Commuting the mean-field and classical limits in quantum mechanics

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The question and its motivation

We focus on two important limits used to derive evolution laws from microscopic dynamics:

- The **classical limit** (sometimes called semiclassical limit) when Planck's constant $\hbar$ (which measures the strength of quantum effects) is small with respect to the scale of observation as discussed later. It is closely related to the high-frequency limit of PDEs.
- The **mean-field limit** (one form of many-body limits, also called sometimes thermodynamical limits) when the number of bodies-particles N is sent to infinity, under some appropriate assumption of low correlations.

In many situations the two regimes are involved: we want to study how they interact. More precisely **we want to quantify the convergence of the mean-field limit uniformly along the classical limit**.

The two limits

Starting point: N-body Schrödinger equation for bosons

$$i\hbar\partial_t\Psi^N = -\frac{\hbar^2}{2m}\sum_{k=1}^{N}\Delta_{x_k}\Psi^N + \sum_{k,l=1}^{N} V(x_k - x_l)\Psi^N$$

on **symmetric** N-body wave function $\Psi^N(t, x_1, \ldots, x_N)$, $x_k \in \mathbb{R}^d$.

Binary interaction potential V: measurable and even on $\mathbb{R}^d$.

Rescaling: $\hat{x} = x/L$, $\hat{t} = t/T$, $\hat{V}(\hat{z}) = (NT^2)/(mL^2)V(z)$

$$i\partial_t\tilde{\Psi}^N_\epsilon = -\frac{\epsilon}{2}\sum_{k=1}^{N}\Delta_{x_k}\tilde{\Psi}^N_\epsilon + \frac{1}{N\epsilon}\sum_{k,l=1}^{N}\tilde{V}(\tilde{x}_k - \tilde{x}_l)\tilde{\Psi}^N_\epsilon, \quad \epsilon := \frac{\hbar T}{mL^2}$$

$\boxed{\text{Classical limit } \epsilon \to 0}$ - $\boxed{\text{Mean-field limit } N \to \infty}$

The quantum mean-field limit (Hartree equation)

$\triangle$ <u>N goes to infinity while ϵ is kept fixed</u> $\triangle$

Assuming initial decorrelation $\Psi^N_{\epsilon,in} \sim \prod_{k=1}^N \psi_{\epsilon,in}(x_k)$ and under various assumptions on V:

$$\int_{x_2,\dots,x_N} \overline{\Psi^N(t,x,x_2,\dots,x_N)}\Psi^N(t,y,x_2,\dots,x_N)\,\mathrm{d}x_2\cdots\mathrm{d}x_N$$

$$\xrightarrow[N\to\infty]{} \overline{\psi(t,x)}\psi(t,y)$$

where ψ solves the **Hartree equation**

$$i\partial_t\psi_\epsilon = -\frac{\epsilon}{2}\Delta_x\psi_\epsilon + \frac{1}{\epsilon}\left(V * |\psi_\epsilon|^2\right)\psi_\epsilon, \qquad (\psi_\epsilon)_{|t=0} = \psi_{\epsilon,in}$$

Coulomb potential V or even more singular (cf. cubic NLS) covered by existing results, but most of the time non-quantitative apart from restricted cases and degenerates as $\epsilon \to 0$

The classical limit (N-body Liouville equation)

$\triangle$ $\underline{\epsilon \text{ goes to zero while } N \text{ is kept fixed}}$ $\triangle$

High-frequency limit $\Rightarrow$ one needs to localise oscillations

Wigner transform at scale ϵ:

$$W_\epsilon[\Phi](X,\Xi) := \frac{1}{(2\pi)^n}\int_Y \overline{\Phi\left(X-\frac{Y}{2\epsilon}\right)}\Phi\left(X+\frac{Y}{2\epsilon}\right)e^{-i\Xi\cdot Y}\,\mathrm{d}Y$$

If the initial conditions satisfy $W_\epsilon[\Psi^N_{\epsilon,in}] \sim F^N_{in}$ as $\epsilon \to 0$, and under appropriate conditions on V:

$$W_\epsilon[\Psi^N_\epsilon(t,\cdot)] \sim F^N(t,\cdot) \quad \text{at later times } t \geq 0$$

where F^N satisfies the N-**body Liouville equation**

$$\partial_t F^N + \sum_{k=1}^N \xi_k \cdot \nabla_{x_k} F^N - \frac{1}{N}\sum_{k,l=1}^N \nabla V(x_k - x_l)\cdot \nabla_{\xi_k} F^N = 0.$$

