mathcal{A}_p$ with $p \in (0,1)$ is that it automatically selects an affine parametrization for its critical points with negative action.

The cost function, $c_p : M\times M \to (-\infty,0]\cup\{+\infty\}$, relative to the p-action $\mathcal{A}_p$ is defined by

$$c_p(x,y) = \inf\{\mathcal{A}_p(\gamma) : \gamma \in \mathrm{AC}([0,1],M), \gamma_0 = x, \gamma_1 = y\}.$$

Note that, if p were 1, then this would be the negative of the Lorentz distance between x and y.

Consider a relatively compact open subset

$$E \subset\subset \mathrm{Int}(\mathcal{C}) \subset TM \text{ and } r \in (0, \mathrm{inj}_g(E)),$$

where $\mathrm{inj}_g(E) > 0$ is the injectivity radius of the exponential map of g restricted to E. If we take $p_{TM\to M} : TM \to M$ to be the canonical projection map then

$$\forall x \in p_{TM\to M}(E) \text{ and } v \in T_xM \cap E \text{ with } g(v,v) = -r^2,$$

we have a maximal geodesic $\gamma_x : [0,1] \to M$ defined by

$$\gamma_x(t) = \exp_x((t-1/2)v) \text{ such that } \gamma_x(1/2) = x.$$

The Ricci curvature, $\mathrm{Ric}_x(v,v)$ at a point $x \in M$ in the direction v, is a trace of the curvature tensor so that intuitively it measures the average way in which geodesics near γ_x bend towards or away from it. See Section 1.3. In Riemannian geometry, the Ricci curvature influences the volumes of balls. Here, instead of balls, we define for any $x \in p_{TM\to M}(E)$

$$B_r^{g,E}(x) := \{\exp_x^g(tw) : w \in T_xM \cap E,\ g(w,w) = -1, t \in [0,r]\}$$

precisely to avoid the null directions.

Rather than considering individual paths between a given pair of points, we will consider distributions of paths between a given pair of distributions of points using the optimal transport approach.

We denote by $\mathcal{P}(M)$ the set of Borel probability measures on M. For any $\mu_1, \mu_2 \in \mathcal{P}(M)$, we say that a Borel probability measure

$$\pi \in \mathcal{P}(M \times M) \text{ is a coupling of } \mu_1 \text{ and } \mu_2$$

if $(p_i)_\sharp \pi = \mu_i, i = 1, 2$, where $p_1, p_2 : M \times M \to M$ are the projections onto the first and second coordinate. Recall that the push-forward $(p_1)_\sharp \pi$ is defined by

$$(p_1)_\sharp \pi(A) := \pi\big(p_1^{-1}(A)\big)$$

for any Borel subset $A \subset M$. The set of couplings of μ_1, μ_2 is denoted by $\mathrm{Cpl}(\mu_1, \mu_2)$. The *$c_p$-cost of a coupling* π is given by

$$\int_{M \times M} c_p(x, y) d\pi(x, y) \in [-\infty, 0] \cup \{+\infty\}.$$

Denote by $C_p(\mu_1, \mu_2)$ the *minimal cost relative to c_p* among all couplings from μ_1 to μ_2, i.e.

$$C_p(\mu_1, \mu_2) := \inf \left\{ \int c_p d\pi \;:\; \pi \in \mathrm{Cpl}(\mu_1, \mu_2) \right\} \in [-\infty, 0] \cup \{+\infty\}.$$

If $C_p(\mu_1, \mu_2) \in \mathbb{R}$, a coupling achieving the infimum is said to be *c_p-optimal.*

For $t \in [0, 1]$ denote by $\mathrm{e}_t : \mathrm{AC}([0, 1], M) \to M$ the evaluation map

$$\mathrm{e}_t(\gamma) := \gamma_t.$$

A *c_p-optimal dynamical plan* is a probability measure Π on $\mathrm{AC}([0, 1], M)$ such that $(\mathrm{e}_0, \mathrm{e}_1)_\sharp \Pi$ is a c_p-optimal coupling from $\mu_0 := (\mathrm{e}_0)_\sharp \Pi$ to $\mu_1 := (\mathrm{e}_1)_\sharp \Pi$. One can naturally associate to Π a curve

$$(\mu_t := (\mathrm{e}_t)_\sharp \Pi)_{t \in [0,1]} \subset \mathcal{P}(M)$$

of probability measures. The condition that Π is a c_p-optimal dynamical plan corresponds to saying that the curve $(\mu_t)_{t \in [0,1]} \subset \mathcal{P}(M)$ is a length minimizing geodesic with respect to C_p, i.e.

$$C_p(\mu_s, \mu_t) = |t - s| C_p(\mu_0, \mu_1) \quad \forall s, t \in [0, 1].$$

We will mainly consider a special class of c_p-optimal dynamical plans, that we call *regular*: roughly, a c_p-optimal dynamical plan is said to be regular if it is obtained by exponentiating the gradient (which is assumed to be time-like) of a smooth Kantorovich potential ϕ:

(1.4)
$$\mu_t = (\Psi^t_{1/2})_\sharp \mu_{1/2},\ \Psi^t_{1/2}(x) := \exp^g_x\Big(-(t - 1/2) |\nabla_g \phi|_g^{q-2} \nabla_g \phi(x) \Big),\ \frac{1}{p} + \frac{1}{q} = 1,$$

and moreover $\mu_t \ll \mathrm{vol}_g$ for all $t \in (0, 1)$, where vol_g denotes the standard volume measure of (M, g). For the precise notions, the reader is referred to Section 2.5.

A key role in our optimal transport formulation of the Einstein equations will be played by the (relative) Boltzmann-Shannon entropy. Denote by vol_g the standard volume measure on (M, g). Given an absolutely continuous probability measure $\mu = \varrho \, \mathrm{vol}_g$ with density $\varrho \in C_c(M)$, its Boltzmann-Shannon entropy (relative to vol_g) is defined as

$$\mathrm{Ent}(\mu | \mathrm{vol}_g) := \int_M \varrho \log \varrho \, d\mathrm{vol}_g. \tag{1.5}$$

We will be proving that the second order derivative of this entropy along a c_p-optimal dynamical plan,
(1.6)
$$\left|\tfrac{4}{r^2}\left[\mathrm{Ent}(\mu_1|\mathrm{vol}_g) - 2\mathrm{Ent}(\mu_{1/2}|\mathrm{vol}_g) + \mathrm{Ent}(\mu_0|\mathrm{vol}_g)\right] - \tilde{T}(v,v)\right| \leq \epsilon(r)$$
is equivalent to the Einstein Equation in Theorem 4.9. See Figure 1. Throughout the paper we will assume the cosmological constant Λ and the energy momentum tensor T to be given, say from physics and/or mathematical general relativity. Given g, Λ and T it is convenient to set

$$\tilde{T} := \frac{2\Lambda}{n-2} g + 8\pi T - \frac{8\pi}{n-2}\mathrm{Tr}_g(T)\, g. \tag{1.7}$$

so that the Einstein Equation can be written as $\mathrm{Ric} = \tilde{T}$, see Lemma 4.1.

Theorem 1.1 (Theorem 4.9)**.** *Let $(M, g, \mathcal{C})$ be a space-time of dimension $n \geq 3$. Then the following assertions are equivalent:*

(1) *$(M, g, \mathcal{C})$ satisfies the Einstein equations* (1.1)*, which can be rewritten in terms of $\tilde{T}$ of (1.7) as* $\mathrm{Ric} = \tilde{T}$.
(2) *For every $p \in (0,1)$ and for every relatively compact open subset $E \subset\subset \mathrm{Int}(\mathcal{C})$ there exist $R = R(E) \in (0,1)$ and a function*
$$\epsilon = \epsilon_E : (0,\infty) \to (0,\infty) \text{ with } \lim_{r\downarrow 0} \epsilon(r) = 0 \text{ such that}$$
$$\forall x \in p_{TM\to M}(E) \text{ and } v \in T_xM \cap E \text{ with } g(v,v) = -R^2$$
the next assertion holds. For every $r \in (0,R)$, setting $y = \exp^g_x(rv)$, there exists a regular c_p-optimal dynamical plan $\Pi = \Pi(x,v,r)$ with associated curve of probability measures
$$(\mu_t := (\mathrm{e}_t)_\sharp \Pi)_{t\in[0,1]} \subset \mathcal{P}(M)$$
such that
- $\mu_{1/2} = \mathrm{vol}_g(B^{g,E}_{r^4}(x))^{-1}\, \mathrm{vol}_g \llcorner B^{g,E}_{r^4}(x)$,
- $\mathrm{supp}(\mu_1) \subset \{\exp^g_y(r^2 w) : w \in T_yM \cap \mathcal{C},\ g(w,w) = -1\}$

and which has convex/concave entropy in the sense described in (1.6).
(3) *There exists $p \in (0,1)$ such that the assertion as in* (2) *holds true.*

Remark 1.2 (A heuristic thermodynamic interpretation of Theorem 1.1)**.** A curve $(\mu_t)_{t\in[0,1]} \subset \mathcal{P}(M)$ associated to a c_p-optimal dynamical plan can be interpreted as the evolution [(a)] of a distribution of gas passing through a given gas distribution $\mu_{1/2}$ (that in Theorem 1.1 is assumed to be concentrated in the space-time near x). Theorem 1.1 says that the Einstein equations can be equivalently formulated in terms of the convexity properties of the Bolzmann-Shannon entropy along such evolutions $(\mu_t)_{t\in[0,1]} \subset \mathcal{P}(M)$. Extrapolating a bit more, we can say that the second law of thermodynamics (i.e. in a natural thermodynamic process, the sum of the entropies of the interacting thermodynamic systems decreases, due to our sign convention) concerns the *first* derivative of the Bolzmann-Shannon entropy; gravitation

[(a)] strictly speaking t is not the proper time, but only a variable parametrizing the evolution

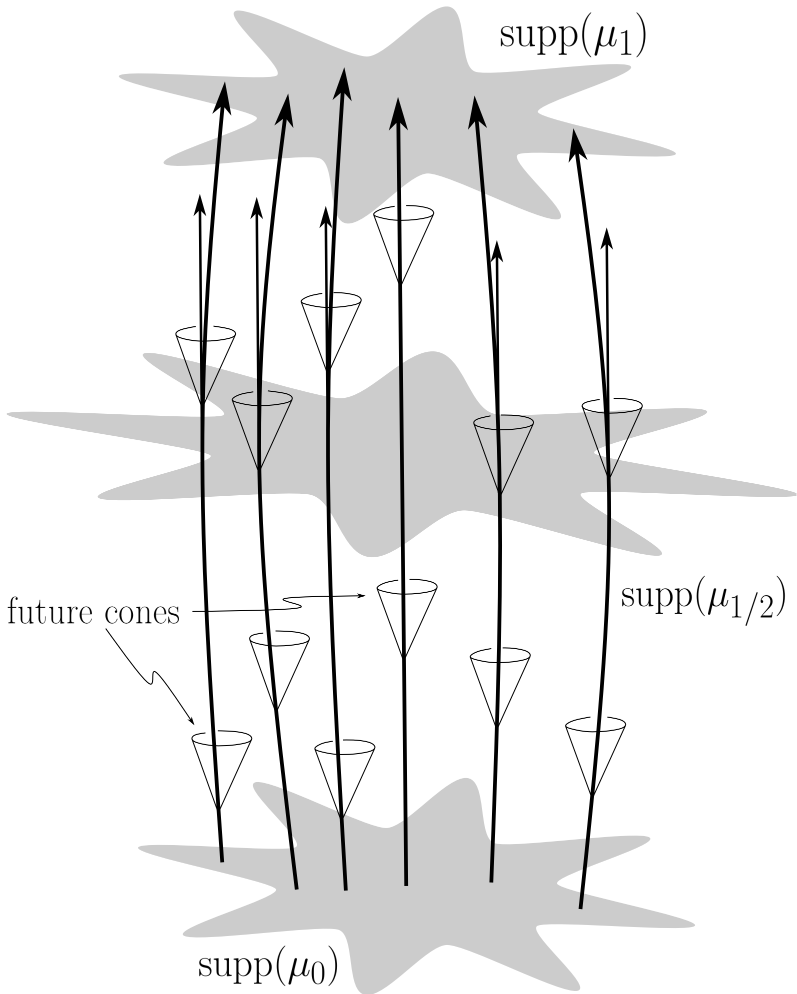

FIGURE 1. The transport in Theorem 1.1

(under the form of Ricci curvature) is instead related to the *second* order derivative of the Bolzmann-Shannon entropy along a natural thermodynamic process.

Remark 1.3 (Disclaimer)**.** In Theorem 1.1 we are *not claiming to solve* the general Einstein Equations via optimal transport; we are instead *proposing a novel formulation/characterization of the solutions of the Einstein Equations based on optimal transport*, assuming the cosmological constant Λ and the energy-momentum tensor T being already given (say from physics and/or mathematical general relativity). The aim is indeed to bridge optimal transport and general relativity, with the goal of stimulating fruitful connections between these two fascinating fields. In particular, optimal transport tools have been very successful to study Ricci curvature bounds in a (low regularity) *Riemannian and metric-measure* framework (see later in the introduction for the related literature) and it is thus natural to expect that optimal transport can be useful also in a low-regularity Lorentzian framework, where singularities correspond to important physical objects (e.g. black holes).

For equivalent formulations of Theorem 1.1, see Remark 4.10 and Remark 4.11.

In the vacuum case $T \equiv 0$ with zero cosmological constant $\Lambda = 0$, the Einstein equations read as

$$\mathrm{Ric} \equiv 0, \tag{1.8}$$

for an n-dimensional space-time $(M, g, \mathcal{C})$. Specializing Theorem 1.1 with the choice $\tilde{T} = 0$ (plus a small extra observation to sharpen the lower bound in (1.9) from $-\epsilon(r)$ to 0; moreover the same proof extends to $n = 2$) gives the following optimal transport formulation of Einstein vacuum equations with zero cosmological constant.

Corollary 1.4. *Let $(M, g, \mathcal{C})$ be a space-time of dimension $n \geq 2$. Then the following assertions are equivalent:*

(1) *$(M, g, \mathcal{C})$ satisfies the Einstein vacuum equations with zero cosmological constant, i.e.* $\mathrm{Ric} \equiv 0$.

(2) *For every $p \in (0,1)$ and for every relatively compact open subset $E \subset\subset \mathrm{Int}(\mathcal{C})$ there exist $R = R(E) \in (0,1)$ and a function*

$$\epsilon = \epsilon_E : (0,\infty) \to (0,\infty) \text{ with } \lim_{r \downarrow 0} \epsilon(r) = 0 \text{ such that}$$

$$\forall x \in p_{TM \to M}(E) \text{ and } v \in T_xM \cap E \text{ with } g(v,v) = -R^2$$

the next assertion holds. For every $r \in (0, R)$, setting $y = \exp_x^g(rv)$, there exists a regular c_p-optimal dynamical plan $\Pi = \Pi(x, v, r)$ with associated curve of probability measures

$$(\mu_t := (\mathrm{e}_t)_\sharp \Pi)_{t \in [0,1]} \subset \mathcal{P}(M)$$

such that

- $\mu_{1/2} = \mathrm{vol}_g(B^{g,E}_{r^4}(x))^{-1}\, \mathrm{vol}_g \llcorner B^{g,E}_{r^4}(x)$,
- $\mathrm{supp}(\mu_1) \subset \{\exp_y^g(r^2 w) : w \in T_yM \cap \mathcal{C},\, g(w,w) = -1\}$

and which has almost affine entropy in the sense that

$$0 \leq \mathrm{Ent}(\mu_1|\mathrm{vol}_g) - 2\mathrm{Ent}(\mu_{1/2}|\mathrm{vol}_g) + \mathrm{Ent}(\mu_0|\mathrm{vol}_g) \leq \epsilon(r). \tag{1.9}$$

(3) *There exists $p \in (0,1)$ such that the assertion as in (2) holds true.*

1.2. **Outline of the argument.** As already mentioned, the Einstein Equations can be written as $\mathrm{Ric} = \tilde{T}$ where $\tilde{T}$ was defined in (1.7), see Lemma 4.1. The optimal transport formulation of the Einstein equations will consist separately of an optimal transport characterization of the two inequalities

$$\mathrm{Ric} \geq \frac{2\Lambda}{n-2} g + 8\pi T - \frac{8\pi}{n-2} \mathrm{Tr}_g(T)\, g \tag{1.10}$$

and

$$\mathrm{Ric} \leq \frac{2\Lambda}{n-2} g + 8\pi T - \frac{8\pi}{n-2} \mathrm{Tr}_g(T)\, g, \tag{1.11}$$

respectively. The optimal transport characterization of the lower bound (1.10) will be achieved in Theorem 4.3 and consists in showing that (1.10) is equivalent to a *convexity* property of the Bolzmann-Shannon entropy along *every* regular c_p-optimal dynamical plan. The optimal transport characterization of the upper bound (1.11) will be achieved in Theorem 4.6 and consists in showing that (1.11) is equivalent to the *existence* of a large family of regular c_p-optimal dynamical plans (roughly the ones given by exponentiating the gradient of a smooth Kantorovich potential with Hessian vanishing at a given point) along which the Bolzmann-Shannon entropy satisfies the corresponding *concavity* condition.

Important ingredients in the proofs will be the following. In Theorem 4.3, for proving that Ricci lower bounds imply convexity properties of the entropy, we will perform Jacobi fields computations relating the Ricci curvature with the Jacobian of the change of coordinates of the optimal transport map (see Proposition 3.2 and Proposition 3.3); in order to establish the converse implication we will argue by contradiction via constructing c_p-optimal dynamical plans very localized in the space-time (Lemma 3.1).
In Theorem 4.3 we will consider the special class of regular c_p-optimal dynamical plans constructed in Lemma 3.1, roughly the ones given by exponentiating the gradient of a smooth Kantorovich potential with Hessian vanishing at a given point $x \in M$. For proving that Ricci upper bounds imply concavity properties of the entropy, we will need to establish the Hamilton-Jacobi equation satisfied by the evolved Kantorovich potentials (Proposition 3.4) and a non-linear Bochner formula involving the p-Box operator (Proposition A.1), the Lorentzian counterpart of the p-Laplacian. In order to show the converse implication we will argue by contradiction using Theorem 4.3.

1.3. **An Example. FLRW Spacetimes.** We illustrate Theorem 1.1 for the class of Friedmann-Lemaître-Robertson-Walker spacetimes (short FLRW spacetimes), a group of cosmological models well known in general relativity. See [53, Chapter 12] for a discussion of the geometry in the case $n = 4$.

FLRW spacetimes are of the form

$$(M, g) = (I \times \Sigma, -ds^2 + a^2(s)\sigma),$$

where $I \subset \mathbb{R}$ is an interval, $a\colon I \to (0, \infty)$ is smooth, and (Σ, σ) is a Riemannian manifold with constant sectional curvature $k \in \{-1, 0, 1\}$. The Ricci and scalar curvature are given by

$$\mathrm{Ric} = -(n-1)\frac{\ddot{a}}{a}ds^2 + \left[\frac{\ddot{a}}{a} + (n-2)\left(\frac{\dot{a}^2 + k}{a^2}\right)\right]a^2\sigma$$

and

$$\mathrm{Scal} = 2(n-1)\frac{\ddot{a}}{a} + (n-1)(n-2)\frac{\dot{a}^2 + k}{a^2},$$

respectively, where $\dot a := \frac{da}{ds}$. The stress-energy tensor is thus (assuming $\Lambda = 0$)

$$\begin{aligned}8\pi T &= \mathrm{Ric} - \frac{1}{2}\mathrm{Scal}\, g\\ &= \frac{(n-1)(n-2)}{2}\frac{\dot a^2 + k}{a^2} ds^2 - \left[(n-2)\frac{\ddot a}{a} + \frac{(n-2)(n-3)}{2}\frac{\dot a^2+k}{a^2}\right] a^2\sigma.\end{aligned}$$

The foliation

$$\mathfrak{O} := \{s \mapsto (s, \underline{x})\}_{\underline{x}\in\Sigma}$$

is a geodesic foliation by c_p-minimal geodesics. The orthogonal complement $\partial_t^\perp = T\Sigma$ with respect to g is integrable. Consider the projection

$$S\colon M = I \times \Sigma \to I$$

and for $r > 0$ the function

$$\phi\colon M \to \mathbb{R}, \quad x \mapsto r^{p-1} S(x).$$

It is easy to see that

$$\nabla_g^q \phi(x) := -|\nabla_g \phi|_g^{q-2} \nabla_g \phi(x) = r\partial_s.$$

For the c_p-transform we have (see Section 2.5)

$$\begin{aligned}\phi^{c_p}(s, \underline{y}) &= \inf_{x\in M} c_p(x, (s,\underline{y})) - \phi(x)\\ &= \inf_{s'<s} c_p((s', \underline{y}), (s, \underline{y})) - \phi(s', \underline{y})\\ &= \inf_{s'<s} -\frac{1}{p}(s-s')^p - r^{p-1}s' = \frac{p-1}{p} r^p - r^{p-1}s,\end{aligned}$$

where the second equality follows from the fact that the geodesics in $\mathfrak{O}$ minimize c_p to the level sets of ϕ. It follows that

$$\phi^{c_p}(s, \underline{y}) + \phi(s-r, \underline{y}) = -\frac{1}{p} r^p = c_p((s-r, \underline{y}), (s, \underline{y})),$$

i.e. $\partial^{c_p}\phi(x) = \{\exp_x(\nabla_g^q \phi(x))\}$ for all $x \in M$ whenever the right hand side is well defined (see Section 2.5 for the definition).

