Global solutions for the cubic non linear wave equation

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Why study PDEs with low regularity initial data?

- Global existence of smooth solutions: local (in time) classical strategy for initial data in a space X such that the norm in X is essentially preserved by the flow. If it is possible to solve the equation between $t = 0$ and $t = T(\|u_0\|_X)$ and if $\|u(T)\|_X \leq \|u_0\|_X$ then it is possible to solve between $t = T$ and $t = 2T$, etc...
- Large time behaviour of solutions with *smooth* initial data (scattering), or large time behaviour of the norms of these solutions (exponential/polynomial increase rate, etc...)
- Informations about the behaviour of the blowing up solutions in some cases.
- ...

Super/sub critical PDEs (in Sobolev spaces)

While solving non linear PDEs, very often a critical threshold of regularities appears, s_c, for which

- If the initial data are smooth enough, $u_0 \in H^s, s > s_c$ then local existence holds (with a time existence depending only on the norm of u_0 in H^s)
- If the initial data are not smooth enough i.e $u_0 \in H^s, s < s_c$ (and not better) then the PDE is unstable, or even ill posed

For example for the Navier-Stokes equation, the critical index is

- $s_c = 0$ in space dimension 2
- $s_c = 1/2$ in space dimension 3

Some unstabilities

- The solution ceases to exist after a finite time: finite time blow up.
- No continuous flow (on any ball in H^s) ill posedness.
- The flow defined by the PDE (if it exists) is not uniformly continuous on the balls of H^s.
- The flow defined by the PDE (if it exists) is not C^k on the balls of H^s.

N.B. This latter type of instabilities say very little about the smooth solutions, but tell essentially that some approaches for solving the PDE will not work.

The d-dimensional wave equation: a model dispersive PDE

Let (M, g) be a d-dimensional riemannian manifold (without boundary) and

$$\mathbf{\Delta} = \sum_{i,j=1,\cdots 3} \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x_i} g^{i,j}(x) \sqrt{\det g} \frac{\partial}{\partial x_j},$$

be the Laplace operator on functions, and

$$(\text{NLW}) \quad \begin{cases} (\frac{\partial^2}{\partial t^2} - \mathbf{\Delta})u + u^3 = 0, \\ (u, \partial_t u)_{t=0} = (u_0(x), u_1(x)) \in H^s(M) \times H^{s-1}(M) \end{cases}$$

the cubic defocusing non linear wave equation.

Critical index: $s_c = (d-2)/2$

Theorem (Strichartz, Ginibre-Velo, Kapitanskii, ...)

Let $s \geq (d-2)/2$. *For any initial data* $(u_0, u_1) \in \mathcal{H}^s(M) = H^s(M) \times H^{s-1}(M)$, *there exists* $T > 0$ *and a unique solution to the system (NLW) in the space*

$$C^0([0,T]; H^s(M)) \cap C^1([0,T]; H^{s-1}(M)) \cap L^4((0,T) \times M)$$

Furthermore, if $s > s_c$, $T = T(\|(u_0, u_1)\|_{\mathcal{H}^s(M)})$.

The super-critical wave equation is *ill posed*

Theorem (Lebeau 01, Christ-Colliander-Tao 04, Burq-Tzvetkov 07)

Assume $s < s_c$.

- *There exists sequences* $(u_{0,n}, u_{1,n}) \in C_0^\infty(M), (t_n) \in \mathbb{R}$ *such that the solution of (NLW) with initial data* (u_0, u_1) *exists on* $[0, 1]$ *but*

$$\lim_{n \to +\infty} \|(u_{0,n}, u_{1,n})\|_{\mathcal{H}^s(M)} = 0,$$

$$\forall \epsilon > 0, \lim_{n \to +\infty} \|u_n\|_{L^\infty((0,\epsilon);H^s(M))} = +\infty$$

- *There exists an initial data* $(u_0, u_1) \in \mathcal{H}^s(M)$ *such that any weak solution of (NLW) associated to this initial data, satisfying the "finite speed of propagation" (or "light cone dependence property") principle ceases instantaneously to belong to* $\mathcal{H}^s(M)$.