The mean-field limit in classical mechanics

$\triangle$ $\underline{N \to \infty \text{ while } \epsilon = 0}$ $\triangle$

Assuming initial decorrelation $F_{in}^N(X,\Xi) \sim \prod_{k=1}^N f_{in}(x_k,\xi_k)$ and under various assumptions on V:

$$F^N(t,X,\Xi) \sim \prod_{k=1}^N f(t,x_k,\xi_k) \quad \text{at later times } t \geq 0 \text{ and}$$

$$\int_{x_2,\xi_2,\ldots,x_N,\xi_N} F^N(t,x,\xi,x_2,\xi_2,\ldots,x_N,\xi_N)\,\mathrm{d}x_2\,\mathrm{d}\xi_2\cdots\,\mathrm{d}x_N\,\mathrm{d}\xi_N$$

$$\xrightarrow[N\to\infty]{} f(t,x,\xi)$$

where f solves the **Vlasov equation**

$$\partial_t f + \xi \cdot \nabla_x f - (\nabla V *_x f)\nabla_\xi f = 0, \qquad f_{|t=0} = f_{in}.$$

Quantitative results for $V \in C^2$, partial results for some singular V, open for Coulomb-Newton potentials

The classical limit in mean-field mechanics

$\triangle$ $\underline{\epsilon \to 0 \text{ while } N = \infty}$ $\triangle$

High frequency limit again $\Rightarrow$ localise oscillations

Wigner transform at scale ϵ:

$$W_\epsilon[\Phi](X,\Xi) := \frac{1}{(2\pi)^n}\int_Y \overline{\Phi\left(X - \frac{Y}{2\epsilon}\right)}\Phi\left(X + \frac{Y}{2\epsilon}\right) e^{-i\Xi\cdot Y}\,\mathrm{d}Y$$

If the initial conditions satisfy $W_\epsilon[\psi_{\epsilon,in}](x,\xi) \sim f_{in}(x,\xi)$ as $\epsilon \to 0$, and under appropriate conditions on V:

$$W_\epsilon[\psi_\epsilon(t,\cdot)] \sim f(t,x,\xi) \quad \text{at later times } t \geq 0$$

with ψ_ϵ satisfies the Hartree equation, where f satisfies the Vlasov equation

$$\partial_t f + \xi\cdot\nabla_x f - (\nabla V *_x f)\cdot\nabla_\xi f = 0, \qquad f_{|t=0} = f_{in}.$$

The diagram of limits

$$\begin{tikzcd}
\boxed{\textbf{Schrödinger}} \arrow[r, "N\to\infty"] \arrow[d, "\epsilon\to 0"'] \arrow[dr, "?"] & \boxed{\textbf{Hartree}} \arrow[d, "\epsilon\to 0"] \\
\boxed{\textbf{Liouville}} \arrow[r, "N\to\infty"'] & \boxed{\textbf{Vlasov}}
\end{tikzcd}$$

Quantum mean-field limit:

Spohn'80, Bardos-Golse-Mauser'90s, Erdös-Schlein-Yau'00s, Fröhlich-Knowles-Schwarz, Rodnianski-Schlein, Pickl...

Classical limit by Wigner transform (finite or infinite N):

Lions-Paul'90s, Gérard-Markowich-Poupaud-Mauser'90s

Classical mean-field limit:

Neunzert-Wick'74, Braun-Hepp'77, Dobrushin'79, Hauray-Jabin'07, Golse-Mouhot-Ricci'13, Mischler-Mouhot-Ricci

The conceptual difficulties

(1) Classical mean-field limit traditionally reframed as the convergence of **empirical measures**

$$\mu^N_{(X,\Xi)} := \sum_{k=1}^{N} \delta_{(x_k,\xi_k)} \rightharpoonup f$$

It is based on the use of *weak topologies* and either compactness arguments or stability estimates in associated metrics (e.g. **Monge-Kantorovich-Wasserstein distances**)

(2) Quantum mean-field limit based most often on the **BBGKY hierarchy** written on the wave function, on compactness arguments and in the topology of the **trace-norm** which corresponds as $\epsilon \to 0$ to **total variation norm**

$\hookrightarrow$ quantitative results rare and restricted at quantum level
$\hookrightarrow$ conflict of topologies (weak vs strong topology)
$\hookrightarrow$ no equivalent of empirical measure at quantum level
$\hookrightarrow$ Schrödinger equation $\sim$ Newton equations not Liouville!