It now follows by standard transportation theory (see for instance [1, Theorem 1.13]) that for a Borel probability measure $\mu_{1/2}$ on $I \times \Sigma$ the family $\left(\mu_t := (\Psi_{1/2}^t)_\sharp \mu_{1/2}\right)_{t\in[0,1]}$, where

$$\Psi_{1/2}^t\colon I\times\Sigma \to I \times \Sigma, \quad (s, \underline{x}) \mapsto \left(s + r\left(t - \frac{1}{2}\right), \underline{x}\right)$$

defines a c_p-optimal dynamical plan as long as it is defined in accordance with the notation in (1.4) and ϕ is a smooth Kantorovich potential for $(\mu_t)_{t\in[0,1]}$.

If $\mu_{1/2} \ll \mathrm{vol}_g$ with density $\rho_{1/2} \in C_c(M)$, we get (compare with the proof of Theorem 4.3)

$$\begin{aligned}\mathrm{Ent}(\mu_t|\mathrm{vol}_g) &= \int_M \log \rho_t\,(y)\; d\mu_t\,(y) = \int_M \log \rho_t(\Psi^t_{1/2}(x))\, d\mu_{1/2}(x)\\ &= \int_M \log[\rho_{1/2}(x)(\mathrm{Det}_g(D\Psi^t_{1/2})(x))^{-1}]\, d\mu_{1/2}(x)\\ &= \mathrm{Ent}(\mu_{1/2}|\mathrm{vol}_g) - \int_M \log[\mathrm{Det}_g(D\Psi^t_{1/2})(x))]\, d\mu_{1/2}(x).\end{aligned}$$

We have $\mathrm{Det}_g(D\Psi^t_{1/2})((s,\underline{x}))) = \frac{a^{n-1}(s+r(t-1/2))}{a^{n-1}(s)}$ and thus obtain

$$\frac{d^2}{dt^2}\log[\mathrm{Det}_g(D\Psi^t_{1/2})(x))] = (n-1)\frac{\ddot{a}a - \dot{a}^2}{a^2}r^2.$$

Neglecting the term $\frac{\dot{a}^2}{a^2} \geq 0$ we conclude (compare with the proof of Proposition 3.3)

$$\begin{aligned}\frac{d^2}{dt^2}\mathrm{Ent}(\mu_t|\mathrm{vol}_g) &= -\frac{d^2}{dt^2}\int_M \log[\mathrm{Det}_g(D\Psi^t_{1/2})(x))]\, d\mu_{1/2}(x)\\ &\geq \int_M \mathrm{Ric}(r\partial_s, r\partial_s)_{\Psi^t_{1/2}(x)}\, d\mu_{1/2}(x),\end{aligned} \tag{1.12}$$

which implies

$$\tfrac{4}{r^2}\left[\mathrm{Ent}(\mu_1|\mathrm{vol}_g) - 2\mathrm{Ent}(\mu_{1/2}|\mathrm{vol}_g) + \mathrm{Ent}(\mu_0|\mathrm{vol}_g)\right] \geq \int_M \mathrm{Ric}(\partial_s,\partial_s)d\mu_{1/2}$$

for $r \to 0$, i.e. one side of (1.6).

Note that the gap in (1.12) is

$$(n-1)r^2\int_M \frac{\dot{a}^2}{a^2}\, d\mu_{1/2}.$$

From this we see that the bound from above

$$\begin{aligned}&\tfrac{4}{r^2}[\mathrm{Ent}(\mu_1|\mathrm{vol}_g) - 2\mathrm{Ent}(\mu_{1/2}|\mathrm{vol}_g) + \mathrm{Ent}(\mu_0|\mathrm{vol}_g)]\\ &\qquad\leq \int_M \mathrm{Ric}(\partial_s,\partial_s)d\mu_{1/2} + \epsilon(r),\end{aligned} \tag{1.13}$$

in (1.6) for the aforementioned transports holds for $r \to 0$ if $\mu_{1/2}$ is concentrated on $\{(s,\underline{x})|\ \dot{a}(s) = 0\}$ or, more generally, if $\mu_{1/2} = \mu^r_{1/2}$ satisfies $\lim_{r\to 0}\int_M \frac{\dot{a}^2}{a^2}\, d\mu^r_{1/2} = 0$.

The Levi-Civita connection ∇ of g satisfies

$$\nabla_X \partial_s = \nabla_{\partial_s} X = \frac{\dot{a}}{a}X$$

for all vector fields X tangent to Σ. Thus the Hessian of ϕ is given by

$$\mathrm{Hess}_\phi = r^{p-1}\mathrm{Hess}_S = r^{p-1}\nabla_{.}\nabla_g S = -r^{p-1}\nabla_{.}\partial_s = -r^{p-1}\frac{\dot{a}}{a}(\mathrm{Id} - ds). \tag{1.14}$$

It vanishes at $(s,\underline{x})$ if and only if $\dot{a}(s) = 0$. Thus the inequality (1.13) follows for these transports if the Hessian of ϕ vanishes on $\mathrm{supp}(\mu_{1/2})$ or, more generally, if $\lim_{r\to 0}\int_M \|r^{1-p}\mathrm{Hess}_\phi\|^2 d\mu^r_{1/2} = 0$.

1.4. **Related literature.**

1.4.1. *Ricci curvature via optimal transport in Riemannian setting.* In the Riemannian framework, a line of research pioneered by McCann [48], Cordero-Erausquin-McCann-Schmuckenschläger [24, 25], Otto-Villani [55] and von Renesse-Sturm, has culminated in a characterization of Ricci-curvature lower bounds (by a constant $K \in \mathbb{R}$) involving only the displacement convexity of certain information-theoretic entropies [62]. This in turn led Sturm [59, 60] and independently Lott-Villani [46] to develop a theory for lower Ricci curvature bounds in a non-smooth metric-measure space setting. The theory of such spaces has seen a very fast development in the last years, see e.g. [2, 4, 5, 6, 7, 18, 21, 22, 29, 34, 35, 51]. An approach to the complementary upper bounds on the Ricci tensor (again by a constant $K' \in \mathbb{R}$) has been recently proposed by Naber [52] (see also Haslhofer-Naber [37]) in terms of functional inequalities on path spaces and martingales, and by Sturm [61] (see also Erbar-Sturm [30]) in terms of contraction/expansion rate estimates of the heat flow and in terms of displacement *concavity* of the Shannon-Bolzmann entropy.

1.4.2. *Optimal transport in Lorentzian setting.* The optimal transport problem in Lorentzian geometry was first proposed by Brenier [13] and further investigated in [12, 63, 41]. An intriguing physical motivation for studying the optimal transport problem in Lorentzian setting called the "early universe reconstruction problem" [15, 33]. The Lorentzian cost C_p, for $p \in (0, 1)$, was proposed by Eckstein-Miller [26] and thoroughly studied by Mc Cann [49] very recently. In the same paper [49], Mc Cann gave an optimal transport formulation of the strong energy condition $\mathrm{Ric} \geq 0$ of Penrose-Hawking [57, 38, 39] in terms of displacement convexity of the Shannon-Bolzmann entropy under the assumption that the space time is globally hyperbolic.

We learned of the work of Mc Cann [49] when we were already in the final stages of writing the present paper. Though both papers (inspired by the aforementioned Riemannian setting) are based on the idea of analyzing convexity properties of entropy functionals on the space of probability measures endowed with the cost C_p, $p \in (0, 1)$, the two approaches are largely independent: while Mc Cann develops a general theory of optimal transportation in globally hyperbolic space times focusing on the strong energy condition $\mathrm{Ric} \geq 0$, in this paper we decided to take the quickest path in order to reach our goal of giving an optimal transport formulation of the full Einstein's equations. Compared to [49], in the present paper we remove the assumption of global hyperbolicity on the space-time, we extend the optimal transport formulation to any lower bound of the type $\mathrm{Ric} \geq \tilde{T}$ for any symmetric bilinear form $\tilde{T}$, and we also characterize general upper bounds $\mathrm{Ric} \leq \tilde{T}$.

1.4.3. *Physics literature.* The existence of strong connections between thermodynamics and general relativity is not new in the physics literature; it has its origins at least in the work Bekenstein [10] and Hawking with collaborators [8] in the mid-1970s about the black hole thermodynamics. These works inspired a new research field in theoretical physics, called entropic gravity (also known as emergent gravity), asserting that gravity is an entropic force rather than a fundamental interaction. Let us give a brief account. In 1995 Jacobson [36] derived the Einstein equations from the proportionality of entropy and horizon area of a black hole, exploiting the fundamental relation $\delta Q = T\,\delta S$ linking heat Q, temperature T and entropy S. Subsequently, other physicists, most notably Padmanabhan (see for instance the recent survey [56]), have been exploring links between gravity and entropy.

More recently, in 2011 Verlinde [65] proposed a heuristic argument suggesting that (Newtonian) gravity can be identified with an entropic force caused by changes in the information associated with the positions of material bodies. A relativistic generalization of those arguments leads to the Einstein equations.

The optimal transport formulation of Einstein equations obtained in the present paper involving the Shannon-Bolzmann entropy can be seen as an additional strong connection between general relativity and thermodynamics/information theory. It would be interesting to explore this relationship further.

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2. Preliminaries

2.1. Some basics of Lorentzian geometry.

Let M be a smooth manifold of dimension $n \geq 2$. It is convenient to fix a complete Riemannian metric h on M. The norm $|\cdot|$ on T_xM and the distance $\mathrm{dist}(\cdot,\cdot) : M \times M \to \mathbb{R}^+$ are understood to be induced by h, unless otherwise specified. Recall that h induces a Riemannian metric on TM. Distances on TM are understood to the induced by such a metric. The metric ball around $x \in M$ with radius r, with respect to h, is denoted by $B^h_r(x)$ or simply by $B_r(x)$.

A *Lorentzian metric* g on M is a smooth $(0,2)$-tensor field such that

$$g|_x : T_xM \times T_xM \to \mathbb{R}$$

is symmetric and non-degenerate with signature $(-,+,\dots,+)$ for all $x \in M$. It is well known that, if M is compact, the vanishing of the Euler characteristic of M is equivalent to the existence of a Lorentzian metric; on the other hand, any *non-compact* manifold admits a Lorentzian metric. A non-zero tangent vector $v \in T_xM$ is said

- *Time-like*: if $g(v,v) < 0$,
- *Light-like* (or *null*): if $g(v,v) = 0$ as well as $v \neq 0$,
- *Spacelike*: if $g(v,v) > 0$ or $v = 0$.

A non-zero tangent vector $v \in T_xM$ which is either time-like or light-like, i.e. $g(v,v) \leq 0$ and $v \neq 0$, is said *causal* (or *non-spacelike*). A Lorentzian manifold (M,g) is said to be *time-oriented* if M admits a continuous nowhere vanishing time-like vector field X. The vector field X induces a partition on the set of causal vectors, into two equivalence classes:

- The *future pointing* tangent vectors v for which $g(X,v) < 0$,
- The *past pointing* tangent vectors v for which $g(X,v) > 0$.

The closure of the set of future pointing time-like vectors is denoted

$$\mathcal{C} = \mathrm{Cl}(\{v \in TM : \ g(v,v) < 0 \text{ and } g(X,v) < 0\}) \subset TM.$$

Note that the fiber $\mathcal{C}_x := \mathcal{C} \cap T_xM$ is a closed convex cone and the open interior $\mathrm{Int}(\mathcal{C})$ is a connected component of $\{v : g(v,v) < 0\}$. A time-oriented Lorentzian manifold $(M, g, \mathcal{C})$ is called a *space-time*.

An absolutely continuous curve $\gamma : I \to M$ is said $(\mathcal{C})$-*causal* if $\dot\gamma_t \in \mathcal{C}$ for every differentiability point $t \in I$. A causal curve $\gamma : I \to M$ is said *time-like* if for every $s \in I$ there exist $\varepsilon, \delta > 0$ such that $\mathrm{dist}(\dot\gamma_t, \partial\mathcal{C}) \geq \varepsilon|\dot\gamma_t|$ for every $t \in I$ for which $\dot\gamma_t$ exists and $|s-t| < \delta$. In [16, Section 2.2] time-like curves are defined in terms of the Clarke differential of a Lipschitz curve. Whereas the definition via the Clarke differential is probably more satisfying from a conceptual point of view, the definition given here is easier to state. All relevant sets and curves used below are independent of the definition, [16, Lemma 2.11] and Proposition 2.4, though.

We denote by $J^+(x)$ (resp. $J^-(x)$) the set of points $y \in M$ such that there exists a causal curve with initial point x (resp. y) and final point y (resp. x), i.e. the causal future (resp. past) of x. The sets $I^\pm(x)$ are defined analogously by replacing causal curves by time-like ones. The sets $I^\pm(p)$ are always open in any space-time, on the other hand the sets $J^\pm(p)$ are in general neither closed nor open.

For a subset $A \subset M$, define $J^\pm(A) := \cup_{x\in A} J^\pm(x)$, moreover set

$$J^+ := \{(x,y) \in M \times M \,:\, y \in J^+(x)\}. \tag{2.1}$$

2.2. **The Lagrangian $\mathcal{L}_p$, the action $\mathcal{A}_p$ and the cost c_p.** On a space-time $(M, g, \mathcal{C})$ consider, for any $p \in (0,1)$, the following Lagrangian on TM:

$$\mathcal{L}_p(v) := \begin{cases} -\frac{1}{p}(-g(v,v))^{\frac{p}{2}} & \text{if } v \in \mathcal{C} \\ +\infty & \text{otherwise.} \end{cases} \tag{2.2}$$

The following fact appears in [49, Lemma 3.1]. We provide a proof for the readers convenience.

Lemma 2.1. *The function $\mathcal{L}_p$ is fiberwise convex, finite (and non-positive) on its domain and positive homogenous of degree p. Moreover $\mathcal{L}_p$ is smooth and fiberwise strictly convex on* $\mathrm{Int}(\mathcal{C})$.

Proof. It is clear from its very definition that the restriction of $\mathcal{L}_p$ to $\mathrm{Int}(\mathcal{C})$ is smooth. A direct computation gives

$$\frac{\partial \mathcal{L}_p}{\partial v^i} = (-g(v,v))^{\frac{p-2}{2}} g_{ik}v^k, \quad i = 1, \dots, n \tag{2.3}$$

$$\frac{\partial^2 \mathcal{L}_p}{\partial v^i \partial v^j} = (-g(v,v))^{\frac{p-4}{2}} \left(-g(v,v)g_{ij} + (2-p)g_{ik}v^k g_{jl}v^l\right), \quad i,j = 1, \dots, n. \tag{2.4}$$

Fix $v \in \mathrm{Int}(\mathcal{C})$. Decompose $w \in T_xM$ into $w^{\parallel}$ the part parallel to v and $w^{\perp}$ the part orthogonal to v, all with respect to g. Then we have

$$D^2_{vv}\mathcal{L}_p(w,w) = (-g(v,v))^{\frac{p-4}{2}}\Big(- g(w^{\perp},w^{\perp})g(v,v) - g(w^{\parallel},w^{\parallel})g(v,v) \tag{2.5}$$

$$+ (2-p)g(v,w^{\parallel})^2\Big) \tag{2.6}$$

$$= (-g(v,v))^{\frac{p-4}{2}}\left(-g(w^{\perp},w^{\perp})g(v,v) + (1-p)g(v,w^{\parallel})^2\right). \tag{2.7}$$

Since $g(w^{\perp},w^{\perp}) \geq 0$ and $p<1$ we have

$$D^2_{vv}\mathcal{L}_p(w,w) > 0$$

for $w \neq 0$. □

We define the Lagrangian action $\mathcal{A}_p$ associated to $\mathcal{L}_p$ as follows:

$$\mathcal{A}_p(\gamma) := \int_0^1 \mathcal{L}_p(\dot\gamma_t)dt \in (-\infty,0] \cup \{+\infty\}. \tag{2.8}$$

Note that if $\mathcal{A}_p(\gamma) \in \mathbb{R}$, then γ is causal. A causal curve $\gamma : [0,1] \to M$ is an $\mathcal{A}_p$*-minimizer* between its endpoints $x, y \in M$ if

$$\mathcal{A}_p(\gamma) = \inf\{\mathcal{A}_p(\eta) \,:\, \eta \in \mathrm{AC}([0,1],M), \eta_0 = x, \eta_1 = y\}.$$

Lemma 2.2. *Any $\mathcal{A}_p$-minimizer with finite action is either a future pointing time-like geodesic of (M,g) or a future pointing light-like pregeodesic of (M,g), i.e. an orientation preserving reparameterization is a future pointing light-like geodesic of (M,g).*

Proof. Let $\gamma\colon [0,1] \to M$ be a $\mathcal{A}_p$-minimizer with finite action. Then $\dot\gamma(t) \in \mathcal{C}$ for a.e. t. By Jensen's inequality we have

$$\int_0^1 -\frac{1}{p}(-g(\dot\eta,\dot\eta))^{\frac{p}{2}}dt \geq -\frac{1}{p}\left(\int_0^1 \sqrt{-g(\dot\eta,\dot\eta)}dt\right)^p,$$

for any causal curve $\eta\colon [0,1] \to M$ with equality if and only if η is parametrized proportionally to arclength.

Recall that the restriction of a minimizer to any subinterval of $[0,1]$ is a minimizer of the restricted action. Since any point in a spacetime admits a globally hyperbolic neighborhood, see [50, Theorem 2.14], the Avez-Seifert Theorem [53, Proposition 14.19] implies that every minimizer of $\mathcal{A}_1$ with finite action is a causal pregeodesic.

Combining both points we see that if the action of γ is negative, the curve is a time-like pregeodesic parameterized with respect to constant arclength, i.e. a time-like geodesic. If the action of γ vanishes, the curve is a light-like pregeodesic. □

Consider the cost function relative to the p-action $\mathcal{A}_p$:

$$c_p : M \times M \to \mathbb{R} \cup \{+\infty\}$$
$$(x,y) \mapsto \inf\{\mathcal{A}_p(\eta) \,:\, \eta \in \mathrm{AC}([0,1],M), \eta_0 = x, \eta_1 = y\}.$$

Remark 2.3. We will always assume that:

(i) The cost function is bounded from below on bounded subsets of $M \times M$. By transitivity of the causal relation this follows from the assumption that $c_p(x,y) > -\infty$ for all $x, y \in M$.

(ii) The cost function is localizable, i.e. every point $x \in M$ has a neighborhood $U \subset M$ such that the cost function of the space-time $(U, g|_U, \mathcal{C}|_U)$ coincides with the global cost function.

Note since the main results of this paper are local in nature, the assumptions can always be satisfied by restricting the space-time to a suitable open subset.