Is instability a generic situation?

- Unstable initial data are very particular:

$$(u_{0,n}, u_{1,n}) = n^{\frac{3}{2}-s}(\phi(nx), n^{-1}\psi(nx)), \quad \phi, \psi \in C_0^\infty(\mathbb{R}^3).$$

- Question: are the initial data exhibiting the pathological behaviour described by Christ-Colliander-Tao's and Lebeau's result rares or on the contrary generic?
- Can we still define solutions for a large class of initial data with super-critical regularity?

A first answer is that in some sense the situation is much better behaved than what CCT and Lebeau's theorems might let think: the phenomenon described above appears to be rare (in some sense). We show that for random initial data, the situation is much better behaved.

Random initial data.

Any function $u \in H^s(M)$ writes with $\Delta e_n = -\lambda_n^2 e_n$

$$u = \sum_n \alpha_n e_n(x), \qquad \sum_n (1 + |\lambda_n|^2)^s |\alpha_n|^2 = \|u\|^2_{H^s(M)} < +\infty.$$

Let $\Omega, \mathcal{A}, p$ be a probabilistic space and $(\mathbf{g_n})$ a sequence of *independent* random random variables *with mean equal to* 0 and super exponential decay at infinity (e.g. Gaussian) :

$$\exists C, \delta > 0; \forall \alpha > 0, \sup_n \mathbb{E}(e^{\alpha|\mathbf{g_n}|}) < Ce^{\delta\alpha^2}$$

a random function in $H^s(M)$ takes the form

$$\mathbf{u_0}(x) = \sum_{n \in \mathbb{Z}^3} \mathbf{g_n} \alpha_n e_n(x), \qquad \sum_n (1 + \lambda_n^2)^s |\alpha_n|^2 < +\infty,$$

with possible symetries to keep real functions (in which case independence is assumed modulo the symetries)

Almost sure local well posedness for random initial data in $\mathcal{H}^s = H^s \times H^{s-1}, \forall s \geq 0$

Theorem (Tzvetkov-B. 2008)

Consider $s \geq 0$, $M = \mathbb{T}^3$, *assume*

$$(u_0, u_1) = (\sum_n \alpha_n e_n(x), \sum_n \beta_n e_n(x)) \in \mathcal{H}^s$$

and a random initial data

$$(\mathbf{u_0}, \mathbf{u_1}) = \Big(\sum_{n\in\mathbb{Z}^3} \mathbf{g_n} \alpha_n e_n(x), \sum_{n\in\mathbb{Z}^3} \widetilde{\mathbf{g}}_\mathbf{n} \beta_n e_n(x)\Big)$$

Notice that a.s. $(\mathbf{u_0}, \mathbf{u_1}) \in \mathcal{H}^s(M)$. *Then a.s. there exists* $\mathbf{T} > 0$ *and a unique solution* $\mathbf{u}(t, x)$ *of (NLW) in a space*

$$X_T \subset C([0, T]; H^s(M)) \cap C^1([0, T]; H^{s-1}(M)).$$

From local to global existence

The result above shows that we have a good Cauchy theory at the probabilistic level in $\mathcal{H}^s(M)$, $s \geq 0$. and we can almost surely solve the non linear wave equation on a maximal time interval $(0, \mathbf{T})$.

Natural question: $\mathbf{T} = +\infty$ a.s.? (global existence).

Theorem (Tzvetkov-B.2011)

Assume $M = \mathbb{T}^3$. For any $0 \leq s$, the solution of (NLW) constructed above exists almost surely globally in time and satisfies:

$$\|(\mathbf{u}(t,\cdot), \partial_t \mathbf{u}(t,\cdot))\|^2_{\mathcal{H}^s(M)} \leq \begin{cases} C((\mathbf{K}+t))^{\frac{(1-s)}{s}+0}, & \textit{if } s > 0 \\ e^{C(\mathbf{K}+t^2)}, & \textit{if } s = 0 \end{cases}$$