Back to microscopic Hamiltonian dynamics

- Binary interactions through a potential V depending only on the distance between two interacting bodies
- External forces with some potential $\phi(\text{time}, \text{position})$
- Hamilton equations (Newton laws)

$$\frac{\mathrm{d}x_k}{\mathrm{d}t} = \frac{\partial H^N}{\partial \xi_k} \qquad \frac{\mathrm{d}\xi_k}{\mathrm{d}t} = -\frac{\partial H^N}{\partial x_k} \quad \text{in } \mathbb{R}^d, \quad 1 \leq k \leq N$$

$$H^N(X, \Xi) = \underbrace{\sum_{k=1}^{N} \frac{\xi_k^2}{2}}_{\text{kinetic energy}} + \underbrace{\sum_{k<l} V(x_k - x_l)}_{\text{interaction energy}} + \underbrace{\sum_{k=1}^{N} \phi(t, x_k)}_{\text{potential energy}}$$

- This corresponds to the set of N second-order ODEs in $\mathbb{R}^d$

$$\dot{x}_k = \xi_k, \quad \dot{\xi}_k = -\sum_{k \neq l} \nabla_x V(x_k - x_l) - \nabla_x \phi(x_k), \quad 1 \leq k \leq N$$

The N-body Liouville equation (I)

Statistical solution to the previous ODEs, i.e. evolution of a *distribution* of trajectories:

$$\frac{\partial F^N}{\partial t} + \sum_{k=1}^{N} \left(\frac{\partial H^N}{\partial \xi_k} \cdot \frac{\partial F^N}{\partial x_k} - \frac{\partial H^N}{\partial x_k} \cdot \frac{\partial F^N}{\partial \xi_k} \right) = 0$$

on joint microscopic probability distribution function $F^N(t, X, \Xi)$

Liouville's theorem

For any $t \in \mathbb{R}$ one has $F^N(t, S_t(X, \Xi)) = F^N(0, X, \Xi)$, where S_t is the flow of the Hamilton equations, and S_t preserves volume

Consequence: statistical Casimir invariants (for $\Theta : \mathbb{R} \mapsto \mathbb{R}$)

$$\int_{\mathbb{R}^{2dN}} \Theta\left(F^N(t, X, \Xi)\right) \mathrm{d}X \, \mathrm{d}V = \int_{\mathbb{R}^{2dN}} \Theta\left(F^N(0, X, \Xi)\right) \mathrm{d}X \, \mathrm{d}V$$

including Boltzmann's entropy for $\Theta(r) = r \log r$

The N-body Liouville equation (II)

Proof: Differentiate in time $F^N(t, S_t(X, \Xi)) = F^N(t, X_t, \Xi_t)$:

$$\left(\frac{\partial}{\partial t} F^N\right)(t, X_t, \Xi_t) + \left(\frac{\partial}{\partial t} X_t\right) \cdot \left(\frac{\partial}{\partial X} F^N\right)(t, X_t, \Xi_t) \\ + \left(\frac{\partial}{\partial t} \Xi_t\right) \cdot \left(\frac{\partial}{\partial \Xi} F^N\right)(t, X_t, \Xi_t) = 0$$

which means, using the equations on X_t and Ξ_t:

$$\left(\frac{\partial}{\partial t} F^N\right)(t, X_t, \Xi_t) + \left(\frac{\partial}{\partial \Xi} H\right) \cdot \left(\frac{\partial}{\partial X} F^N\right)(t, X_t, \Xi_t) \\ - \left(\frac{\partial}{\partial X} H\right) \cdot \left(\frac{\partial}{\partial \Xi} F^N\right)(t, X_t, \Xi_t) = 0$$

which is the desired equation at the point (t, X_t, Ξ_t).