Proposition 2.4. *Fix $p \in (0,1)$ and let $(M, g, \mathcal{C})$ be a space-time. Then every point has a neighborhood U such that the following holds for the space-time $(U, g|_U, \mathcal{C}|_U)$. For every pair of points $x, y \in U$ with $(x,y) \in J^+_U$, the causal relation of $(U, g|_U, \mathcal{C}|_U)$, there exists a curve $\gamma : [0,1] \to U$ with $\gamma_0 = x$, $\gamma_1 = y$, and minimizing $\mathcal{A}_p$ among all curves $\eta \in \mathrm{AC}([0,1], M)$ with $\eta_0 = x$ and $\eta_1 = y$. Moreover γ is a constant speed geodesic for the metric g, $\dot\gamma \in \mathcal{C}$ whenever the tangent vector exists, and $\mathcal{A}_p(\gamma) \in \mathbb{R}$.*

Proof. It is well known that in a space-time every point has a globally hyperbolic neighborhood. Let U be such a neighborhood. If $(x,y) \in J^+_U$ there exists a curve with finite action $\mathcal{A}_p$ between x and y. At the same time the action is bounded from below, e.g. by a steep Lyapunov function, see [16]. Therefore any minimizer $\gamma \colon [0,1] \to U$ has finite action, i.e. $\dot\gamma(t) \in \mathcal{C}$ for almost all t. By Jensen's inequality we have

$$\int_0^1 -\frac{1}{p}(-g(\dot\eta,\dot\eta))^{\frac{p}{2}} dt \geq -\frac{1}{p}\left(\int_0^1 \sqrt{-g(\dot\eta,\dot\eta)} dt\right)^p,$$

for any causal curve $\eta \colon [0,1] \to U$ with equality if and only if η is parametrized proportionally to arclength. By the Avez-Seifert Theorem [53, Proposition 14.19] every minimizer of the right hand side is a causal pregeodesic. Combining both it follows that every $\mathcal{A}_p$-minimizer is a causal geodesic. $\square$

2.3. **Ricci curvature and Jacobi equation.** We now fix the notation regarding curvature for a Lorentzian manifold (M,g) of dimension $n \geq 2$. Called ∇ the Levi-Civita connection of (M,g), the Riemann curvature tensor is defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X X - \nabla_{[X,Y]} Z, \tag{2.9}$$

where X, Y, Z are smooth vector fields on M and $[X,Y]$ is the Lie bracket of X and Y.

For each $x \in M$, the Ricci curvature is a symmetric bilinear form $\mathrm{Ric}_x : T_xM \times T_xM \to \mathbb{R}$ defined by

$$\mathrm{Ric}_x(v,w) := \sum_{i=1}^n g(e_i,e_i) g(R(e_i,w)v, e_i), \tag{2.10}$$

where $\{e_i\}_{i=1,\dots,n}$ is an orthonormal basis of T_xM, i.e. $|g(e_i,e_j)| = \delta_{ij}$ for all $i, j = 1\dots,n$.

Given a endomorphism $\mathcal{U} : T_xM \to T_xM$ and a g-orthonormal basis $\{e_i\}_{i=1,\dots,n}$ of T_xM, we associate to $\mathcal{U}$ the matrix

$$(\mathcal{U}_{ij})_{i,j=1,\dots,n}, \quad \mathcal{U}_{ij} := g(e_i, e_j)\, g(\mathcal{U}e_i, e_j). \tag{2.11}$$

The trace $\mathrm{Tr}_g(\mathcal{U})$ and the determinant $\mathrm{Det}_g(\mathcal{U})$ of the endomorphism $\mathcal{U}$ with respect to the Lorentzian metric g are by definition the trace $\mathrm{tr}(\mathcal{U}_{ij})$ and the determinant $\det(\mathcal{U}_{ij}))$ of the matrix $(\mathcal{U}_{ij})_{i,j=1,\dots,n}$, respectively. It is standard to check that such a definition is independent of the chosen orthonormal basis of T_xM. Note that $\mathrm{Ric}_x(v,w)$ is the trace of curvature endomorphism $R(\cdot,w)v : T_xM \to T_xM$.

A smooth curve $\gamma : I \to M$ is called a *geodesic* if $\nabla_{\dot\gamma}\dot\gamma = 0$. A vector field J along a geodesic γ is said to be a *Jacobi field* if it satisfies the *Jacobi equation*:

$$\nabla_{\dot\gamma}(\nabla_{\dot\gamma} J) + R(J,\dot\gamma)\dot\gamma = 0. \tag{2.12}$$

2.4. **The q-gradient of a function.** Finally let us recall the definition of gradient and hessian. Given a smooth function $f : M \to \mathbb{R}$, the *gradient of* f denoted by $\nabla_g f$ is defined by the identity

$$g(\nabla_g f, Y) = df(Y), \quad \forall Y \in TM,$$

where df is the differential of f. The *Hessian of* f, denoted by Hess_f is defined to be the covariant derivative of df:

$$\mathrm{Hess}_f := \nabla(df).$$

It is related to the gradient through the formula

$$\mathrm{Hess}_f(X,Y) = g(\nabla_X \nabla_g f, Y), \quad \forall X, Y \in TM,$$

and satisfies the symmetry

$$\mathrm{Hess}_f(X,Y) = \mathrm{Hess}_f(Y,X) \quad \forall X, Y \in TM. \tag{2.13}$$

Next we recall some notions for the causal character of functions.

- A function $f\colon M \to \mathbb{R} \cup \{\pm\infty\}$ is a *causal function* if $f(x) \le f(y)$ for all $(x,y) \in J^+$;
- it is a *time function* if $f(x) < f(y)$ for all $(x,y) \in J^+ \setminus \Delta$, where Δ denotes the diagonal in $M \times M$.
- Following [16] we call a differentiable (C^k-) function $f\colon M \to \mathbb{R}$ ($k \in \mathbb{N} \cup \{\infty\}$) a *($C^k$-) Lyapunov* or *($C^k$-) temporal* function if $df_x|_{\mathcal{C}_x \setminus \{0\}} > 0$ for all $x \in M$.

Let q be the conjugate exponent to p, i.e.

$$\frac{1}{p} + \frac{1}{q} = 1, \quad \text{or equivalently } (p-1)(q-1) = 1.$$

Notice that, since p ranges in $(0,1)$ then q ranges in $(-\infty, 0)$. In order to describe the optimal transport maps later in the paper, it is useful to introduce the q-gradient (cf. [40])

$$\nabla_g^q \phi := -|g(\nabla_g \phi, \nabla_g \phi)|^{\frac{q-2}{2}} \nabla_g \phi \tag{2.14}$$

for differentiable Lyapunov functions $\phi\colon M \to \mathbb{R}$; in particular, $\nabla_g^q \phi(x) \in \mathcal{C}_x \setminus \{0\}$. Notice that,

$$\text{For } v \in \mathcal{C}_x \setminus \{0\},\ \nabla_g \phi(x) = -|g(v,v)|^{\frac{p-2}{2}} v \text{ if and only if } \nabla_g^q \phi(x) = v.$$

Moreover

$$x \mapsto \nabla_g^q \phi(x) \text{ is continuous (resp. } C^k,\ k \geq 1) \text{ on } U \subset \{|g(\nabla_g^q \phi, \nabla_g^q \phi)| > 0\}$$

if and only if

$$x \mapsto \nabla_g \phi(x) \text{ is continuous (resp. } C^k,\ k \geq 1) \text{ on } U \subset \{|g(\nabla_g \phi, \nabla_g \phi)| > 0\}.$$

The motivation for the use of the q-gradient comes from the Hamiltonian formulation of the dynamics; let us briefly mention a few key facts that will play a role later in the paper. For $\alpha \in T_x^*M$, let

$$\mathcal{H}_p(\alpha) = \sup_{v \in T_x M} [\alpha(v) - \mathcal{L}_p(v)] \tag{2.15}$$

be the Legendre transform of $\mathcal{L}_p$. Denote with g^* the dual Lorentzian metric on T^*M and $\mathcal{C}^* \subset T^*M$ the dual cone field to $\mathcal{C}$. Then $\mathcal{H}_p$ satisfies

$$\mathcal{H}_p(\alpha) := \begin{cases} -\frac{1}{q}(-g^*(\alpha,\alpha))^{\frac{q}{2}} & \text{if } \alpha \in \mathcal{C}^* \setminus T^{*,0}M \\ +\infty & \text{otherwise} \end{cases}, \tag{2.16}$$

for $(p-1)(q-1) = 1$. By analogous computations as performed in the proof of Lemma 2.1, one can check that

$$\nabla_g^q \phi(x) = D\mathcal{H}_p(-d\phi(x)). \tag{2.17}$$

By well known properties of the Legendre transform (see for instance [19, Theorem A.2.5]) it follows that $D\mathcal{H}_p$ is invertible on $\mathrm{Int}(\mathcal{C}^*)$ with inverse given by $D\mathcal{L}_p$. Thus (2.17) is equivalent to

$$D\mathcal{L}_p(\nabla_g^q \phi(x)) = -d\phi(x). \tag{2.18}$$

2.5. **c_p-concave functions and regular c_p-optimal dynamical plans.** We denote by $\mathcal{P}(M)$ the set of Borel probability measures on M. For any $\mu_1, \mu_2 \in \mathcal{P}(M)$, we say that a Borel probability measure

$$\pi \in \mathcal{P}(M \times M) \text{ is a coupling of } \mu_1 \text{ and } \mu_2$$

if $(p_i)_\sharp \pi = \mu_i, i = 1, 2$, where $p_1, p_2 : M \times M \to M$ are the projections onto the first and second coordinate. Recall that the push-forward $(p_1)_\sharp \pi$ is defined by

$$(p_1)_\sharp \pi(A) := \pi\big(p_1^{-1}(A)\big)$$

for any Borel subset $A \subset M$. The set of couplings of μ_1, μ_2 is denoted by $\mathrm{Cpl}(\mu_1, \mu_2)$. The *$c_p$-cost of a coupling* π is given by

$$\int_{M \times M} c_p(x, y) d\pi(x, y) \in [-\infty, 0] \cup \{+\infty\}.$$

Denote by $C_p(\mu_1, \mu_2)$ the *minimal cost relative to* c_p among all couplings from μ_1 to μ_2, i.e.

$$C_p(\mu_1, \mu_2) := \inf \left\{ \int c_p d\pi \ :\ \pi \in \mathrm{Cpl}(\mu_1, \mu_2) \right\} \in [-\infty, 0] \cup \{+\infty\}.$$

If $C_p(\mu_1, \mu_2) \in \mathbb{R}$, a coupling achieving the infimum is said to be *c_p-optimal.*

We next define the notion of *c_p-optimal dynamical plan.* To this aim, it is convenient to consider the set of $\mathcal{A}_p$-minimizing curves, denoted by Γ_p. The set Γ_p is endowed with the sup metric induced by the auxiliary Riemannian metric h. It will be useful to consider the maps for $t \in [0,1]$:

$$\begin{aligned} \mathrm{e}_t : \Gamma_p \to M, &\quad \mathrm{e}_t(\gamma) := \gamma_t \\ \partial\mathrm{e}_t : \Gamma_p \to TM, &\quad \partial\mathrm{e}_t(\gamma) := \dot{\gamma}_t \in T_{\gamma_t}M. \end{aligned}$$

A *c_p-optimal dynamical plan* is a probability measure Π on Γ_p such that $(\mathrm{e}_0, \mathrm{e}_1)_\sharp \Pi$ is a c_p-optimal coupling from $\mu_0 := (\mathrm{e}_0)_\sharp\Pi$ to $\mu_1 := (\mathrm{e}_1)_\sharp\Pi$.
We will mostly be interested in c_p-optimal dynamical plans obtained by "exponentiating the q-gradient of a c_p-concave function", what we will call *regular c_p-optimal dynamical plans*. In order to define them precisely, let us first recall some basics of Kantorovich duality (we adopt the convention of [1]).

Fix two subsets $X, Y \subset M$. A function $\phi : X \to \mathbb{R} \cup \{-\infty\}$ is said *c_p-concave* (with respect to (X, Y)) if it is not identically $-\infty$ and there exists $u : Y \to \mathbb{R} \cup \{-\infty\}$ such that

$$\phi(x) = \inf_{y \in Y} c_p(x, y) - u(y), \qquad \text{for every } x \in X.$$

Then, its *c_p-transform* is the function $\phi^{c_p} : Y \to \mathbb{R} \cup \{-\infty\}$ defined by

$$\phi^{c_p}(y) := \inf_{x \in X} c_p(x, y) - \phi(x), \tag{2.19}$$

and its *c_p-superdifferential* $\partial^{c_p}\phi(x)$ at a point $x \in X$ is defined by

$$\partial^{c_p}\phi(x) := \{y \in Y \,:\, \phi(x) + \phi^{c_p}(y) = c_p(x, y)\}. \tag{2.20}$$

Note that

$$\begin{cases} \phi(x) = c_p(x, y) - \phi^{c_p}(y), & \text{for all } x \in X, y \in \partial^{c_p}\phi(x) \\ \phi(x) \le c_p(x, y) - \phi^{c_p}(y), & \text{for all } x \in X, y \in Y. \end{cases} \tag{2.21}$$

From the definition it follows readily that if ϕ is c_p-concave, then for $(x, z) \in J^+ \cap (X \times X)$ we have

$$\phi(z) = \inf_{y \in Y} c_p(z, y) - u(y) \ge \inf_{y \in Y} c_p(x, y) - u(y) = \phi(x),$$

i.e. ϕ is a causal function. The same argument gives that $-\phi^{c_p}$ is a causal function as well.

Definition 2.5 (Regular c_p-optimal dynamical plan)**.** A c_p-optimal dynamical plan $\Pi \in \mathcal{P}(\Gamma_p)$ is *regular* if the following holds.
There exists $U, V \subset M$ relatively compact open subsets and a smooth c_p-concave (with respect to (U, V)) function $\phi_{1/2} : U \to \mathbb{R}$ such that

- $\nabla_g^q \phi_{1/2}(x) \in \mathcal{K} \subset\subset \mathrm{Int}(\mathcal{C})$ for every $x \in U$ and
$$\big(-g(\nabla_g^q \phi_{1/2}, \nabla_g^q \phi_{1/2})\big)^{1/2} \le \mathrm{Inj}_g(U),$$
where $\mathrm{Inj}_g(U)$ is the injectivity radius of g on U;
- Setting
$$\Psi_{1/2}^t(x) = \exp_x^g((t - 1/2)\nabla_g^q \phi_{1/2}(x))$$
and $\mu_t := (\mathrm{e}_t)_\sharp\Pi$ for every $t \in [0,1]$, it holds that
$$\mathrm{supp}(\mu_{1/2}) \subset U,\ \mu_t = (\Psi_{1/2}^t)_\sharp \mu_{1/2} \text{ and } \mu_t \ll \mathrm{vol}_g \, \forall t \in [0,1].$$

Roughly, the above notion of regularity asks that the $\mathcal{A}_p$-minimizing curves performing the optimal transport from $\mu_0 := (\mathrm{e}_0)_\sharp \Pi$ to $\mu_1 := (\mathrm{e}_1)_\sharp \Pi$ have velocities contained in $\mathcal{K}$, i.e. they are all "uniformly" time-like future pointing. Moreover it also implies that $\cup_{t\in[0,1]} \mathrm{supp}(\mu_t) \subset M$ is compact; in addition the optimal transport is assumed to be driven by a smooth potential $\phi_{1/2}$. Even if these conditions may appear a bit strong, we will prove in Lemma 3.1 that there are a lot of such regular plans; moreover in the paper we will show that it is enough to consider such particular optimal transports in order to characterize upper and lower bounds on the (causal-)Ricci curvature and thus characterize the solutions of Einstein equations.

3. Existence, regularity and evolution of Kantorovich potentials

We first show that for every point $\bar{x} \in M$ and every $v \in \mathcal{C}_{\bar{x}}$ "small enough" we can find a smooth c_p-concave function ϕ defined on a neighbourhood of $\bar{x}$, such that $\nabla^q_g \phi = v$ and the hessian of ϕ vanishes at $\bar{x}$. This is well known in the Riemannian setting (e.g. [66, Theorem 13.5]) and should be compared with the recent paper by Mc Cann [49] in the Lorentzian framework. The second part of the next lemma shows that the class of regular c_p-optimal dynamical plans is non-empty, and actually rather rich.

Lemma 3.1. *Let $(M, g, \mathcal{C})$ be a space-time, fix $\bar{x} \in M$ and $v \in \mathcal{C}_{\bar{x}}$ with $g(v, v) < 0$. Then*

1. *There exists $\varepsilon = \varepsilon(\bar{x}, v) > 0$ with the following property: for every $t \in (0, \varepsilon)$, for every C^2 function $\phi : M \to \mathbb{R}$ satisfying*
$$\nabla^q_g \phi(\bar{x}) = tv, \quad \mathrm{Hess}_\phi(\bar{x}) = 0, \tag{3.1}$$
there exists a neighbourhood $U_{\bar{x}}$ of $\bar{x}$ and a neighbourhood $U_{\bar{y}}$ of $\bar{y} := \exp^g_{\bar{x}}(tv)$ such that ϕ is c_p-concave relatively to $(U_{\bar{x}}, U_{\bar{y}})$.
2. *Let*
$$\Psi^t_{1/2}(x) := \exp_x((t - 1/2)\nabla^q_g \phi(x)), \quad \forall x \in U_{\bar{x}}$$
$$\tilde{\Psi} : U_{\bar{x}} \to \mathrm{AC}([0,1], M), \quad x \mapsto \Psi^{(\cdot)}_{1/2}(x).$$
Then, for every $\mu_{1/2} \in \mathcal{P}(M)$ with $\mathrm{supp}(\mu_{1/2}) \subset U_{\bar{x}}$, the measure $\Pi := (\tilde{\Psi})_\sharp \mu_{1/2}$ is a c_p-optimal dynamical plan.

Proof. (1) Calling $\bar{y} = \bar{y}(tv) := \exp^g_{\bar{x}}(tv)$, notice that $\nabla^q_g \phi(\bar{x}) = tv$ is equivalent to
$$d\phi(\bar{x}) = D_x c_p(\bar{x}, \bar{y}), \tag{3.2}$$
where $D_x c_p(\bar{x}, \bar{y})$ denotes the differential at $\bar{x}$ of the function $x \mapsto c_p(x, \bar{y})$. Indeed, a computation shows that $D_x c_p(\bar{x}, \bar{y}) = -D\mathcal{L}_p(tv)$ and thus the claim follows from (2.18).

Let $\phi : M \to \mathbb{R}$ be any smooth function satisfying
$$\nabla^q_g \phi(\bar{x}) = tv, \quad \mathrm{Hess}_\phi(\bar{x}) = 0. \tag{3.3}$$

In what follows we denote with $\mathrm{Hess}_{x,c_p}(\bar{x}, \bar{y})$(resp. $\mathrm{Hess}_{v,\mathcal{L}_p}(tv)$ the Hessian of the function $x \mapsto c_p(x, \bar{y})$ evaluated at $x = \bar{x}$ (resp. the Hessian of the function $T_{\bar{x}}M \ni w \mapsto \mathcal{L}_p(tv+w))$. By taking normal coordinates centred at $\bar{x}$ one can check that the operator norm

$$\|\mathrm{Hess}_{x,c_p}(\bar{x},\bar{y}) - \mathrm{Hess}_{v,\mathcal{L}_p}(tv)\| \to 0 \qquad \text{as } t \to 0.$$

Recalling that from (2.7) there exists $C_{p,v} > 0$ such that $\mathrm{Hess}_{v,\mathcal{L}_p}(tv) \geq C_{p,v} t^{-2+p}$ as quadratic forms, we infer

(3.4)

$$\mathrm{Hess}_{x,c_p}(\bar{x},\bar{y}) - \mathrm{Hess}_\phi(\bar{x}) > 0 \quad \text{as quadratic forms, for every } t \in (0,\varepsilon),$$

for some $\varepsilon = \varepsilon(\bar{x}, v) > 0$ small enough. Since by construction we have $D_x c_p(\bar{x},\bar{y}) - d\phi(\bar{x}) = 0$, by the Implicit Function Theorem there exists a neighbourhood $U_{\bar{x}} \times U_{\bar{y}}$ of $(\bar{x},\bar{y}) \in M \times M$ and a smooth function $F : U_{\bar{y}} \to U_{\bar{x}}$ such that $F(\bar{y}) = \bar{x}$ and

$$D_x c_p(F(y), y) - d\phi(F(y)) = 0, \quad \text{for every } y \in U_{\bar{y}}.$$

Differentiating the last equation in y at $\bar{y}$ and using that $\mathrm{Hess}_\phi(\bar{x}) = 0$, we obtain

$$D^2_{yx} c_p(\bar{x},\bar{y}) + \mathrm{Hess}_{x,c_p}(\bar{x},\bar{y}) DF(\bar{y}) = 0. \tag{3.5}$$

Using normal coordinates centred at $\bar{x}$ and (2.4) one can check that the operator norm

$$\left\| D^2_{yx} c_p(\bar{x},\bar{y}) - (-g(tv,tv))^{\frac{p-4}{2}} \left(-g(tv,tv)g + (2-p)(tv)^* \otimes (tv)^*\right) \right\| \to 0$$

as $t \to 0$, where $(tv)^* = g(tv, \cdot)$ is the covector associated to tv.

Since by assumption $g(v,v) < 0$ and $p \in (0,1)$, it follows from the reverse Cauchy-Schwartz inequality that $\det[D^2_{yx} c_p(\bar{x},\bar{y})] > 0$ for $t \in (0,\varepsilon)$. Recalling that $\det[\mathrm{Hess}_{x,c_p}(\bar{x},\bar{y})] \neq 0$, from (3.5) we infer that $\det(DF(\bar{y})) \neq 0$. By the Inverse Function Theorem, up to reducing the neighbourhoods, we get that $F : U_{\bar{y}} \to U_{\bar{x}}$ is a smooth diffeomorphism. Define now

$$u : U_{\bar{y}} \to \mathbb{R}, \quad u(y) := c_p(F(y), y) - \phi(F(y)).$$

For every fixed $y \in U_{\bar{y}}$, the function $U_{\bar{x}} \ni x \mapsto c_p(x,y) - \phi(x) - u(y)$ vanishes at $x = F(y)$; moreover, from (3.4), it follows that $x = F(y)$ is the strict global minimum of such a function on $U_{\bar{x}}$. In other words, the function $U_{\bar{x}} \times U_{\bar{y}} \ni (x,y) \mapsto c_p(x,y) - \phi(x) - u(y)$ is always non-negative and vanishes exactly on the graph of F. It follows that

$$\phi(x) = \inf_{y \in U_{\bar{y}}} c_p(x,y) - u(y), \quad \text{for every } x \in U_{\bar{x}}, \tag{3.6}$$

i.e. $\phi : U_{\bar{x}} \to \mathbb{R}$ is a smooth c_p-concave function relative to $(U_{\bar{x}}, U_{\bar{y}})$ satisfying (3.1).

Proof of (2). We have to show that $(\mathrm{e}_{1/2}, \mathrm{e}_t)_\sharp \Pi$ (respectively $(\mathrm{e}_t, \mathrm{e}_{1/2})_\sharp \Pi$) is a c_p-optimal coupling for $(\mu_{1/2}, \mu_t)$ and every $t \in [1/2, 1]$ (resp. for $(\mu_t, \mu_{1/2})$ and every $t \in [0, 1/2]$); we discuss the case $t \in [1/2, 1]$, the other being analogous.

It is convenient to define

$$\Psi'_t(x) := \exp_x(t \nabla^q_g \phi(x)), \quad \forall x \in U_{\bar{x}}$$
$$\tilde{\Psi}' : U_{\bar{x}} \to \mathrm{AC}([0,1], M), \quad x \mapsto \Psi'_{(\cdot)}(x).$$

Setting $\Pi' := (\tilde{\Psi}')_\sharp \mu_{1/2}$, we have $(\mathrm{e}_t)_\sharp \Pi' = (\mathrm{e}_{t+1/2})_\sharp \Pi$ for every $t \in [0, 1/2]$. If we show that

$$(\mathrm{e}_0, \mathrm{e}_1)_\sharp \Pi' \text{ is a } c_p\text{-optimal coupling for } \big((\mathrm{e}_0)_\sharp \Pi', (\mathrm{e}_1)_\sharp \Pi'\big), \tag{3.7}$$

then, by the triangle inequality, it will follow that $(\mathrm{e}_0, \mathrm{e}_t)_\sharp \Pi'$ is a c_p-optimal coupling for $\big((\mathrm{e}_0)_\sharp \Pi', (\mathrm{e}_t)_\sharp \Pi'\big)$ for every $t \in [0, 1]$; in particular our claim that $(\mathrm{e}_{1/2}, \mathrm{e}_t)_\sharp \Pi$ is a c_p-optimal coupling for $(\mu_{1/2}, \mu_t)$, $t \in [1/2, 1]$, will be proved. Thus, the rest of the proof will be devoted to establish (3.7).

Since by construction $c_p : U_{\bar{x}} \times U_{\bar{y}} \to \mathbb{R}$ is smooth, by classical optimal transport theory it is well know that the c_p-superdifferential $\partial^{c_p}\phi \subset U_{\bar{y}}$ is c_p-cyclically monotone (see for instance [1, Theorem 1.13]). Therefore, in order to have (3.7), it is enough to prove that

$$\partial^{c_p}\phi(x) = \{\exp_x(\nabla^q_g \phi(x))\} \text{ for every } x \in U_{\bar{x}}. \tag{3.8}$$

Let us first show that $\partial^{c_p}\phi(x) \neq \emptyset$, for every $x \in U_{\bar{x}}$. From the proof of (1), there exists a smooth diffeomorphism $F : U_{\bar{y}} \to U_{\bar{x}}$ such that

$$\begin{cases} \phi(F(y)) = c_p(F(y), y) - u(y), & \text{for all } y \in U_{\bar{y}} \\ \phi(x) \le c_p(x, y) - u(y), & \text{for all } x \in U_{\bar{x}}, y \in U_{\bar{y}}. \end{cases} \tag{3.9}$$

From the definition of ϕ^{c_p} in (2.19), it is readily seen that $\phi^{c_p} = u$ on $U_{\bar{y}}$. Thus (3.9) combined with (2.21) gives that $y \in \partial^{c_p}\phi(F(y))$ for every $y \in U_{\bar{y}}$ or, equivalently, $F^{-1}(x) \in \partial^{c_p}\phi(x)$ for every $x \in U_{\bar{x}}$. In particular, $\partial^{c_p}\phi(x) \neq \emptyset$, for every $x \in U_{\bar{x}}$.

Now fix $x \in U_{\bar{x}}$ and pick $y \in \partial^{c_p}\phi(x) \subset U_{\bar{y}}$. Since $z \mapsto c_p(z, y)$ is differentiable on $U_{\bar{x}}$, we get

$$c_p(z, y) = c_p(x, y) + (D_x c_p(x, y))[(\exp^h_x)^{-1}(z)] + o(\mathrm{d}_h(z, x)), \quad \text{for every } z \in U_{\bar{x}}. \tag{3.10}$$

From $y \in \partial^{c_p}\phi(x)$, we have

$$\begin{aligned} \phi(z) - \phi(x) &\overset{(2.21)}{\le} c_p(z, y) - c_p(x, y) \\ &\overset{(3.10)}{=} (D_x c_p(x, y))[(\exp^h_x)^{-1}(z)] + o(\mathrm{d}_h(z, x)), \text{ for every } z \in U_{\bar{x}}. \end{aligned}$$

Since ϕ is differentiable at $x \in U_{\bar{x}}$, it follows that

$$d\phi(x) = D_x c_p(x, y) = -D\mathcal{L}_p(w),$$

where $w \in \mathrm{Int}(\mathcal{C}_x)$ is such that $y = \exp^g_x(w)$, which by (2.17) is equivalent to

$$w = D\mathcal{H}_p(-d\phi(x)) = \nabla^q_g \phi(x),$$

which yields $y = \exp^g_x(w) = \exp^g_x(\nabla^q_g \phi(x))$, concluding the proof of (3.8). □

We next establish some basic properties of c_p-optimal dynamical plans which will turn out to be useful for the OT-characterization of Lorentzian Ricci curvature upper and lower bounds.

Proposition 3.2. *Let $(M, g, \mathcal{C})$ be a space-time and let Π be a regular c_p-optimal dynamical plan with*

$$(\mathrm{e}_t)_\sharp\Pi = \mu_t = (\Psi^t_{1/2})_\sharp\mu_{1/2} \ll \mathrm{vol}_g,$$
$$\Psi^t_{1/2}(x) = \exp^g_x((t-1/2)\nabla^q_g\phi(x)).$$

Then

(1) *$\nabla\nabla^q_g\phi(x) : T_xM \to T_xM$ is a symmetric endomorphism, i.e.*

$$g(\nabla_X\nabla^q_g\phi, Y) = g(\nabla_Y\nabla^q_g\phi, X), \quad \forall X, Y \in T_xM,\ \forall x \in \mathrm{supp}(\mu_s).$$

(2) *Calling $\mu_t = \rho_t\mathrm{vol}_g$, the following Monge-Ampère inequality holds true:*

$$\rho_{1/2}(x) \le \mathrm{Det}_g[D\Psi^t_{1/2}(x)]\rho_t(\Psi^t_{1/2}(x)), \quad \mu_{1/2}\text{-a.e. } x,\ \forall t \in [0,1]. \tag{3.11}$$

In particular $\Psi^t_{1/2}$ is $\mu_{1/2}$-a.e. non-degenerate. Moreover (3.11) *holds with equality if $t \in (0,1)$.*

Proof. (1) By construction, $\phi_{1/2}$ is smooth on U and $g(\nabla_g\phi_{1/2}, \nabla_g\phi_{1/2}) < 0$. Thus also $\nabla^q_g\phi : M \to TM$ is a smooth section of the tangent bundle and the symmetry of the endomorphism $\nabla\nabla^q_g\phi(x) : T_xM \to T_xM$ follows by Schwartz's Lemma.

(2) This part should be compared with [42, Lemma 3.9]. Since by construction $(\Psi^t_{1/2})_\sharp\mu_{1/2} = \mu_t$, it follows that for an arbitrary Borel subset $A \subset M$ it holds

$$\mu_{1/2}(A) \le \mu_{1/2}\left((\Psi^t_{1/2})^{-1}(\Psi^t_{1/2}(A))\right) = \mu_t\left(\Psi^t_{1/2}(A)\right). \tag{3.12}$$

Equality holds for $t \in (0,1)$ as the map $\Psi^t_{1/2}$ is $\mu_{1/2}$-essentially injective. By the area formula we infer that

$$\begin{aligned} \mu_t\left(\Psi^t_{1/2}(A)\right) &= \int_{\Psi^t_{1/2}(A)} \rho_t\, d\mathrm{vol}_g \\ &\le \int_{\Psi^t_{1/2}(A)} \rho_t(y)\mathcal{H}^0((\Psi^t_{1/2}\llcorner A)^{-1}(y))\, d\mathrm{vol}_g(y) \\ &= \int_A \rho_t(\Psi^t_{1/2}(x))\mathrm{Det}_g\left[D\Psi^t_{1/2}(x)\right]\, d\mathrm{vol}_g(x), \end{aligned} \tag{3.13}$$

with equality if $t \in (0,1)$ as the map $\Psi^t_{1/2}$ is $\mu_{1/2}$-essentially injective. The combination of (3.12) and (3.13) gives that for an arbitrary Borel subset $A \subset M$ it holds

$$\int_A \rho_{1/2}\, d\mathrm{vol}_g = \mu_{1/2}(A) \le \mu_t(\Psi^t_{1/2}(A)) \le \int_A \rho_t(\Psi^t_{1/2})\mathrm{Det}_g\left[D\Psi^t_{1/2}\right]\, d\mathrm{vol}_g.$$

The Monge-Ampère inequality (3.11) follows, with equality for $t \in (0,1)$. □

It will be convenient to consider the matrix of Jacobi fields

$$\mathcal{B}_t(x) := D\Psi^t_{1/2}(x) : T_xM \to T_{\Psi^t_{1/2}(x)}M, \quad \text{for } \mu_{1/2}\text{-a.e. } x, \tag{3.14}$$

along the geodesic $t \mapsto \gamma_t := \Psi^t_{1/2}(x)$; recalling (2.12), $\mathcal{B}_t(x)$ satisfies the Jacobi equation

$$\nabla_t\nabla_t\mathcal{B}_t(x) + R(\mathcal{B}_t(x), \dot\gamma_t)\dot\gamma_t = 0, \tag{3.15}$$

where we denoted $\nabla_t := \nabla_{\dot\gamma_t}$ for short.

Since by Proposition 3.2 we know that $\mathcal{B}_t$ is non-singular for $\mu_{1/2}$-a.e. x, we can define

$$\mathcal{U}_t(x) := \nabla_t \mathcal{B}_t \circ \mathcal{B}_t^{-1} : T_{\gamma_t}M \to T_{\gamma_t}M, \quad \text{for } \mu_{1/2}\text{-a.e. } x. \tag{3.16}$$

The next proposition will be key in the proof of the lower bounds on causal Ricci curvature. It is well known in Riemannian and Lorentzian geometry, see for instance [25, Lemma 3.1] and [27]; in any case we report a proof for the reader's convenience.

Proposition 3.3. *Let $\mathcal{U}_t$ be defined in* (3.16)*. Then $\mathcal{U}_t$ is a symmetric endomorphism of $T_{\gamma_t}M$ (i.e. the matrix $(\mathcal{U}_t)_{ij}$ with respect to an orthonormal basis is symmetric) and it holds*

$$\nabla_t \mathcal{U}_t + \mathcal{U}_t^2 + R(\cdot, \dot\gamma_t)\dot\gamma_t = 0.$$

Taking the trace with respect to g yields

$$\mathrm{Tr}_g(\nabla_t \mathcal{U}_t) + \mathrm{Tr}_g(\mathcal{U}_t^2) + \mathrm{Ric}(\dot\gamma_t, \dot\gamma_t) = 0. \tag{3.17}$$

Setting $y(t) := \log \mathrm{Det}_g \mathcal{B}_t$, it holds

$$y''(t) + \frac{1}{n}(y'(t))^2 + \mathrm{Ric}(\dot\gamma_t, \dot\gamma_t) \le 0. \tag{3.18}$$

Proof. Using (3.15) we get

$$\begin{aligned}\nabla_t \mathcal{U}_t &= (\nabla_t \nabla_t \mathcal{B}_t)\, \mathcal{B}_t^{-1} + \nabla_t \mathcal{B}_t \nabla_t (\mathcal{B}_t^{-1}) = -R(\cdot, \dot\gamma_t)\dot\gamma_t - (\nabla_t \mathcal{B}_t)\mathcal{B}_t^{-1}(\nabla_t \mathcal{B}_t)\mathcal{B}_t^{-1} \\ &= -R(\cdot, \dot\gamma_t)\dot\gamma_t - \mathcal{U}_t^2.\end{aligned}$$

Taking the trace with respect to g yields the second identity.
The rest of the proof is devoted to show (3.18). Let $(e_i(t))_{i=1,\dots,n}$ be an orthonormal basis of $T_{\gamma_t}M$ parallel along γ. Setting $y(t) = \log \det \mathcal{B}_t$, we have that

$$\begin{aligned}y'(t_0) &= \frac{d}{dt}\Big|_{t=t_0} \log \mathrm{Det}_g\big(\mathcal{B}_t \mathcal{B}_{t_0}^{-1}\big) \\ &= \frac{d}{dt}\Big|_{t=t_0} \log \det \Big[\big(g(e_i(t), e_j(t)) g(\mathcal{B}_t \mathcal{B}_{t_0}^{-1} e_i(t), e_j(t))\big)_{i,j}\Big] \\ &= \mathrm{Tr}_g \left[(\nabla_t \mathcal{B}_t)\mathcal{B}_{t_0}^{-1}\right]|_{t=t_0} = \mathrm{Tr}_g(\mathcal{U}_{t_0}).\end{aligned} \tag{3.19}$$

We next show that $\mathcal{U}_t$ is a symmetric endomorphism of $T_{\gamma_t}M$, i.e. the matrix $(\mathcal{U}_t)_{ij}$ is symmetric. To this aim, calling $\mathcal{U}_t^*$ the adjoint, we observe that

$$\mathcal{U}_t^* - \mathcal{U}_t = (\mathcal{B}_t^*)^{-1}\left[(\nabla_t \mathcal{B}_t^*)\mathcal{B}_t - \mathcal{B}_t^*(\nabla_t \mathcal{B}_t)\right]\mathcal{B}_t^{-1}, \tag{3.20}$$

and that

$$\nabla_t\left[(\nabla_t \mathcal{B}_t^*)\mathcal{B}_t - \mathcal{B}_t^*(\nabla_t \mathcal{B}_t)\right] = (\nabla_t \nabla_t \mathcal{B}_t^*)\mathcal{B}_t - \mathcal{B}_t^*(\nabla_t \nabla_t \mathcal{B}_t). \tag{3.21}$$

Now the Jacobi equation (3.15) reads

$$\nabla_t \nabla_t \mathcal{B}_t = -R(\mathcal{B}_t, \dot\gamma_t)\dot\gamma_t = -\mathcal{R}(t)\mathcal{B}_t, \tag{3.22}$$

where

$$\mathcal{R}(t) : T_{\gamma_t}M \to T_{\gamma_t}M, \quad \mathcal{R}(t)[v] := R(v, \dot\gamma_t)\dot\gamma_t$$

is symmetric; indeed, in the orthonormal basis $(e_i(t))_{i=1,\dots,n}$, it is represented by the symmetric matrix

$$\Big(g(e_i(t), e_j(t))\, g(R(e_i(t), \dot\gamma_t)\dot\gamma_t, e_j(t))\Big)_{i,j=1,\dots,n}.$$

Plugging (3.22) into (3.21), we obtain that

$$(\nabla_t \mathcal{B}_t^*)\mathcal{B}_t - \mathcal{B}_t^*(\nabla_t \mathcal{B}_t)$$

is constant in t. But $\mathcal{B}_0 = \mathrm{Id}_{T_{\gamma_{1/2}}M}$ and $\nabla_t \mathcal{B}_t|_{t=1/2} = -\nabla\nabla_g^q \phi$ is symmetric by assertion (1) in Proposition 3.2. Taking into account (3.20), we conclude that $\mathcal{U}_t$ is symmetric for every $t \in [0,1]$.
Using that $\mathcal{U}_t$ is symmetric, by Cauchy-Schwartz inequality, we have that

$$\mathrm{Tr}_g\big[\mathcal{U}_t^2\big] = \mathrm{tr}\big[(\mathcal{U}^2)_{ij}\big] \geq \frac{1}{n}\big(\mathrm{tr}\big[\mathcal{U}_{ij}\big]\big)^2 = \frac{1}{n}\big(\mathrm{Tr}_g\big[\mathcal{U}_{ij}\big]\big)^2. \tag{3.23}$$

The desired estimate (3.18) then follows from the combination of (3.17), (3.19) and (3.23). □

In order to characterize Lorentzian Ricci curvature upper bounds, it will be useful the next proposition concerning the evolution of Kantorovich potentials along a regular $\mathcal{A}_p$-minimizing curve of probability measures $(\mu_t)_{t\in[0,1]}$ given by exponentiating the q-gradient of a smooth c_p-concave function with time-like gradient. To this aim it is convenient to consider, for $0 \leq s < t \leq 1$, the restricted minimal action

$$c_p^{s,t}(x,y) := \inf\left\{ \int_s^t \mathcal{L}_p(\dot\gamma(\tau))d\tau \,\middle|\, \gamma \in \mathrm{AC}([s,t], M),\, \gamma(s) = x,\, \gamma(t) = y \right\}.$$