The N-body Liouville equation (III)

Then compute time derivative of $J(t, X, \Xi) := \det \nabla_{X,\Xi} S_t(X, \Xi)$:

$$\frac{\mathrm{d}}{\mathrm{d}t} J(t, X, \Xi) = \left[\sum_i \left(\frac{\partial^2 H^N}{\partial x_k \partial \xi_k} - \frac{\partial^2 H^N}{\partial \xi_k \partial x_k} \right) \right] J(t, X, \Xi) = 0$$

Together with $J(0, X, \Xi) = \det \mathrm{Id} = 1$, it yields $J(t, X, \Xi) \equiv 1$
One deduces by change of variable

$$\int_{\mathbb{R}^{2dN}} \Theta \left(F^N(t, X, \Xi) \right) \mathrm{d}X \, \mathrm{d}V = \int_{\mathbb{R}^{2dN}} \Theta \left(F^N(0, X, \Xi) \right) \mathrm{d}X \, \mathrm{d}V$$

$\to$ conservation of Lebesgue norms, Boltzmann entropy...
This reflects the time-reversibility of the Liouville equation:
invariance under the change of variable $(t, X, \Xi) \mapsto (-t, X, -V)$
Cf. reversibility of Newton laws at microscopic level

The BBGKY hierarchy (I)

- N-particle Liouville equation allows for considering superpositions of all trajectories at once, still contains same amount of information as the Newton equations
- Desirable to simplify description of the system by throwing away information: (Hopefully) the system is described by a one-particle distribution (first marginal):

  $$f_1^N(t,x,v) := \int_{\mathbb{R}^{2d(N-1)}} F^N(t,X,\Xi)\,\mathrm{d}x_2\,\mathrm{d}x_3\ldots\mathrm{d}x_N\,\mathrm{d}v_2\ldots\mathrm{d}v_N$$

  (Observe that it <u>still</u> depends on N)
- Why the marginal according to the <u>first</u> variable? No loss of generality since F^N symmetric (invariant under permutations) by indistinguability of the particles

The BBGKY hierarchy (II)

- How can we interpret this equation?
- Binary collisions $\Rightarrow$ evolution of first marginal (f_1^N) depends on second marginal f_2^N: interactions $\Rightarrow$ correlations
- Similarly f_2^N's evolution depends on f_3^N and so on:

$$\begin{aligned}
\frac{\partial f_1^N}{\partial t} &= \mathcal{L}_1(f_1^N) + \mathcal{B}_1(f_2^N) \\
&\dots \\
\frac{\partial f_k^N}{\partial t} &= \mathcal{L}_k(f_k^N) + \mathcal{B}_k(f_{k+1}^N) \\
&\dots \\
\frac{\partial f_N^N}{\partial t} &= \frac{\partial F^N}{\partial t} = \left\{ H^N, F^N \right\}
\end{aligned}$$

- This is the **BBGKY hierarchy** (Bogoliubov, Born, Green, Kirkwood, Yvon) for

$$f_1^N, f_3^N, \dots, f_k^N, \dots, f_N^N = F_N.$$

The Many-particle or "Thermodynamic" Limit

- Goal of thermodynamical limit: perform $N \to \infty$ and recover closed equations on reduced distribution $f_1^N \sim f_1$ as $N \sim \infty$
- Natural to ask whether (low correlations)

$$f_2^N = f_1^N \otimes f_1^N := f_1^N(t,x,v) f_1^N(t,y,w)?$$

- However the probability independence assumption not preserved along time for interacting particle systems
- Boltzmann discovered (and Kac formulated mathematically...) that this could hold in the limit $N \to \infty$

$$f_2^N \sim f_1^N \otimes f_1^N \quad \text{as } N \to +\infty \qquad \text{("near-product structure")}$$

  $\to$ this is the idea of molecular chaos
- Formally chaos $\Rightarrow$ closed equation on f_1 as $N \to \infty$ (Vlasov in mean-field scaling, Boltzmann with $Nr(N)^2 = 1$)

Empirical distribution solutions to the Vlasov equation (I)

Crucial property uncovered by Dobrushin: the empirical distribution following the microscopic trajectories is a weak solution to the nonlinear Vlasov equation

Let $Z_t^N = (X_t, \Xi_t)$ be the solutions to the microscopic equations with initial data Z_0^N, then the corresponding empirical distribution μ_t^N satisfies

$$\frac{\partial \mu_t^N}{\partial t} + v \cdot \nabla_x \mu_t^N - [\nabla_x V *_{x,\xi} \mu_t^n](t,x) \cdot \nabla_v \mu_t^N = 0$$

in the weak sense with

$$[V *_{x,\xi} \mu_t^n](t,x) := \int_{y,\xi} V(x-y) \, \mathrm{d}\mu_t^N(y,\xi)$$