Proposition 3.4. *Let $(M, g, \mathcal{C})$ be a space-time, fix $p \in (0,1)$ and let $q \in (-\infty, 0)$ be the Hölder conjugate exponent, i.e. $\frac{1}{p} + \frac{1}{q} = 1$ or equivalently $(p-1)(q-1) = 1$. Let $U, V \subset M$ be relatively compact open subsets and $\phi_{1/2}$ be a smooth c_p-concave function relative to (U, V) such that*

- *$\phi_{1/2}$ is a smooth Lyapunov function on U,*
- *$\big(-g(\nabla_g^q \phi_{1/2}, \nabla_g^q \phi_{1/2})\big)^{1/2} \leq \mathrm{Inj}_g(U)$, the injectivity radius of g on U.*

For $t \in [0,1]$, let

$$\Psi_{1/2}^t : U \to M,\ \Psi_{1/2}^t(x) := \exp_x((t - 1/2)\nabla_g^q \phi_{1/2}(x))$$

be the c_p-optimal transport map associated to $\phi_{1/2}$. For every $x \in U$, define

$$\phi_t(\Psi_{1/2}^t(x)) = \phi(t, \Psi_{1/2}^t(x)) := \begin{cases} \phi_{1/2}(x) - c_p^{1/2,t}(x, \Psi_{1/2}^t(x)) & \text{for } t \in [1/2, 1] \\ \phi_{1/2}(x) + c_p^{t,1/2}(\Psi_{1/2}^t(x), x) & \text{for } t \in [0, 1/2) \end{cases}. \tag{3.24}$$

Then the map $(t,y) \mapsto \phi(t,y)$ defined on $\bigcup_{t\in[0,1]} \{t\} \times \Psi_{1/2}^t(U)$ is C^∞ and satisfies the Hamilton-Jacobi equation

$$\partial_t \phi_t(t,y) + \frac{1}{q}(-g(\nabla_g \phi_t(y), \nabla_g \phi_t(y)))^{q/2} = 0,\ \forall (t,y) \in \bigcup_{t\in[0,1]} \{t\} \times \Psi_{1/2}^t(U) \tag{3.25}$$

with

$$\frac{d}{dt}\Psi^t_{1/2}(x) = \nabla^q_g \phi_t(\Psi^t_{1/2}(x)), \quad \forall (t,x) \in [0,1]\times U. \tag{3.26}$$

Proof. **Step 1**: smoothness of ϕ.
The fact that $t \mapsto \Psi^t_{1/2}$ is a smooth 1-parameter family of maps performing c_p-optimal transport gives that ϕ defined in (3.24) satisfies (cf. [19, Theorem 6.4.6])

$$\phi_t(\Psi^t_{1/2}(x)) = \phi_s(\Psi^s_{1/2}(x)) - c^{s,t}_p(\Psi^s_{1/2}(x), \Psi^t_{1/2}(x)), \quad \forall x \in U,\ 0 \le s < t \le 1. \tag{3.27}$$

In particular it holds

$$\phi_t(\Psi^t_{1/2}(x)) = \phi_0(\Psi^0_{1/2}(x)) - c^{0,t}_p(\Psi^0_{1/2}(x), \Psi^t_{1/2}(x)) \quad \forall x \in U, t \in [0,1], \tag{3.28}$$

$$\phi_t(\Psi^t_{1/2}(x)) = \phi_1(\Psi^1_{1/2}(x)) + c^{t,1}_p(\Psi^t_{1/2}(x), \Psi^1_{1/2}(x)) \quad \forall x \in U, t \in [0,1]. \tag{3.29}$$

Since by construction everything is defined inside the injectivity radius and all the transports rays are non-constant, from (3.28) (respectively (3.29)) it is manifest that the map $(t,y) \mapsto \phi(t,y)$ is C^∞ on $\bigcup_{t\in(0,1]}\{t\}\times \Psi^t_{1/2}(U)$ (resp. $\bigcup_{t\in[0,1)}\{t\}\times \Psi^t_{1/2}(U)$). The smoothness of ϕ on $\bigcup_{t\in[0,1]}\{t\}\times \Psi^t_{1/2}(U)$ follows.

Step 2: validity of the Hamilton-Jacobi equation (3.25).
We consider $t \in (1/2, 1]$, the case $t \in [0, 1/2]$ being analogous. Fix $y = \Psi^t_{1/2}(x)$ for some arbitrary $x \in U$ and $t \in (1/2, 1]$, and let $\gamma\colon [0,s] \to M$ be a smooth curve with $\dot\gamma(0) = v \in T_yM$. From (3.24) we have

$$\phi(t+s, \gamma_s) \ge -\int_0^s \mathcal{L}_p(\dot\gamma_\tau)\, d\tau + \phi(t, \gamma_0),$$

with equality for $\gamma(\tau) = \Psi_{t+\tau}(x)$ for all $\tau \in [0,s]$. Dividing by s and taking the limit for $s \to 0$, we obtain

$$\lim_{s\to 0} \frac{\phi(t+s,\gamma_s) - \phi(t,\gamma_0)}{s} \ge -\mathcal{L}_p(v),$$

which in turn implies

$$\partial_t \phi_t(y) \ge -d\phi_t(v) - \mathcal{L}_p(v), \quad \text{for every } v \in T_yM.$$

Note that equality holds for $v = \nabla^q_g \phi_{1/2}(x)$. For $\alpha \in T^*_yM$, let

$$\mathcal{H}_p(\alpha) = \sup_{v\in T_yM} [\alpha(v) - \mathcal{L}_p(v)]$$

denote the Legendre transform of $\mathcal{L}_p$. Thus we get

$$\partial_t \phi_t(y) = \mathcal{H}_p(-d(\phi_t)_y). \tag{3.30}$$

Recalling that $\mathcal{H}_p$ has the representation (2.16), we have

$$\mathcal{H}_p(-d(\phi_t)_y) = -\frac{1}{q}\left(-g(\nabla_g\phi_t(y), \nabla_g\phi_t(y))\right)^{q/2},$$

which, together with (3.30), implies (3.25).

Step 3: validity of (3.26).
Since $\Psi^t_{1/2}$ is a smooth 1-parameter family of maps performing c_p-optimal

transport and the function ϕ defined in (3.24) is smooth, it coincides with the viscosity solution (resp. backward solution)

$$\phi_t(y) = \begin{cases} \sup_{z\in\Psi^s_{1/2}(U)} \phi_s(z) - c_p^{s,t}(z,y) & \text{for } t\in[s,1] \\ \inf_{z\in\Psi^s_{1/2}(U)} \phi_s(z) + c_p^{t,s}(y,z) & \text{for } t\in[0,s) \end{cases}, \tag{3.31}$$

for every $s\in(0,1)$, $y\in\Psi^t_{1/2}(U)$.

Let us discuss the case $t\in(s,1]$, the other is analogous. From (3.27) it follows that $\Psi^s_{1/2}(x)$ is a maximum point in the right hand side of (3.31) corresponding to $y=\Psi^t_{1/2}(x)$. Thus

$$d\phi_s(\Psi^s_{1/2}(x)) = d\left[c_p^{s,t}(\cdot,\Psi^t_{1/2}(x))\right](\Psi^s_{1/2}(x)) = -D\mathcal{L}_p\left(\frac{d}{ds}\Psi^s_{1/2}(x)\right).$$

By construction $\frac{d}{ds}\Psi^s_{1/2}(x)\in \mathrm{Int}(\mathcal{C})$ and, as already observed, $D\mathcal{L}_p$ is invertible on $\mathrm{Int}(\mathcal{C})$ with inverse given by $D\mathcal{H}_p$. We conclude that

$$\frac{d}{ds}\Psi^s_{1/2}(x) = D\mathcal{H}_p\left(-d\phi_s(\Psi^s_{1/2}(x))\right) = \nabla^q_g\phi_s(\Psi^s_{1/2}(x)).$$

□

4. Optimal transport formulation of the Einstein equations

The Einstein equations of General Relativity for an n-dimensional space-time $(M^n,g,\mathcal{C})$, $n\geq 3$, read as

$$\mathrm{Ric} - \frac{1}{2}\mathrm{Scal}\, g + \Lambda g = 8\pi T, \tag{4.1}$$

where Scal is the scalar curvature, $\Lambda\in\mathbb{R}$ is the cosmological constant, and T is the energy-momentum tensor.

Lemma 4.1. *The space-time $(M^n,g,\mathcal{C})$, $n\geq 3$, satisfies the Einstein Equation* (4.1) *if and only if*

$$\mathrm{Ric} = \frac{2\Lambda}{n-2}g + 8\pi T - \frac{8\pi}{n-2}\mathrm{Tr}_g(T)\, g. \tag{4.2}$$

Proof. Taking the trace of (4.1), one can express the scalar curvature as

$$\mathrm{Scal} = \frac{2n\Lambda}{n-2} - \frac{16\pi}{n-2}\mathrm{Tr}_g(T). \tag{4.3}$$

Plugging (4.3) into 4.1 gives the equivalent formulation (4.2) of Einstein equations just in terms of the metric, the Ricci and the energy-momentum tensors. □

The optimal transport formulation of the Einstein equations will consist separately of an optimal transport characterization of the two inequalities

$$\mathrm{Ric} \geq \frac{2\Lambda}{n-2}g + 8\pi T - \frac{8\pi}{n-2}\mathrm{Tr}_g(T)\, g$$

and

$$\mathrm{Ric} \leq \frac{2\Lambda}{n-2}g + 8\pi T - \frac{8\pi}{n-2}\mathrm{Tr}_g(T)\, g,$$

respectively. Subsection 4.1 will be devoted to the lower bound and Subsection 4.2 to the upper bound on the Ricci tensor.

A key role in such an optimal transport formulation will be played by the (relative) Boltzmann-Shannon entropy defined below. Denote by vol_g the standard volume measure on (M, g). Given an absolutely continuous probability measure $\mu = \varrho\, \mathrm{vol}_g$ with density $\varrho \in C_c(M)$, define its Boltzmann-Shannon entropy (relative to vol_g) as

$$\mathrm{Ent}(\mu|\mathrm{vol}_g) := \int_M \varrho \log \varrho \, d\mathrm{vol}_g. \tag{4.4}$$

4.1. **OT-characterization of** $\mathrm{Ric} \geq \frac{2\Lambda}{n-2} g + 8\pi T - \frac{8\pi}{n-2} \mathrm{Tr}_g(T)\, g$**.** In establishing the Ricci curvature lower bounds, the next elementary lemma will be key (for the proof see for instance [66, Chapter 16] or [6, Lemma 9.1])

Lemma 4.2. *Define the function* $\mathrm{G} : [0,1] \times [0,1] \to [0,1]$ *by*

$$\mathrm{G}(s,t) := \begin{cases} (1-t)s & \text{if } s \in [0,t], \\ t(1-s) & \text{if } s \in [t,1], \end{cases} \tag{4.5}$$

so that for all $t \in (0,1)$ *one has*

$$-\frac{\partial^2}{\partial s^2} \mathrm{G}(s,t) = \delta_t \quad \text{in } \mathscr{D}'(0,1), \qquad \mathrm{G}(0,t) = \mathrm{G}(1,t) = 0. \tag{4.6}$$

If $u \in C([0,1], \mathbb{R})$ *satisfies* $u'' \geq f$ *in* $\mathscr{D}'(0,1)$ *for some* $f \in L^1(0,1)$ *then*

$$u(t) \leq (1-t)u(0) + tu(1) - \int_0^1 \mathrm{G}(s,t)\, f(s)\, ds, \quad \forall t \in [0,1]. \tag{4.7}$$

In particular, if $f \equiv c \in \mathbb{R}$ *then*

$$u(t) \leq (1-t)u(0) + tu(1) - c\frac{t(1-t)}{2}, \quad \forall t \in [0,1]. \tag{4.8}$$

The characterization of Ricci curvature lower bounds (i.e. $\mathrm{Ric} \geq Kg$ for some constant $K \in \mathbb{R}$) via displacement convexity of the entropy is by now classical in the Riemannian setting, let us briefly recall the key contributions. Otto & Villani [55] gave a nice heuristic argument for the implication "$\mathrm{Ric} \geq Kg \Rightarrow K$-convexity of the entropy"; this implication was proved for $K = 0$ by Cordero-Erausquin, McCann & Schmuckenschläger [24]; the equivalence for every $K \in \mathbb{R}$ was then established by Sturm & von Renesse [62]. Our optimal transport characterization of $\mathrm{Ric} \geq \frac{2\Lambda}{n-2} g + 8\pi T - \frac{8\pi}{n-2} \mathrm{Tr}_g(T)\, g$ is inspired by such fundamental papers (compare also with [42] for the implication (3)⇒(1)). Let us also mention that the characterization of $\mathrm{Ric} \geq Kg$ for $K \geq 0$ via displacement convexity in the globally hyperbolic Lorentzian setting has recently been obtained independently by Mc Cann [49]. Note that Corollary 4.4 extends such a result to any lower bounds $K \in \mathbb{R}$ and to the case of general (possibly non globally hyperbolic) space times.

The next general result will be applied with $n \geq 3$ and

$$\tilde{T} = \frac{2\Lambda}{n-2} g + 8\pi T - \frac{8\pi}{n-2} \mathrm{Tr}_g(T)\, g.$$

Theorem 4.3 (OT-characterization of $\mathrm{Ric} \geq \tilde{T}$)**.** *Let* $(M, g, \mathcal{C})$ *be a space-time of dimension* $n \geq 2$ *and let* $\tilde{T}$ *be a quadratic form on* M*. Then the following are equivalent:*

(1) $\mathrm{Ric}(v,v) \geq \tilde{T}(v,v)$ *for every causal vector* $v \in \mathcal{C}$*.*

(2) *For every $p \in (0,1)$, for every regular dynamical c_p-optimal plan Π it holds*

$$\mathrm{Ent}(\mu_t|\mathrm{vol}_g) \leq (1-t)\mathrm{Ent}(\mu_0|\mathrm{vol}_g)+t\mathrm{Ent}(\mu_1|\mathrm{vol}_g)-\int\int_0^1 \mathrm{G}(s,t)\tilde{T}(\dot{\gamma}_s,\dot{\gamma}_s)ds\, d\Pi(\gamma), \tag{4.9}$$

where we denoted $\mu_t := (\mathrm{e}_t)_\sharp\Pi$, $t \in [0,1]$, the curve of probability measures associated to Π.

(3) *There exists $p \in (0,1)$ such that for every regular dynamical c_p-optimal plan Π the convexity property (4.9) holds.*

Proof. **(1)⇒ (2)**
Fix $p \in (0,1)$. Let Π be a regular dynamical c_p-optimal coupling and let $(\mu_t := (\mathrm{e}_t)_\sharp\Pi)_{t\in[0,1]}$ be the corresponding curve of probability measures with $\mu_t = \rho_t \,\mathrm{vol}_g \ll \mathrm{vol}_g$ compactly supported. By definition of regular dynamical c_p-optimal coupling there exists a smooth function $\phi_{1/2}$ such that, calling

$$\Psi^t_{1/2}(x) = \exp^g_x((t-1/2)\nabla^q_g\phi_{1/2}(x)),$$

it holds $\mu_t = (\Psi^t_{1/2})_\sharp\mu_{1/2}$ for every $t \in [0,1]$. Moreover the Jacobian $D\Psi^t_{1/2}$ is non-singular for every $t \in [0,1]$ on $\mathrm{supp}(\mu_{1/2})$. Recall the definition of $\mathcal{B}_t$ and $\mathcal{U}_t$ along the geodesic $t \mapsto \gamma_t := \Psi^t_{1/2}(x)$.

$$\begin{aligned}
\mathcal{B}_t(x) &:= D\Psi^t_{1/2}(x) : T_xM \to T_{\Psi^t_{1/2}(x)}M, &&\text{for } \mu_{1/2}\text{-a.e. } x,\\
\mathcal{U}_t(x) &:= \nabla_t\mathcal{B}_t \circ \mathcal{B}_t^{-1} : T_{\gamma_t}M \to T_{\gamma_t}M, &&\text{for } \mu_{1/2}\text{-a.e. } x.
\end{aligned}$$

Calling $y_x(t) := \log \mathrm{Det}_g\mathcal{B}_t(x)$ and $\gamma^x_t := \Psi^t_{1/2}(x)$, from Proposition 3.3 we get

$$y''_x(t) + \tilde{T}(\dot{\gamma}^x_t,\dot{\gamma}^x_t) \leq y''_x(t) + \mathrm{Ric}(\dot{\gamma}^x_t,\dot{\gamma}^x_t) \leq 0, \quad \mu_{1/2}\text{-a.e. } x. \tag{4.10}$$

Now, for $t \in (0,1)$ we have

$$\begin{aligned}
\mathrm{Ent}(\mu_t|\mathrm{vol}_g) &= \int \log\rho_t(y)\, d\mu_t(y) = \int_M \log\rho_t(\Psi^t_{1/2}(x))\, d\mu_{1/2}(x)\\
&= \int \log[\rho_{1/2}(x)(\mathrm{Det}_g(D\Psi^t_{1/2})(x))^{-1}]\, d\mu_{1/2}(x)\\
&= \mathrm{Ent}(\mu_{1/2}|\mathrm{vol}_g) - \int y_x(t)\, d\mu_{1/2}(x), \qquad (4.11)
\end{aligned}$$

where the second to last equality follows from Proposition 3.2(2). Using (4.10) we obtain

$$\frac{d^2}{dt^2}\mathrm{Ent}(\mu_t|\mathrm{vol}_g) \geq \int \tilde{T}(\dot{\gamma}^x_t,\dot{\gamma}^x_t)d\mu_{1/2}(x) = \int \tilde{T}(\dot{\gamma}_t,\dot{\gamma}_t)d\Pi(\gamma), \quad \forall t \in (0,1). \tag{4.12}$$

Using Lemma 4.2, we get (4.9).

(2)⇒ (3): trivial.

(3)⇒ (1)
We argue by contradiction. Assume there exist $x_0 \in M$ and $v \in T_{x_0}M \cap \mathcal{C}$ with $g(v,v) < 0$ such that the Ricci curvature at x_0 in the direction of $v \in \mathcal{C}_x$ satisfies

$$\mathrm{Ric}(v,v) \leq (\tilde{T} + 3\epsilon g)(v,v), \tag{4.13}$$

for some $\epsilon > 0$. Thanks to Lemma 3.1, for $\eta \in (0, \bar{\eta}(x_0, v)]$ small enough, there exists $\bar{\delta} > 0$ and a c_p-convex function $\phi_{1/2}$, smooth on $B_{\bar{\delta}}(x_0)$ and satisfying

$$\nabla_g^q \phi_{1/2}(x_0) = \eta v \neq 0 \quad \text{ and } \mathrm{Hess}_{\phi_{1/2}}(x_0) = 0. \tag{4.14}$$

From now on we fix $\eta \in \big(0, \min(\bar{\eta}(x_0, v), \mathrm{inj}_g(B_{\bar{\delta}}(x_0)))\big]$, where $\mathrm{inj}_g(B_{\bar{\delta}}(x_0))$ is the injectivity radius of $B_{\bar{\delta}}(x_0)$ with respect to the metric g. It is easily checked that, for $\delta \in (0, \bar{\delta})$, small enough the map

$$x \mapsto \Psi_{1/2}^t(x) = \exp_x^g((t - 1/2)\nabla_g^q \phi_{1/2}(x))$$

is a diffeomorphism from $B_\delta(x_0)$ onto its image for any $t \in [0, 1]$. Moreover, since $\nabla_g^q \phi_{1/2}(x_0) \in \mathrm{Int}(\mathcal{C})$ and arguing by continuity and by parallel transport along the geodesics $t \mapsto \Psi_{1/2}^t(x), x \in B_\delta(x_0)$, for $\delta > 0$ small enough we have that

$$\bigcup_{t\in[0,1]} \bigcup_{x\in B_\delta(x_0)} \frac{d}{dt} \Psi_{1/2}^t(x) \subset\subset \mathrm{Int}(\mathcal{C}). \tag{4.15}$$

Define $\mu_{\frac{1}{2}} := \mathrm{vol}_g(B_\delta(x_0))^{-1} \, \mathrm{vol}_g \llcorner B_\delta(x_0)$. Let Π be the c_p-optimal dynamical plan representing the curve of probability measures $\Big(\mu_t := (\Psi_{1/2}^t)_\sharp \mu_{\frac{1}{2}}\Big)_{t\in[0,1]}$. Note that (4.15) together with (4.14) ensures that Π is regular, for $\delta > 0$ small enough.
Calling $\gamma_t^x := \Psi_{1/2}^t(x) = \exp_x^g\big((t - 1/2)\nabla_g^q \phi_{1/2}(x)\big)$ for $x \in B_\delta(x_0)$ the geodesic performing the transport, note that by continuity there exists $\delta > 0$ small enough such that

$$\mathrm{Ric}(\dot\gamma_t^x, \dot\gamma_t^x) < (\tilde{T} + 2\epsilon g)(\dot\gamma_t^x, \dot\gamma_t^x), \quad \forall x \in B_\delta(x_0), \quad \forall t \in [0, 1]. \tag{4.16}$$

The identity (3.17) proved in Proposition 3.3 reads as

$$[\mathrm{Tr}_g(\mathcal{U}_t^x)]' + \mathrm{Tr}_g[(\mathcal{U}_t^x)^2] + \mathrm{Ric}(\dot\gamma_t^x, \dot\gamma_t^x) = 0, \quad \forall x \in B_\delta(x_0), \quad \forall t \in [0, 1]. \tag{4.17}$$

Since by construction $\mathcal{U}_{1/2}^{x_0} := \nabla_t \mathcal{B}_{1/2}^{x_0}(\mathcal{B}_{1/2}^{x_0})^{-1} = \nabla\nabla_g^q \phi_{1/2}(x_0) = 0$ and $g(\nabla_g^q \phi_{1/2}(x_0), \nabla_g^q \phi_{1/2}(x_0)) < 0$, again by continuity we can choose $\delta > 0$ even smaller so that

$$\mathrm{Tr}_g[(\mathcal{U}_t^x)^2] < -\epsilon g(\dot\gamma_t^x, \dot\gamma_t^x), \quad \forall x \in B_\delta(x_0), \quad \forall t \in [0, 1]. \tag{4.18}$$

The combination of (4.16), (4.17) and (4.18) yields

$$[\mathrm{Tr}_g(\mathcal{U}_t^x)]' + (\tilde{T} + \epsilon g)(\dot\gamma_t^x, \dot\gamma_t^x) > 0, \quad \forall x \in B_\delta(x_0), \quad \forall t \in [0, 1].$$

Recalling (3.19), the last inequality can be rewritten as

$$y_x(t)'' + (\tilde{T} + \epsilon g)(\dot\gamma_t^x, \dot\gamma_t^x) > 0, \quad \forall x \in B_\delta(x_0), \quad \forall t \in [0, 1].$$

The combination of the last inequality with (4.11) gives

$$\frac{d^2}{dt^2} \mathrm{Ent}(\mu_t | \mathrm{vol}_g) < \int (\tilde{T} + \epsilon g)(\dot\gamma_t^x, \dot\gamma_t^x) \, d\mu_{1/2}(x), \quad \forall t \in (0, 1). \tag{4.19}$$

By applying Lemma 4.2 we get that

$$
\begin{aligned}
\operatorname{Ent}(\mu_t|\mathrm{vol}_g) &\geq (1-t)\operatorname{Ent}(\mu_0|\mathrm{vol}_g) + t\operatorname{Ent}(\mu_1|\mathrm{vol}_g) \\
&\qquad - \int\int_0^1 \mathrm{G}(s,t)(\tilde{T}+\epsilon g)(\dot\gamma_s,\dot\gamma_s)ds\, d\Pi(\gamma), \\
&= (1-t)\operatorname{Ent}(\mu_0|\mathrm{vol}_g) + t\operatorname{Ent}(\mu_1|\mathrm{vol}_g) - \int\int_0^1 \mathrm{G}(s,t)\tilde{T}(\dot\gamma_s,\dot\gamma_s)ds\, d\Pi(\gamma) \\
&\qquad - \epsilon\frac{t(1-t)}{2}\int g(\dot\gamma,\dot\gamma)d\Pi(\gamma),
\end{aligned}
$$

where, in the equality we used that for every fixed $x \in B_\delta(x_0)$ the function $t \mapsto g(\dot\gamma_t^x, \dot\gamma_t^x)$ is constant (as $t \mapsto \gamma_t^x$ is by construction a g-geodesic).

This clearly contradicts (4.9), as $\int g(\dot\gamma,\dot\gamma)d\Pi(\gamma) < 0$. □

In the vacuum case, i.e. $T \equiv 0$, the inequality $\operatorname{Ric} \geq \frac{2\Lambda}{n-2}g + 8\pi T - \frac{8\pi}{n-2}\operatorname{Tr}_g(T)\, g$ reads as $\operatorname{Ric} \geq Kg$ with $K = \frac{2\Lambda}{n-2} \in \mathbb{R}$. Note that for $v \in \mathcal{C}$ it holds $g(v,v) \leq 0$ so, when comparing the next result with its Riemannian counterparts [55, 24, 62], the sign of the lower bound K is reversed.

Corollary 4.4 (The vacuum case $T \equiv 0$)**.** *Let $(M, g, \mathcal{C})$ be a space-time of dimension $n \geq 2$ and let $K \in \mathbb{R}$. Then the following are equivalent:*

(1) $\operatorname{Ric}(v,v) \geq Kg(v,v)$ *for every causal vector* $v \in \mathcal{C}$.

(2) *For every $p \in (0,1)$, for every regular dynamical c_p-optimal plan* Π *it holds*

$$
\begin{aligned}
\operatorname{Ent}(\mu_t|\mathrm{vol}_g) &\leq (1-t)\operatorname{Ent}(\mu_0|\mathrm{vol}_g) + t\operatorname{Ent}(\mu_1|\mathrm{vol}_g) - K\frac{t(1-t)}{2}\int g(\dot\gamma,\dot\gamma)d\Pi(\gamma) \\
&= (1-t)\operatorname{Ent}(\mu_0|\mathrm{vol}_g) + t\operatorname{Ent}(\mu_1|\mathrm{vol}_g) - Kt(1-t)\int \mathcal{A}_2(\gamma)d\Pi(\gamma), \qquad (4.20)
\end{aligned}
$$

where we denoted $\mu_i := (\mathrm{e}_i)_\sharp\Pi$, $i = 0,1$, the endpoints of the curve of probability measures associated to Π.

(3) *There exists $p \in (0,1)$ such that for every regular dynamical c_p-optimal plan Π the convexity property* (4.20) *holds.*

Remark 4.5 (The strong energy condition)**.** The strong energy condition asserts that, called T the energy-momentum tensor, it holds $T(v,v) \geq \frac{1}{2}\operatorname{Tr}_g(T)$ for every time-like vector $v \in TM$ satisfying $g(v,v) = -1$. Assuming that the space-time $(M, g, \mathcal{C})$ satisfies the Einstein equations (4.1) with zero cosmological constant $\Lambda = 0$, the strong energy condition is equivalent to $\operatorname{Ric}(v,v) \geq 0$ for every time-like vector $v \in TM$. This corresponds to the case $K = 0$ in Corollary 4.4.

The strong energy condition, proposed by Hawking and Penrose [57, 38, 39], plays a key role in general relativity. For instance, in the presence of trapped surfaces, it implies that the space-time has singularities (e.g. black holes) [28, 68].

4.2. **OT-characterization of** $\operatorname{Ric} \leq \frac{2\Lambda}{n-2}g + 8\pi T - \frac{8\pi}{n-2}\operatorname{Tr}_g(T)\, g$**.** The goal of the present section is to provide an optimal transport formulation of upper bounds on *causal* Ricci curvature in the *Lorentzian setting.* More precisely, given a quadratic form $\tilde{T}$ (which will later be chosen to be equal to the right hand side of Einstein equations, i.e. $\frac{2\Lambda}{n-2}g + 8\pi T - \frac{8\pi}{n-2}\mathrm{Tr}_g(T)\,g)$, we aim to find an optimal transport formulation of the condition

$$\text{“}\mathrm{Ric}(v,v) \le \tilde{T}(v,v) \text{ for every causal vector } v \in \mathcal{C}\text{”}.$$

The Riemannian counterpart, in the special case of Ric $\le Kg$ for some constant $K \in \mathbb{R}$, has been recently established by Sturm [61].

In order to state the result, let us fix some notation. Given a relatively compact open subset $E \subset\subset \mathrm{Int}(\mathcal{C})$ let $p_{TM\to M} : TM \to M$ be the canonical projection map and $\mathrm{inj}_g(E) > 0$ be the injectivity radius of the exponential map of g restricted to E. For $x \in p_{TM\to M}(E)$ and $r \in (0, \mathrm{inj}_g(E))$ we denote

$$B_r^{g,E}(x) := \{\exp_x^g(tw) : w \in T_xM \cap E,\ g(w,w) = -1, t \in [0,r]\}.$$

The next general result will be applied with $n \ge 3$ and

$$\tilde{T} = \frac{2\Lambda}{n-2}g + 8\pi T - \frac{8\pi}{n-2}\mathrm{Tr}_g(T)\,g.$$

Theorem 4.6 (OT-characterization of Ric $\le \tilde{T}$)**.** *Let $(M, g, \mathcal{C})$ be a space-time of dimension $n \ge 2$ and let $\tilde{T}$ be a quadratic form on M. Then the following assertions are equivalent:*

(1) $\mathrm{Ric}(v,v) \le \tilde{T}(v,v)$ *for every causal vector $v \in \mathcal{C}$.*

(2) *For every $p \in (0,1)$ and for every relatively compact open subset $E \subset\subset \mathrm{Int}(\mathcal{C})$ there exist $R = R(E) \in (0,1)$ and a function*

$$\epsilon = \epsilon_E : (0,\infty) \to (0,\infty) \text{ with } \lim_{r\downarrow 0}\epsilon(r) = 0 \text{ such that}$$

$$\forall x \in p_{TM\to M}(E) \text{ and } v \in T_xM \cap E \text{ with } g(v,v) = -R^2$$

the next assertion holds. For every $r \in (0,R)$, setting $y = \exp_x^g(rv)$, there exists a regular c_p-optimal dynamical plan $\Pi = \Pi(x,v,r)$ with associated curve of probability measures

$$(\mu_t := (\mathrm{e}_t)_\sharp \Pi)_{t\in[0,1]} \subset \mathcal{P}(M)$$

such that

- $\mu_{1/2} = \mathrm{vol}_g(B_{r^4}^{g,E}(x))^{-1}\,\mathrm{vol}_g \llcorner B_{r^4}^{g,E}(x)$,
- $\mathrm{supp}(\mu_1) \subset \{\exp_y^g(r^2 w) : w \in T_yM \cap \mathcal{C},\ g(w,w) = -1\}$

and which has $\tilde{T}(v,v)$-concave entropy in the sense that

$$\frac{4}{r^2}\left[\mathrm{Ent}(\mu_1|\mathrm{vol}_g) - 2\mathrm{Ent}(\mu_{1/2}|\mathrm{vol}_g) + \mathrm{Ent}(\mu_0|\mathrm{vol}_g)\right] \le \tilde{T}(v,v) + \epsilon(r). \tag{4.21}$$

(3) *There exists $p \in (0,1)$ such that the analogous assertion as in* (2) *holds true.*

Remark 4.7. Given an auxiliary Riemannian metric h on M, with analogous arguments as in the proof below, the condition (2) in Theorem 4.6 can be replaced by

(2)' For every $p \in (0,1)$ and for every $x \in M$ there exist $R = R(x) > 0$ and a function

$$\epsilon = \epsilon_x : (0,\infty) \to (0,\infty) \text{ with } \lim_{r\downarrow 0}\epsilon(r) = 0 \text{ such that}$$

$$\forall v \in \mathrm{Int}(\mathcal{C}_x) \text{ with } h(v,v) \le R^2$$

the next assertion holds. For every $r \in (0, R)$, setting $y = \exp_x^g(rv)$, there exists a regular c_p-optimal dynamical plan $\Pi = \Pi(x, v, r)$ with associated curve of probability measures

$$(\mu_t := (\mathrm{e}_t)_\sharp \Pi)_{t\in[0,1]} \subset \mathcal{P}(M)$$

such that

- $\mu_{1/2} = \mathrm{vol}_g(B^{g,E}_{r^4}(x))^{-1}\, \mathrm{vol}_g \llcorner B^{g,E}_{r^4}(x)$,
- $\mathrm{supp}(\mu_1) \subset \{\exp^g_y(r^2 w) : w \in T_yM \cap \mathcal{C},\, g(w,w) = -1\}$

and satisfying (4.21).

Moreover, both in (2) and (2') one can replace $B^{g,E}_{r^4}(x)$ (resp. $\{\exp^g_y(r^2 w) : w \in T_yM \cap \mathcal{C},\, g(w,w) = -1\}$) by $B^h_{r^4}(x)$ (resp. $B^h_{r^2}(y)$).

Proof. **(1)⇒ (2)**
Let $(M, g, \mathcal{C})$ be a space time and let h be an auxiliary Riemannian metric on M such that

$$\frac{1}{4} h(w,w) \leq |g(w,w)| \leq 4h(w,w), \quad \forall w \in E.$$

We denote with d^{TM}_h the distance on TM induced by the auxiliary Riemannian metric h. Once the compact subset $E \subset\subset \mathrm{Int}(\mathcal{C})$ is fixed, thanks to Lemma 3.1 there exist a constant

$$R = R(E) \in (0, \min(1, \mathrm{inj}_g(E)))$$

and a function

$$\epsilon = \epsilon_E : (0, \infty) \to (0, \infty) \text{ with } \lim_{r\downarrow 0} \epsilon(r) = 0$$

such that

$$\forall r \in (0, R/10),\, x \in p_{TM\to M}(E),\, v \in T_xM \cap E \text{ with } g(v,v) = -R^2$$

we can find a c_p-convex function $\phi : M \to \mathbb{R}$ with the following properties:

(1) ϕ is smooth on $B^h_{100r}(x)$, $\nabla^q_g\phi(x) = v$, $\nabla^q_g\phi \in E$ on $B^h_{10r}(x)$, $\mathrm{d}^{TM}_h(\nabla^q_g\phi, v) \leq \epsilon(r)$ on $B^h_{10r}(x)$;
(2) $|\mathrm{Hess}_\phi|_h \leq \epsilon(r)$ on $B^h_{10r}(x)$.

For $t \in [0,1]$, consider the map

$$\Psi^t_{1/2} : z \mapsto \exp_z(r(t - 1/2)\nabla^q_g\phi(z)).$$

Notice that

$$\Psi^t_{1/2}(B^{g,E}_{r^4}(x)) \subset B^h_{10r}(x), \quad \forall t \in [0,1].$$

Let

$$\mu_{1/2} = \mathrm{vol}_g(B^{g,E}_{r^4}(x))^{-1}\, \mathrm{vol}_g \llcorner B^{g,E}_{r^4}(x)$$

and define

$$\mu_t := (\Psi^t_{1/2})_\sharp(\mu_{1/2}) \quad \forall t \in [0,1].$$

By the properties of ϕ, the plan Π representing the curve of probability measures $(\mu_t)_{t\in[0,1]}$ is a regular c_p-optimal dynamical plan and

$$\mathrm{supp}(\mu_1) \subset \{\exp^g_y(r^2 w) : w \in T_yM \cap \mathcal{C},\, g(w,w) = -1\}.$$

By Proposition 3.4 we can find a smooth family of functions $(\phi_t)_{t\in[0,1]}$ defined on $\bigcup_{t\in[0,1]}\{t\}\times \mathrm{supp}(\mu_t)$ with $\phi_{1/2}=\phi$ satisfying

$$(\partial_t\phi_t)(\gamma_t)+\frac{r}{q}\left(-g(\nabla_g\phi_t(\gamma_t),\nabla_g\phi_t(\gamma_t))\right)^{q/2}=0 \quad \text{for } \Pi\text{-a.e. } \gamma, \text{ for all } t\in[0,1], \tag{4.22}$$

$$r\nabla_g^q\phi_t(\gamma_t)-\dot\gamma_t=0 \quad \text{for } \Pi\text{-a.e. } \gamma, \text{ for all } t\in[0,1]. \tag{4.23}$$

Moreover, using the properties of $\phi_{1/2}=\phi$ and the smoothness of the family $(\phi_t)_{t\in[0,1]}$, we have

$$\mathrm{d}_h^{TM}(\nabla_g^q\phi_t(\gamma_t),v)\le\epsilon(r), \quad |\mathrm{Hess}_{\phi_t}(\gamma_t)|_h\le\epsilon(r), \quad \Pi\text{-a.e. } \gamma, \text{ for all } t\in[0,1], \tag{4.24}$$

up to renaming $\epsilon(r)$ with a suitable function

$$\epsilon=\epsilon_E:(0,\infty)\to(0,\infty)$$

with

$$\lim_{r\downarrow 0}\epsilon(r)=0.$$

The curve $[0,1]\ni t\mapsto \mathrm{Ent}(\mu_t|\mathrm{vol}_g)\in\mathbb{R}$ is smooth and, in virtue of (4.23), it satisfies

$$\frac{d}{dt}\mathrm{Ent}(\mu_t|\mathrm{vol}_g)=r\int_M g(\nabla_g^q\phi_t,\nabla_g\rho_t)d\mathrm{vol}_g=r\int_M \Box_g^q\phi_t\,d\mu_t, \quad \text{for all } t\in[0,1], \tag{4.25}$$

where

$$\rho_t:=\frac{d\mu_t}{d\mathrm{vol}_g}$$

is the density of μ_t and

$$\Box_g^q\phi_t:=\mathrm{div}(-\nabla_g^q\phi_t)$$

is the q-Box of ϕ_t (the Lorentzian analog of the q-Laplacian).

For what follows it is useful to consider the linearization of the q-Box at a smooth function f, denoted by L_f^q and defined by the following relation:

$$\left.\frac{d}{dt}\right|_{t=0}\Box_g^q(f+tu)=L_f^q u, \quad \forall u\in C_c^\infty(M). \tag{4.26}$$

The map $[0,1]\ni t\mapsto\int_M\Box_g^q\phi_t\,d\mu_t\in\mathbb{R}$ is smooth and, in virtue of (4.22) and (4.23), it satisfies

$$\frac{d}{dt}r\int_M\Box_g^q\phi_t\,d\mu_t=-r^2\int_M L_\phi^q\left(\frac{1}{q}(-g(\nabla_g\phi_t,\nabla_g\phi_t))^{q/2}\right)+g(\nabla_g\Box_g^q\phi_t,-\nabla_g^q\phi_t)\,d\mu_t,$$

for every $t\in[0,1]$. Using the q-Bochner identity (A.2) together with the assumption $\mathrm{Ric}(w,w)\le\tilde T(w,w)$ for any $w\in\mathcal{C}$ and the estimates (4.24) on

ϕ_t, we can rewrite the last formula as

$$
\begin{aligned}
&\frac{d}{dt}\frac{1}{r}\int_M \Box_g^q \phi_t \, d\mu_t = \\
&\quad \int_M |g(\nabla_g\phi_t, \nabla_g\phi_t)|^{q-2} \Big[\mathrm{Ric}(\nabla_g\phi_t, \nabla_g\phi_t) + g(\mathrm{Hess}_{\phi_t}, \mathrm{Hess}_{\phi_t}) \\
&\qquad\qquad + \left((q-2)\frac{\mathrm{Hess}_{\phi_t}(\nabla_g\phi_t, \nabla_g\phi_t)}{|g(\nabla_g\phi_t, \nabla_g\phi_t)|}\right)^2 \\
&\qquad\qquad -2(q-2)\frac{\mathrm{Hess}_{\phi_t}\big((\nabla_g\phi_t, \mathrm{Hess}_{\phi_t}(\nabla_g\phi_t)\big)}{|g(\nabla_g\phi_t, \nabla_g\phi_t)|}\Big] d\mu_t \\
&\quad \le \int_M \big(\tilde{T}(v,v) + \epsilon(r)\big) d\mu_t = \tilde{T}(v,v) + \epsilon(r) \quad \text{for all } t \in [0,1],
\end{aligned}
$$

up to renaming $\epsilon(r)$ with a suitable function

$$\epsilon = \epsilon_E : (0,\infty) \to (0,\infty) \text{ with } \lim_{r\downarrow 0} \epsilon(r) = 0.$$

Thus

$$
\begin{aligned}
&\mathrm{Ent}(\mu_1|\mathrm{vol}_g) - 2\mathrm{Ent}(\mu_{1/2}|\mathrm{vol}_g) + \mathrm{Ent}(\mu_0|\mathrm{vol}_g) \\
&\quad = \int_0^{1/2} \left(\frac{d}{dt}\mathrm{Ent}(\mu_t|\mathrm{vol}_g)|_{t=s+1/2} - \frac{d}{dt}\mathrm{Ent}(\mu_t|\mathrm{vol}_g)|_{t=s}\right) ds \\
&\quad = \int_0^{1/2}\int_s^{s+1/2} \frac{d^2}{dt^2}\mathrm{Ent}(\mu_t|\mathrm{vol}_g) dt\, ds \\
&\quad \le \frac{\tilde{T}(v,v) + \epsilon(r)}{4} r^2.
\end{aligned}
$$

(2)⇒ (3): trivial.