Empirical distribution solutions to the Vlasov equation (II)

Proof: in the sense of distribution for a test function $\varphi \in C_c^\infty(E)$

$$\partial_t \langle \mu_t^N, \varphi \rangle = \partial_t \left(\frac{1}{N} \sum_{k=1}^{N} \varphi(x_k, \xi_k) \right)$$

$$= \frac{1}{N} \sum_{k=1} \left(\frac{\partial x_k}{\partial t} \cdot \nabla_x \varphi(x_k, \xi_k) + \frac{\partial \xi_k}{\partial t} \cdot \nabla_\xi \varphi(x_k, \xi_k) \right)$$

$$= \frac{1}{N} \sum_{k=1}^{N} \left(\xi_k \cdot \nabla_x \varphi(x_k, \xi_k) - \frac{1}{N} \sum_{l=1}^{N} \nabla_x V(x_k - x_l) \cdot \nabla_\xi \varphi(x_k, \xi_k) \right)$$

$$= \left\langle \mu_t^N, \xi \cdot \nabla_x \varphi \right\rangle - \left\langle \mu_t^N, \left[\nabla_x V * \mu_t^N \right] \nabla_\xi \varphi \right\rangle$$

$$= -\left\langle \xi \cdot \nabla_x \mu_t^N, \varphi \right\rangle + \left\langle \left[\nabla_x V * \mu_t^N \right] \cdot \nabla_\xi \mu_t^N, \varphi \right\rangle$$

Proof of classical mean-field limit (I)

Convergence of μ_t^N by compactness and weak-strong uniqueness stability of Vlasov equation
$\Rightarrow$ Frontier between classical and statistical mechanics is in the topology M^1 / L^1 (strong distance is too crude for handling Dirac masses $\|\delta_x - \delta_y\| = 2\,1_{x \neq y} \ldots$)

$$W_p(\mu, \nu) = \left(\inf_{\pi \in \Pi(\mu,\nu)} \int_{E \times E} |Z - Z'|^p \, \mathrm{d}\pi(Z, Z') \right)^{1/p}$$

$\Pi(\mu, \nu)$ set of probability with marginals μ and ν ("coupling")

$$W_p(\mu, \nu) = \left(\inf_{(Z,Z') \sim \pi \in \Pi(\mu,\nu)} \mathbb{E}\left(|Z - Z'|^p \right) \right)^{1/p}$$

Observe that $W_p(\delta_x, \delta_y) = |x - y|_p$ (sensitive to the distance)

Proof of classical mean-field limit (II)

To avoid using empirical measure: new Eulerian proof of Dobrushin's estimate in W_p on the BBGKY hierarchy

Start from an optimal coupling π_{in}^N between f^N and g^N at time zero, and derive at later time π_t^N by the evolution

$$\partial_t \pi^N + \left\{ (H^{MF})_1^{\otimes N} + H_2^N, \pi^N \right\}_{2N} = 0$$

Study $D^N(t) := \frac{1}{N} \int \left(\sum_{k=1}^{N} |x_k - y_k|^p + |\xi_k - \eta_k|^p \right) \mathrm{d}\pi_t^N$

$$\frac{\mathrm{d}}{\mathrm{d}t} D^N(t) = -\frac{p}{N} \sum_{j=1}^{N} \int (\xi_j - \eta_j) \cdot (x_j - y_j) |x_j - y_j|^{p-2} \, \mathrm{d}\pi_t^N$$

$$-\frac{p}{N} \sum_{j=1}^{N} \int (\xi_j - \eta_j) \cdot \left([\nabla V * f](x_j) - \frac{1}{N} \sum_{k=1}^{N} \nabla V(y_j - y_k) \right) |\xi_j - \eta_j|^{p-2} \, \mathrm{d}\pi_t^N$$

Proof of classical mean-field limit (III)

Use Young inequality to reduce RHS to $D^N(t)$ and

$$\frac{1}{N}\sum_{j=1}^{N}\int\left|[\nabla V * f](x_j) - \frac{1}{N}\sum_{k=1}^{N}\nabla V(y_j - y_k)\right| \mathrm{d}\pi_t^N$$

Use Lipschitz constant on V to reduce it to $D^N(t)$ and

$$\frac{1}{N}\sum_{j=1}^{N}\int\left|[\nabla V * f](x_j) - \frac{1}{N}\sum_{k=1}^{N}\nabla V({\color{red}x_j} - {\color{red}x_k})\right| \mathrm{d}\pi_t^N$$

Use quantitative law of large number at each time on π_t^N

$$\mathbf{E}_{\pi_t^N}\left[[\nabla V * f](x_k) - \frac{1}{N}\sum_{l=1}^{N}\nabla V(x_k - x_l)\right] = O\left(N^{-\min(1/2,p/d)}\right)$$

[Fournier-Guillin PTRF to appear]

Proof of classical mean-field limit (IV)

Finally differential inequality of the form:

$$\frac{\mathrm{d}}{\mathrm{d}t} D^N(t) \lesssim D^N(t) + O\left(N^{-\min(1/2, p/d)}\right)$$

$\Rightarrow$ control over time by Gronwall lemma

Then use that

$$W_p\left(\mu_t^N, f_t^{\otimes N}\right)^p \leq D^N(t)$$

(particular coupling) to conclude

The quantum N-body Von Neumann-Liouville equation

Functional setting: $\mathfrak{H} = L^2(\mathbb{R}^d)$, $\mathfrak{H}^N = L^2(\mathbb{R}^{dN})$
$\mathcal{L}(\mathfrak{H})$ bounded linear operators
$\mathcal{D}(\mathfrak{H})$ subset where $A^* = A$ and trace$(A) = 1$

Von-Neumann-Liouville equation

$$i\partial_t \rho_\epsilon^N = \left[-\frac{\epsilon}{2} \sum_{k=1}^{N} \Delta_k + \frac{1}{N\epsilon} \sum_{k,l=1}^{N} V(x_k - x_l) \ , \ \rho_\epsilon^N \right]$$

(commutator bracket) with $\rho_\epsilon^N \in \mathcal{D}(\mathfrak{H})$

In the mean-field limit

$$i\partial_t \rho_\epsilon = \left[-\frac{\epsilon}{2}\Delta + \frac{1}{\epsilon}\mathcal{V}(\rho_\epsilon) \ , \ \rho_\epsilon \right], \quad \mathcal{V}(\rho_\epsilon) := \int_z V(x-z)\rho_\epsilon(t,z,z)$$

Concept of marginal replaced by **partial traces** $\rho_\epsilon^{N,n} \in \mathcal{D}(\mathfrak{H}^n)$:

$$\mathrm{trace}_{\mathfrak{H}^n}\left(A\rho_\epsilon^{N,n} \right) = \mathrm{trace}\left((A \otimes I_{N-n})\rho^N \right) \qquad \text{where} \quad A \in \mathcal{L}(\mathfrak{H}^n)$$

A semi-classical Monge-Kantorovich quasi-distance

Concept of quantum coupling between ρ_1 and ρ_2:
$R \in \mathcal{D}(\mathfrak{H}^2)$ with partial traces respectively ρ_1 and ρ_2

Monge-Kantorovich quantum quasi-distance:

$$MK_2\left(\rho_1, \rho_2\right) = \inf_R \operatorname{trace}_{\mathfrak{H}^2}\left((Q^*Q + P^*P)R\right)^{1/2}$$

with $Q\psi = (x_1 - x_2)\psi(x_1, x_2)$ and $P\psi = -i\epsilon(\nabla_{x_1} - \nabla_{x_2})\psi$

Properties:
(i) $MK_2(\rho_1, \rho_2) \geq 2d\epsilon$
(ii) If $\rho_{1/2}$ Töplitz operators at scale ϵ with symbols $(2\pi\epsilon)^d\mu_{1/2}$ then

$$MK_2(\rho_1, \rho_2) \leq W_2(\mu_1, \mu_2) + 2d\epsilon$$

(iii) Husimi transforms at scale ϵ: $\tilde{W}_\epsilon[\rho_{1/2}]$ then

$$MK_2(\rho_1, \rho_2) \geq W_2(\tilde{W}_\epsilon[\rho_1], \tilde{W}_\epsilon[\rho_2]) - 2d\epsilon$$

The main result

Similar Gronwall estimate on quantum total cost

$$D^N(t) := \mathrm{trace}_{\mathfrak{H}^2}\left((Q^*Q + P^*P)R_t\right)$$

where the coupling R_t evolves according to

$$\partial_t R_t + \left[(H^{MF})_1^{\otimes N} + H_2^N, R_t\right]_{2N} = 0$$

and use the same other ingredients. . .