(3)⇒ (1)
Fix $p \in (0,1)$ given by (3) and assume by contradiction that there exists $x \in M$, $\epsilon > 0$ and $v \in T_xM \cap \mathcal{C}$ with $-g(v,v) = 1$ such that

$$\mathrm{Ric}(v,v) \ge (\tilde{T} - 2\epsilon g)(v,v).$$

Then, by continuity, we can find a relatively compact neighbourhood $E \subset\subset \mathrm{Int}(\mathcal{C})$ of v in TM such that

$$\mathrm{Ric}(w,w) \ge (\tilde{T} - \epsilon g)(w,w), \quad \forall w \in E. \tag{4.27}$$

By Lemma 3.1 we can construct a c_p-convex function $\phi : M \to \mathbb{R}$ such that ϕ is smooth on a neighbourhood of x and

$$\nabla_g^q \phi(x) = Rv.$$

For $t \in [0,1]$, define

$$\Psi_{1/2}^t(z) := \exp_z^g(2r(t-1/2)\nabla_g^q\phi(z)).$$

By continuity, for $r \in (0,R)$ small enough, we have that

$$\frac{1}{r}\frac{d}{dt}\Psi_{1/2}^t(z) \in E, \quad \mathrm{d}_h^{TM}\left(\frac{d}{dt}\Psi_{1/2}^t(z), rv\right) \le \epsilon r, \quad \forall z \in B_{r^4}^{g,E}(x), \quad \forall t \in [0,1]. \tag{4.28}$$

Moreover $\Psi^1_{1/2}(B^{g,E}_{r^4}(x)) \subset B^{g,E}_{r^2}(y)$.
Set $\mu_{1/2} := \mathrm{vol}_g(B^{g,E}_{r^4}(x))^{-1}\, \mathrm{vol}_g \llcorner B^{g,E}_{r^4}(x)$ and consider $\mu_t := (\Psi^t_{1/2})_\sharp \mu_{1/2}$. Notice that

$$\mathrm{supp}(\mu_1) \subset B^{g,E}_{r^2}(y) \subset \{\exp^g_y(r^2 w) : w \in T_yM \cap \mathcal{C},\ g(w,w) = -1\}.$$

By the above construction, we get that $(\mu_t)_{t\in[0,1]}$ can be represented by a regular c_p-optimal dynamical plan Π such that

$$\mathrm{supp}((\partial \mathrm{e})_\sharp \Pi) \subset E.$$

Therefore (4.27) together with Theorem 4.3 yields

$$\begin{aligned}
\mathrm{Ent}(\mu_{1/2}|\mathrm{vol}_g) &\le \frac{1}{2}\mathrm{Ent}(\mu_0|\mathrm{vol}_g) + \frac{1}{2}\mathrm{Ent}(\mu_1|\mathrm{vol}_g) \\
&\qquad - \int\int_0^1 \mathrm{G}(s,1/2)(\tilde{T} - \epsilon g)(\dot\gamma_s, \dot\gamma_s)ds\, d\Pi(\gamma), \\
&\le \frac{1}{2}\mathrm{Ent}(\mu_0|\mathrm{vol}_g) + \frac{1}{2}\mathrm{Ent}(\mu_1|\mathrm{vol}_g) - \frac{r^2(\tilde{T}(v,v) - \epsilon g(v,v) + C\epsilon r)}{8}, \qquad (4.29)
\end{aligned}$$

where in the second inequality we used (4.28) and that $C > 0$ is a constant independent of r and ϵ. Note that $\epsilon > 0$ in (4.29) is fixed independently of $r > 0$. Clearly (4.29) contradicts the existence of $\epsilon_E(r) \to 0$ as $r \to 0$ so that (4.21) holds. □

In the vacuum case when $T \equiv 0$, the inequality

$$\mathrm{Ric} \le \frac{2\Lambda}{n-2} g + 8\pi T - \frac{8\pi}{n-2}\mathrm{Tr}_g(T)\, g$$

reads as

$$\mathrm{Ric} \le Kg \text{ with } K = \frac{2\Lambda}{n-2} \in \mathbb{R}.$$

Note that for $v \in \mathcal{C}$ it holds $g(v,v) \le 0$ so, when comparing the next result with its Riemannian counterpart [61], the sign of the lower bound K is reversed.

Corollary 4.8. *Let $(M, g, \mathcal{C})$ be a space-time of dimension $n \ge 2$ and let $K \in \mathbb{R}$. Then the following assertions are equivalent:*

1. $\mathrm{Ric}(v,v) \le Kg(v,v)$ *for every causal vector* $v \in \mathcal{C}$.
2. *For every $p \in (0,1)$ and for every relatively compact open subset $E \subset\subset \mathrm{Int}(\mathcal{C})$ there exist $R = R(E) \in (0,1)$ and a function*

   $$\epsilon = \epsilon_E : (0,\infty) \to (0,\infty) \text{ with } \lim_{r\downarrow 0} \epsilon(r) = 0 \text{ such that}$$

   $$\forall x \in p_{TM\to M}(E) \text{ and } v \in T_xM \cap E \text{ with } g(v,v) = -R^2$$

   the next assertion holds. For every $r \in (0,R)$, setting $y = \exp^g_x(rv)$, there exists a regular c_p-optimal dynamical plan $\Pi = \Pi(x,v,r)$ with associated curve of probability measures

   $$(\mu_t := (\mathrm{e}_t)_\sharp \Pi)_{t\in[0,1]} \subset \mathcal{P}(M)$$

   such that
   - $\mu_{1/2} = \mathrm{vol}_g(B^{g,E}_{r^4}(x))^{-1}\, \mathrm{vol}_g \llcorner B^{g,E}_{r^4}(x)$,
   - $\mathrm{supp}(\mu_1) \subset \{\exp^g_y(r^2 w) : w \in T_yM \cap \mathcal{C},\ g(w,w) = -1\}$

and which has $-K$-concave entropy in the sense that

$$\frac{4}{r^2}\left[\mathrm{Ent}(\mu_1|\mathrm{vol}_g)-2\mathrm{Ent}(\mu_{1/2}|\mathrm{vol}_g)+\mathrm{Ent}(\mu_0|\mathrm{vol}_g)\right]\leq -K+\epsilon(r). \tag{4.30}$$

(3) *There exists $p\in(0,1)$ such that the analogous assertion as in* (2) *holds true.*

4.3. **Optimal transport formulation of the Einstein equations.** Recall that Einstein equations of general relativity, with cosmological constant equal to $\Lambda\in\mathbb{R}$ and energy-momentum tensor T, read as

$$\mathrm{Ric}=\frac{2\Lambda}{n-2}g+8\pi T-\frac{8\pi}{n-2}\mathrm{Tr}_g(T)\,g, \tag{4.31}$$

for an n-dimensional space-time $(M,g,\mathcal{C})$. Combining Theorem 4.3 with Theorem 4.6, both with the choice

$$\tilde{T}=\frac{2\Lambda}{n-2}g+8\pi T-\frac{8\pi}{n-2}\mathrm{Tr}_g(T)\,g,$$

we obtain the following optimal transport formulation of (4.31).

Theorem 4.9. *Let $(M,g,\mathcal{C})$ be a space-time of dimension $n\geq 2$ and let $\tilde{T}$ be a quadratic form on M. Then the following assertions are equivalent:*

(1) $\mathrm{Ric}(v,v)=\tilde{T}(v,v)$ *for every $v\in T_xM$*
(2) $\mathrm{Ric}(v,v)=\tilde{T}(v,v)$ *for every causal vector $v\in\mathcal{C}$.*
(3) *For every $p\in(0,1)$ the following holds. For every relatively compact open subset $E\subset\subset\mathrm{Int}(\mathcal{C})$ there exist $R=R(E)\in(0,1)$ and a function*

$$\epsilon=\epsilon_E:(0,\infty)\to(0,\infty)\ \textit{with}\ \lim_{r\downarrow 0}\epsilon(r)=0$$

such that

$$\forall x\in p_{TM\to M}(E)\ \textit{and}\ \forall v\in T_xM\cap E\ \textit{with}\ g(v,v)=-R^2$$

the next assertion holds. For every $r\in(0,R)$, setting $y=\exp_x^g(rv)$, there exists a regular c_p-optimal dynamical plan $\Pi=\Pi(x,v,r)$ with associated curve of probability measures

$$(\mu_t:=(\mathrm{e}_t)_\sharp\Pi)_{t\in[0,1]}\subset\mathcal{P}(M)$$

such that

- $\mu_{1/2}=\mathrm{vol}_g(B^{g,E}_{r^4}(x))^{-1}\,\mathrm{vol}_g\llcorner B^{g,E}_{r^4}(x)$,
- $\mathrm{supp}(\mu_1)\subset\{\exp_y^g(r^2w):w\in T_yM\cap\mathcal{C},\ g(w,w)=-1\}$

and satisfying

$$\tilde{T}(v,v)-\epsilon(r)\leq\frac{4}{r^2}\left[\mathrm{Ent}(\mu_1|\mathrm{vol}_g)-2\mathrm{Ent}(\mu_{1/2}|\mathrm{vol}_g)+\mathrm{Ent}(\mu_0|\mathrm{vol}_g)\right]\leq\tilde{T}(v,v)+\epsilon(r). \tag{4.32}$$

(4) *There exists $p\in(0,1)$ such that the analogous assertion as in* (3) *holds true.*

Remark 4.10 ($\mu_{1/2}$ can be chosen more general)**.** From the proof of Theorem 4.9 it follows that one can replace (2) (and analogously (3)) with the following (a priori stronger, but a fortiori equivalent) statement. For every $p \in (0,1)$ the following holds. For every relatively compact open subset $E \subset\subset \mathrm{Int}(\mathcal{C})$ there exist $R = R(E) \in (0,1)$ and a function

$$\epsilon = \epsilon_E : (0,\infty) \to (0,\infty) \text{ with } \lim_{r\downarrow 0} \epsilon(r) = 0$$

such that

$$\forall x \in p_{TM\to M}(E), \forall v \in T_xM \cap E \text{ with } g(v,v) = -R^2$$

the next assertion holds. For every $r \in (0,R)$ and every

$$\mu_{1/2} \in \mathcal{P}(M) \text{ with } \mu_{1/2} \ll \mathrm{vol}_g \text{ and } \mathrm{supp}(\mu_{1/2}) \subset B^{g,E}_{r^4}(x),$$

setting $y = \exp^g_x(rv)$, there exists a regular c_p-optimal dynamical plan $\Pi = \Pi(\mu_{1/2}, v, r)$ with associated curve of probability measures

$$(\mu_t := (\mathrm{e}_t)_\sharp \Pi)_{t\in[0,1]} \subset \mathcal{P}(M)$$

such that

$$\mathrm{supp}(\mu_1) \subset \{\exp^g_y(r^2 w) : w \in T_yM \cap \mathcal{C},\ g(w,w) = -1\}$$

and satisfying (4.32).

Remark 4.11 (An equivalent statement via an auxiliary Riemannian metric h)**.** Given an auxiliary Riemannian metric h on M, with analogous arguments as in the proof below, the condition (3) in Theorem 4.9 can be replaced by

(3)' For every $p \in (0,1)$ the following holds. For every $x \in M$ there exist $R = R(x) > 0$ and a function

$$\epsilon = \epsilon_x : (0,\infty) \to (0,\infty) \text{ with } \lim_{r\downarrow 0} \epsilon(r) = 0$$

such that

$$\forall v \in \mathrm{Int}(\mathcal{C}_x) \text{ with } h(v,v) \le R^2$$

the next assertion holds. For every $r \in (0,R)$, setting $y = \exp^g_x(rv)$, there exists a regular c_p-optimal dynamical plan $\Pi = \Pi(x,v,r)$ with associated curve of probability measures

$$(\mu_t := (\mathrm{e}_t)_\sharp \Pi)_{t\in[0,1]} \subset \mathcal{P}(M)$$

such that
- $\mu_{1/2} = \mathrm{vol}_g(B^{g,E}_{r^4}(x))^{-1}\, \mathrm{vol}_g \llcorner B^{g,E}_{r^4}(x)$,
- $\mathrm{supp}(\mu_1) \subset \{\exp^g_y(r^2 w) \,:\, w \in T_yM \cap \mathcal{C},\ g(w,w) = -1\}$

and satisfying (4.32).

Moreover, both in (3) and (3') one can replace $B^{g,E}_{r^4}(x)$ by $B^h_{r^4}(x)$ and

$$\{\exp^g_y(r^2 w) \,:\, w \in T_yM \cap \mathcal{C},\ g(w,w) = -1\}$$

by $B^h_{r^2}(y)$.

Remark 4.12 (The tensor $\tilde{T}$)**.** As mentioned above, we will assume the cosmological constant Λ and the energy momentum tensor T to be given, say from physics and/or mathematical general relativity. Given g, Λ and T, for convenience of notation we set $\tilde{T}$ to be defined in (1.7).
Let us stress that not any symmetric bilinear form $\tilde{T}$ would correspond to a physically meaningful situation; in order to be physically relevant, it is crucial that $\tilde{T}$ is given by (1.7) where T is a physical energy-momentum tensor (in particular T has to satisfy $\nabla^a T_{ab} = 0$, i.e. be "freely gravitating", it has to satisfy some suitable energy condition like the "dominant energy condition", etc.).

Proof of Theorem 4.9. **(1)$\Rightarrow$ (2)**: trivial.

(2)$\Rightarrow$ (1): Follows by the identity theorem for polynomials and the fact that $\mathcal{C}$ has non-empty open interior.

(2)$\Rightarrow$ (3): From the implication (1)$\Rightarrow$ (2) in Theorem 4.6, we get a regular c_p-optimal dynamical plan $\Pi = \Pi(x, v, r)$ as in (3) such that the upper bound in (4.32) holds. Moreover, from (4.24) it holds that

$$\mathrm{d}_h^{TM}(\dot{\gamma}_t, rv) \leq r\epsilon(r), \quad \text{for } \Pi\text{-a.e. } \gamma, \text{ for all } t \in [0, 1]. \tag{4.33}$$

Recalling that the implication (1)$\Rightarrow$ (2) in Theorem 4.3 gives the convexity property (4.9) of the entropy along *every* regular c_p-optimal dynamical plan, and using (4.33), we conclude that also the lower bound in (4.32) holds.

(3)$\Rightarrow$ (4): trivial.

(4)$\Rightarrow$ (2).
The fact that

$$\mathrm{Ric}(v, v) \leq \tilde{T}(v, v) \quad \forall v \in \mathcal{C}$$

follows directly from Theorem 4.6. The fact that

$$\mathrm{Ric}(v, v) \geq \tilde{T}(v, v) \quad \forall v \in \mathcal{C}$$

can be showed following arguments already used in the paper, let us briefly discuss it. Fix $p \in (0, 1)$ given by (4) and assume by contradiction that

$$\exists x \in M,\ \delta > 0 \text{ and } v \in T_x M \cap \mathcal{C} \text{ with } -g(v, v) > 0$$

such that

$$\mathrm{Ric}(v, v) \leq (\tilde{T} + 3\delta g)(v, v).$$

Thanks to Lemma 3.1, up to replacing v with tv for some $t \in (0, 1)$ small enough, we know that we can construct a c_p-convex function $\phi : M \to \mathbb{R}$, smooth in a neighbourhood of x and satisfying

$$\nabla_g^q \phi(x) = v \qquad \mathrm{Hess}_\phi(x) = 0.$$

Then, by continuity, we can find

- a relatively compact open neighbourhood $E \subset\subset \mathrm{Int}(\mathcal{C})$ of v in TM such that

$$\mathrm{Ric}(w, w) \leq (\tilde{T} + 2\delta g)(w, w), \quad \forall w \in E; \tag{4.34}$$

- ϕ is smooth on $B^h_{100r}(x)$,
$$\nabla^q_g \phi \in E \text{ on } B^h_{10r}(x),$$
$$\mathrm{d}_h^{TM}(\nabla^q_g\phi, v) \le \epsilon(r) \text{ on } B^h_{10r}(x);$$
- $|\mathrm{Hess}_\phi|_h \le \epsilon(r)$ on $B^h_{10r}(x)$;

where
$$\epsilon(r) \to 0 \text{ as } r \to 0.$$
For $t \in [0,1]$, consider the map
$$\Psi^t_{1/2} : z \mapsto \exp_z(r(t-1/2)\nabla^q_g\phi(z)).$$
Notice that
$$\Psi^t_{1/2}(B^{g,E}_{r^4}(x)) \subset B^h_{10r}(x) \quad \forall t \in [0,1].$$
Call
$$\mu_{1/2} = \mathrm{vol}_g(B^{g,E}_{r^4}(x))^{-1}\, \mathrm{vol}_g \llcorner B^{g,E}_{r^4}(x)$$
and define
$$\mu_t := (\Psi^t_{1/2})_\sharp(\mu_{1/2}) \text{ for } t \in [0,1].$$
By the properties of ϕ, the plan Π representing the curve of probability measures $(\mu_t)_{t\in[0,1]}$ is a regular c_p-optimal dynamical plan and
$$\mathrm{supp}(\mu_1) \subset \{\exp^g_y(r^2 w) : w \in T_yM \cap \mathcal{C},\, g(w,w) = -1\}.$$
Moreover
$$\frac{1}{r}\frac{d}{dt}\Psi^t_{1/2}(z) \in E \quad \forall z \in B^h_r(x), \quad \forall t \in [0,1]. \tag{4.35}$$
We can now follow verbatim the arguments in (1)⇒(2) of Theorem 4.6 by using (4.34) and (4.35), obtaining a function $\epsilon(r) \to 0$ as $r \to 0$ such that
$$\mathrm{Ent}(\mu_1|\mathrm{vol}_g) - 2\mathrm{Ent}(\mu_{1/2}|\mathrm{vol}_g) + \mathrm{Ent}(\mu_0|\mathrm{vol}_g) \le \frac{(\tilde{T} + \delta g)(v,v) + \epsilon(r)}{4}\, r^2.$$
The last inequality clearly contradicts the lower bound in (4.32).

□

In the vacuum case $T \equiv 0$ with cosmological constant $\Lambda \in \mathbb{R}$, the Einstein equations read as
$$\mathrm{Ric} \equiv \frac{\Lambda}{\frac{n}{2} - 1} g, \tag{4.36}$$
for an n-dimensional space-time $(M, g, \mathcal{C})$. Specializing Theorem 4.9 with the choice
$$\tilde{T} = \frac{\Lambda}{\frac{n}{2} - 1} g$$
and using Corollary 4.4 to sharpen the lower bound in (4.37) for the constant case, we obtain the following optimal transport formulation of Einstein vacuum equations.

Theorem 4.13. *Let $(M, g, \mathcal{C})$ be a space-time of dimension $n \ge 3$ and let $\Lambda \in \mathbb{R}$. Then the following assertions are equivalent:*

(1) *The space-time $(M, g, \mathcal{C})$ satisfies Einstein vacuum equations of General Relativity* (4.36) *corresponding to cosmological constant equal to Λ.*

(2) *For every $p \in (0,1)$ the following holds. For every relatively compact open subset $E \subset\subset \mathrm{Int}(\mathcal{C})$ there exist $R = R(E) \in (0,1)$ and a function*

$$\epsilon = \epsilon_E : (0,\infty) \to (0,\infty) \text{ with } \lim_{r\downarrow 0} \epsilon(r) = 0$$

such that

$$\forall x \in p_{TM\to M}(E) \text{ and } v \in T_xM \cap E \text{ with } g(v,v) = -R^2$$

the next assertion holds.
For every $r \in (0,R)$, setting $y = \exp_x^g(rv)$, there exists a regular c_p-optimal dynamical plan $\Pi = \Pi(x,v,r)$ with associated curve of probability measures

$$(\mu_t := (\mathrm{e}_t)_\sharp \Pi)_{t\in[0,1]} \subset \mathcal{P}(M)$$

such that
- $\mu_{1/2} = \mathrm{vol}_g(B^{g,E}_{r^4}(x))^{-1}\, \mathrm{vol}_g \llcorner B^{g,E}_{r^4}(x)$,
- $\mathrm{supp}(\mu_1) \subset \{\exp_y^g(r^2 w) : w \in T_yM \cap \mathcal{C},\ g(w,w) = -1\}$

and satisfying

(4.37)
$$-\frac{\Lambda}{\frac{n}{2}-1} \le \frac{4}{r^2}\left[\mathrm{Ent}(\mu_1|\mathrm{vol}_g) - 2\mathrm{Ent}(\mu_{1/2}|\mathrm{vol}_g) + \mathrm{Ent}(\mu_0|\mathrm{vol}_g)\right] \le -\frac{\Lambda}{\frac{n}{2}-1} + \epsilon(r).$$

(3) *There exists $p \in (0,1)$ such that the analogous assertion as in* (3) *holds true.*

It is worth to isolate the case of zero cosmological constant.

Corollary 4.14. *Let $(M,g,\mathcal{C})$ be a space-time of dimension $n \ge 2$. Then the following assertions are equivalent:*

(1) *The space-time $(M,g,\mathcal{C})$ satisfies Einstein vacuum equations of General Relativity with zero cosmological constant, i.e.* $\mathrm{Ric} \equiv 0$.
(2) *For every $p \in (0,1)$ the following holds. For every relatively compact open subset $E \subset\subset \mathrm{Int}(\mathcal{C})$ there exist $R = R(E) \in (0,1)$ and a function*

$$\epsilon = \epsilon_E : (0,\infty) \to (0,\infty) \text{ with } \lim_{r\downarrow 0} \epsilon(r)/r^2 = 0$$

such that

$$\forall x \in p_{TM\to M}(E) \text{ and } v \in T_xM \cap E \text{ with } g(v,v) = -R^2$$

the next assertion holds. For every $r \in (0,R)$, setting $y = \exp_x^g(rv)$, there exists a regular c_p-optimal dynamical plan $\Pi = \Pi(x,v,r)$ with associated curve of probability measures

$$(\mu_t := (\mathrm{e}_t)_\sharp \Pi)_{t\in[0,1]} \subset \mathcal{P}(M)$$

such that
- $\mu_{1/2} = \mathrm{vol}_g(B^{g,E}_{r^4}(x))^{-1}\, \mathrm{vol}_g \llcorner B^{g,E}_{r^4}(x)$,
- $\mathrm{supp}(\mu_1) \subset \{\exp_y^g(r^2 w) : w \in T_yM \cap \mathcal{C},\ g(w,w) = -1\}$

and satisfying

$$0 \le \mathrm{Ent}(\mu_1|\mathrm{vol}_g) - 2\mathrm{Ent}(\mu_{1/2}|\mathrm{vol}_g) + \mathrm{Ent}(\mu_0|\mathrm{vol}_g) \le \epsilon(r). \tag{4.38}$$

(3) *There exists $p \in (0,1)$ such that the analogous assertion as in* (3) *holds true.*

Appendix A. A q-Bochner identity in Lorentzian setting

In this section we prove a Bochner type identity in Lorentzian setting for the linearization of the q-Box operator, the Lorentzian analog of the q-Laplacian; let us mention that related results have been obtained in the Riemannian [47, 64] and Finsler settings [69, 70] but at best our knowledge this section is original in the Lorentzian $\mathcal{L}_p$ framework.

Throughout the section, $(M, g, \mathcal{C})$ is a space-time, $U \subset M$ is an open subset and $q \in (-\infty, 0)$ is fixed. Let $\phi \in C^3(U)$ satisfy

$$-\nabla_g \phi \in \mathrm{Int}(\mathcal{C}) \text{ on } U.$$

Denote by

$$\nabla_g^q \phi := -|g(\nabla_g \phi, \nabla_g \phi)|^{\frac{q-2}{2}} \nabla_g \phi$$

the q-gradient, by

$$\Box_g^q \phi := \mathrm{div}(-\nabla_g^q \phi)$$

the q-Box operator of ϕ and by L^q_ϕ the linearization of the q-Box operator at ϕ defined by the following relation:

$$\left.\frac{d}{dt}\right|_{t=0} \Box_g^q(\phi + tu) = L^q_\phi u, \quad \forall u \in C_c^\infty(M). \tag{A.1}$$

The ultimate goal of the section is to prove the following result.

Proposition A.1. *Under the above notation, the following q-Bochner identity holds:*

$$\begin{aligned}
-L^q_\phi\left(\frac{(-g(\nabla_g\phi,\nabla_g\phi))^{\frac{q}{2}}}{q}\right) &= g(\nabla_g \Box_g^q \phi, -\nabla_g^q \phi) \\
&\quad + (q-2)^2 |g(\nabla_g\phi,\nabla_g\phi)|^{q-2}\left(\frac{\mathrm{Hess}_\phi(\nabla_g\phi,\nabla_g\phi)}{|g(\nabla_g\phi,\nabla_g\phi)|}\right)^2 \\
&\quad + |g(\nabla_g\phi,\nabla_g\phi)|^{q-2}\big(\mathrm{Ric}(\nabla_g\phi,\nabla_g\phi) + g(\mathrm{Hess}_\phi,\mathrm{Hess}_\phi)\big) \\
&\quad - 2(q-2)\,|g(\nabla_g\phi,\nabla_g\phi)|^{q-3}\mathrm{Hess}_\phi\left(\nabla_g\phi, \mathrm{Hess}_\phi(\nabla_g\phi)\right).
\end{aligned} \tag{A.2}$$

The proof of Proposition A.1 requires some preliminary lemmas. First of all we derive an explicit expression for the operator L^q_ϕ.

Lemma A.2. *Under the above notation, it holds*

$$\begin{aligned}
L^q_\phi u =& \big(-g(\nabla_g\phi,\nabla_g\phi)\big)^{\frac{q-2}{2}}\left(\Box_g u - (q-2)\frac{\mathrm{Hess}_u(\nabla_g\phi,\nabla_g\phi)}{-g(\nabla_g\phi,\nabla_g\phi)}\right) \\
&+ (2-q)\big(-g(\nabla_g\phi,\nabla_g\phi)\big)^{-1} g(\nabla_g\phi,\nabla_g u)\Box_g^q\phi \\
&+ 2(2-q)\big(-g(\nabla_g\phi,\nabla_g\phi)\big)^{\frac{q-4}{2}}\mathrm{Hess}_\phi\left(\nabla_g\phi, \nabla_g u + \frac{g(\nabla_g\phi,\nabla_g u)}{-g(\nabla_g\phi,\nabla_g\phi)}\nabla_g\phi\right).
\end{aligned} \tag{A.3}$$

Proof. By the very definitions of $L^q_\phi u$ and $\Box^q_g \phi$, we have

$$
\begin{aligned}
L^q_\phi u &= \operatorname{div}\left(\frac{d}{dt}\Big|_{t=0} (-g(\nabla_g(\phi+tu), \nabla_g(\phi+tu)))^{\frac{q-2}{2}} \nabla_g(\phi+tu) \right) \\
&= \operatorname{div}\Bigg((2-q)\big(-g(\nabla_g\phi, \nabla_g\phi)\big)^{\frac{q-4}{2}} g(\nabla_g\phi, \nabla_g u)\, \nabla_g\phi \\
&\qquad + \big(-g(\nabla_g\phi, \nabla_g\phi)\big)^{\frac{q-2}{2}} \nabla_g u \Bigg).
\end{aligned} \tag{A.4}
$$

In order to explicit the last formula, compute

$$
\nabla_g\big(-g(\nabla_g\phi,\nabla_g\phi)\big)^{\alpha} = -2\alpha\,\big(-g(\nabla_g\phi,\nabla_g\phi)\big)^{\alpha-1}\mathrm{Hess}_\phi(\nabla_g\phi) \tag{A.5}
$$

$$
\nabla_g\big(g(\nabla_g\phi,\nabla_g u)\big) = \mathrm{Hess}_u(\nabla_g\phi) + \mathrm{Hess}_\phi(\nabla_g u). \tag{A.6}
$$

Plugging (A.5) and (A.6) into (A.4) gives (A.3). □

We next show a q-Bochner identity for the operator A^q_ϕ defined as

$$
A^q_\phi(u) := \big(-g(\nabla_g\phi,\nabla_g\phi)\big)^{\frac{q-2}{2}} \left(\Box_g u - (q-2)\frac{\mathrm{Hess}_u(\nabla_g\phi,\nabla_g\phi)}{-g(\nabla_g\phi,\nabla_g\phi)} \right). \tag{A.7}
$$

Lemma A.3. *Under the above notation, the following identity holds:*

$$
\begin{aligned}
&-L^q_\phi\left(\frac{(-g(\nabla_g\phi,\nabla_g\phi))^{\frac{q}{2}}}{q} \right) + g(\nabla_g \Box^q_g\phi, \nabla^q_g\phi) = \\
&\quad = (q-2)|g(\nabla_g\phi,\nabla_g\phi)|^{\frac{q-4}{2}} \Box^q_g\phi\, \mathrm{Hess}_\phi(\nabla_g\phi,\nabla_g\phi) \\
&\qquad + |g(\nabla_g\phi,\nabla_g\phi)|^{q-2}\Bigg(q(q-2)\left(\frac{\mathrm{Hess}_\phi(\nabla_g\phi,\nabla_g\phi)}{|g(\nabla_g\phi,\nabla_g\phi)|} \right)^2 \\
&\qquad\qquad + \mathrm{Ric}(\nabla_g\phi,\nabla_g\phi) + g(\mathrm{Hess}_\phi,\mathrm{Hess}_\phi) \Bigg).
\end{aligned} \tag{A.8}
$$

Proof. We perform the computation at an arbitrary point $x_0 \in U$. In order to simplify the computations, we consider normal coordinates (x^i) in a neighbourhood of x_0 with $\frac{\partial}{\partial x^1} \in \mathcal{C}$. It holds

$$
\begin{aligned}
-\frac{\Box_g(-g(\nabla_g\phi,\nabla_g\phi))^{\frac{q}{2}}}{q} &= g^{ij}\partial_i\left((-g(\nabla_g\phi,\nabla_g\phi))^{\frac{q-2}{2}} g^{kl}\partial_{jk}\phi\,\partial_l\phi \right) \\
&= (-g(\nabla_g\phi,\nabla_g\phi))^{\frac{q-2}{2}}\Bigg(-\frac{q-2}{-g(\nabla_g\phi,\nabla_g\phi)} g^{ij}g^{mn}\partial_{im}\phi\partial_n\phi g^{kl}\partial_{jk}\phi\,\partial_l\phi \\
&\qquad + g^{ij}g^{kl}\partial_{ijk}\phi\,\partial_l\phi + g^{ij}g^{kl}\partial_{jk}\phi\,\partial_{li}\phi \Bigg).
\end{aligned} \tag{A.9}
$$

Now, from the symmetry of second order derivatives and the very definition of the the Riemann tensor (2.9), we have

$$
\partial_{ijk}\phi = \partial_{ikj}\phi = g(R(\partial_{x^i},\partial_{x^k})\nabla_g\phi, \partial_{x^j}) + \partial_{kij}\phi. \tag{A.10}
$$

Thus

$$
g^{ij}g^{kl}\partial_{ijk}\phi\,\partial_l\phi = g(\nabla_g\Box_g\phi, \nabla_g\phi) + \mathrm{Ric}(\nabla_g\phi,\nabla_g\phi),
$$

and we can rewrite (A.9) as

$$-\frac{\Box_g(-g(\nabla_g\phi,\nabla_g\phi))^{\frac{q}{2}}}{q} = (-g(\nabla_g\phi,\nabla_g\phi))^{\frac{q-2}{2}}\Big((2-q)\frac{g(\mathrm{Hess}_\phi(\nabla_g\phi),\mathrm{Hess}_\phi(\nabla_g\phi))}{-g(\nabla_g\phi,\nabla_g\phi)} + g(\nabla_g\Box_g\phi,\nabla_g\phi)+\mathrm{Ric}(\nabla_g\phi,\nabla_g\phi) + g(\mathrm{Hess}_\phi,\mathrm{Hess}_\phi)\Big). \tag{A.11}$$

We now compute the second part of $-L^q_\phi\left(\frac{(-g(\nabla_g\phi,\nabla_g\phi))^{\frac{q}{2}}}{q}\right)$. To this aim observe that

$$\nabla_g\left(\frac{\big(-g(\nabla_g\phi,\nabla_g\phi)\big)^{\frac{q}{2}}}{q}\right) = -\big(-g(\nabla_g\phi,\nabla_g\phi)\big)^{\frac{q-2}{2}}\,\mathrm{Hess}_\phi(\nabla_g\phi) \tag{A.12}$$

$$\mathrm{Hess}_{\frac{(-g(\nabla_g\phi,\nabla_g\phi))^{\frac{q}{2}}}{q}}(\nabla_g\phi,\nabla_g\phi) = (q-2)\big(-g(\nabla_g\phi,\nabla_g\phi)\big)^{\frac{q-4}{2}}[\mathrm{Hess}_\phi(\nabla_g\phi,\nabla_g\phi)]^2 - \big(-g(\nabla_g\phi,\nabla_g\phi)\big)^{\frac{q-2}{2}} g(\nabla_{\nabla_g\phi}\nabla_{\nabla_g\phi}\nabla_g\phi,\nabla_g\phi). \tag{A.13}$$

It is useful to express $\Box^q_g$ in terms of $\Box_g$:

$$\begin{aligned}\Box^q_g\phi &:= \mathrm{div}\Big(\big(-g(\nabla_g\phi,\nabla_g\phi)\big)^{\frac{q-2}{2}}\nabla_g\phi\Big)\\ &= \big(-g(\nabla_g\phi,\nabla_g\phi)\big)^{\frac{q-2}{2}}\left(\Box_g\phi-(q-2)\frac{\mathrm{Hess}_\phi(\nabla_g\phi,\nabla_g\phi)}{-g(\nabla_g\phi,\nabla_g\phi)}\right).\end{aligned} \tag{A.14}$$

Using (A.14), we can write

$$\begin{aligned} g(\nabla_g\Box^q_g\phi,\nabla_g\phi) &= \\ &= g\left(\nabla_g\left(\big(-g(\nabla_g\phi,\nabla_g\phi)\big)^{\frac{q-2}{2}}\left(\Box_g\phi-(q-2)\frac{\mathrm{Hess}_\phi(\nabla_g\phi,\nabla_g\phi)}{-g(\nabla_g\phi,\nabla_g\phi)}\right)\right),\nabla_g\phi\right)\\ &= \big(-g(\nabla_g\phi,\nabla_g\phi)\big)^{\frac{q-2}{2}} g(\nabla_g\Box_g\phi,\nabla_g\phi)-(q-2)\frac{\mathrm{Hess}_\phi(\nabla_g\phi,\nabla_g\phi)}{-g(\nabla_g\phi,\nabla_g\phi)}\Box^q_g\phi\\ &\quad -(q-2)\big(-g(\nabla_g\phi,\nabla_g\phi)\big)^{\frac{q-4}{2}}\Big(g(\nabla_{\nabla_g\phi}\nabla_{\nabla_g\phi}\nabla_g\phi,\nabla_g\phi)\\ &\qquad + g(\mathrm{Hess}_\phi(\nabla_g\phi),\mathrm{Hess}_\phi(\nabla_g\phi))\Big)\\ &\quad -2(q-2)\big(-g(\nabla_g\phi,\nabla_g\phi)\big)^{\frac{q-6}{2}}[\mathrm{Hess}_\phi(\nabla_g\phi,\nabla_g\phi)]^2.\end{aligned} \tag{A.15}$$

Plugging (A.11) and (A.13) into (A.7), and simplifying using (A.15), gives the desired (A.8). $\square$

Proof of Proposition A.1 Combining the expression of $L^q_\phi u$ as in (A.3) with the definition of $A^q_\phi u$ as in (A.7) we can write

$$\begin{aligned} L^q_\phi u =& A^q_\phi u + (2-q)\big(-g(\nabla_g\phi,\nabla_g\phi)\big)^{-1} g(\nabla_g\phi,\nabla_g u)\Box^q_g\phi\\ &+2(2-q)\big(-g(\nabla_g\phi,\nabla_g\phi)\big)^{\frac{q-4}{2}}\mathrm{Hess}_\phi\left(\nabla_g\phi,\nabla_g u+\frac{g(\nabla_g\phi,\nabla_g u)}{-g(\nabla_g\phi,\nabla_g\phi)}\nabla_g\phi\right).\end{aligned} \tag{A.16}$$

Specializing (A.16) with $u = \frac{(-g(\nabla_g\phi,\nabla_g\phi))^{\frac{q}{2}}}{q}$ and using (A.12) gives

$$\begin{aligned}
-L^q_\phi\left(\frac{(-g(\nabla_g\phi,\nabla_g\phi))^{\frac{q}{2}}}{q}\right) &= -A^q_\phi\left(\frac{(-g(\nabla_g\phi,\nabla_g\phi))^{\frac{q}{2}}}{q}\right)\\
&\quad -(q-2)\big(-g(\nabla_g\phi,\nabla_g\phi)\big)^{\frac{q-2}{2}}\frac{\mathrm{Hess}_\phi(\nabla_g\phi,\nabla_g\phi)}{-g(\nabla_g\phi,\nabla_g\phi)}\Box^q_g\phi\\
&\quad -2(q-2)\big(-g(\nabla_g\phi,\nabla_g\phi)\big)^{q-3}\mathrm{Hess}_\phi\left(\nabla_g\phi,\mathrm{Hess}_\phi(\nabla_g\phi)\right)\\
&\quad -2(q-2)\big(-g(\nabla_g\phi,\nabla_g\phi)\big)^{q-2}\left(\frac{\mathrm{Hess}_\phi\left(\nabla_g\phi,\nabla_g\phi\right)}{-g(\nabla_g\phi,\nabla_g\phi)}\right)^2.
\end{aligned} \tag{A.17}$$

Now, the combination of (A.17) and (A.8) yields

$$\begin{aligned}
-L^q_\phi\left(\frac{(-g(\nabla_g\phi,\nabla_g\phi))^{\frac{q}{2}}}{q}\right) &= -g(\nabla_g\Box^q_g\phi,\nabla^q_g\phi)\\
&\quad +(q-2)^2|g(\nabla_g\phi,\nabla_g\phi)|^{q-2}\left(\frac{\mathrm{Hess}_\phi(\nabla_g\phi,\nabla_g\phi)}{|g(\nabla_g\phi,\nabla_g\phi)|}\right)^2\\
&\quad +|g(\nabla_g\phi,\nabla_g\phi)|^{q-2}\big(\mathrm{Ric}(\nabla_g\phi,\nabla_g\phi)+g(\mathrm{Hess}_\phi,\mathrm{Hess}_\phi)\big)\\
&\quad -2(q-2)\,|g(\nabla_g\phi,\nabla_g\phi)|^{q-3}\mathrm{Hess}_\phi\left(\nabla_g\phi,\mathrm{Hess}_\phi(\nabla_g\phi)\right).
\end{aligned} \tag{A.18}$$

□

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