with $\mathcal{P}(\mathbf{K} > \Lambda) \leq Ce^{-c\Lambda^\delta}$

Rk 1. Almost surely, the initial data $(\mathbf{u}_0, \mathbf{u}_1) \in \mathcal{H}^s(M)$, $s \geq 0$, but as soon as

$$\sum_{n\in\mathbb{Z}^3} (1+\lambda_n^2)^{s'}|\alpha_n|^2 + (1+\lambda_n^2)^{s'-1}|\beta_n|^2 = +\infty$$

and the random variables $g_n, \widetilde{g}_n$ do not accumulate at 0 (say they are i.i.d. non trivial), then almost surely

$$(\mathbf{u}_0, \mathbf{u}_1) \notin \mathcal{H}^{s'}(M)$$

and the result provides many initial data for which the classical Cauchy theory does not apply (even locally in time)

Rk 2. In the deterministic setting, global well posedness below H^1 iniciated by Bourgain using high/low decomposition. Then global well posedness obtained for $s > \frac{3}{4}$ by Kenig-Ponce-Vega (see also Gallagher-Planchon), and then for $s = \frac{3}{4}$ by Bahouri-Chemin, and for $s = \frac{2}{3}$ by Hani

Rk 3. We also can show that the flow is uniformly continuous in the following Hadamard-probabilistic sense: recall

$$\mathcal{P}(A|B) = \mathcal{P}(A \cap B)/\mathcal{P}(B)$$

Theorem (N.B, N. Tzvetkov, 2011)

Denote by $\mathbf{U} = (\mathbf{u_0}, \mathbf{u_1})$, *and* $\Phi(t)\mathbf{U}$ *the solution of the NLW with initial data* $\mathbf{U}$, *which exists a.s., then* $\forall T, A, \epsilon > 0$,

$$\lim_{\eta \to 0} \mathcal{P}\{(\mathbf{U}, \mathbf{V}); \|\Phi(t)\mathbf{U} - \Phi(t)\mathbf{V}\|_{L^\infty(0,T);\mathcal{H}^s} > \epsilon \Big| \|\mathbf{U} - \mathbf{V}\|_{\mathcal{H}^s} \leq \eta \text{ and } \|\mathbf{U} + \mathbf{V}\|_{\mathcal{H}^s} \leq A\} = 0 \tag{0.1}$$

In other words, among the couples of initial data (U, V), which are A-bounded, and η-close, most of them (the residual probability is arbitrarily small if $\eta > 0$ is small) generate solutions to NLW which remain ϵ-close to each other.

Rk 4. In the continuity property above, one cannot eliminate the probabilistic side: the property is known to be false otherwise. Actually, it is possible to show that there exists $\epsilon, A > 0$ such that for any $\eta > 0$ the probability above is non zero (and consequently the set is non empty!)
Rk 5 This result is linked to results by Colliander-Oh where global existence for one dimensional NLS.

Higher dimensions, other manifolds

Theorem (Thomann, Tzvetkov-B.2012, Lebeau-B 2012)

Here we need additional assumptions on the coefficients of the functions (u_0, u_1) *used to build our measures, to avoid* lacunary series *phenomena*

- *Assume* $dim(M) = 3$. *Then the previous results hold*
- *Assume* $d \geq 4$. *For any* $0 \leq s$, *there exists almost surely a global weak solution of (NLW) which is obtain as a weak limit of the solutions to the truncated systems*

$$(\partial_t^2 - \Delta)u_k + P_k((P_k u_k)^3) = 0$$

(P_k is a (smooth projector on the k first modes of the Laplace operator)

Rk 6 This result of existence of weak solutions is very much linked to previous results by Albeveiro-Cruzeiro, Kuksin-Shirikyan, Da Prato-Debussche and the analog result by Nahmod-Pavlovic-Staffilani on Navier Stokes.
Rk 7 We actually have similar weak-type results for other model equations as

- NLS on S^3
- Benjamin-Ono equation
- The derivative NLS

Deterministic theory: local Cauchy theory in H^1. The case of the dimension 3.

Theorem

Assume that

$$\|u_0\|_{H^1} + \|u_1\|_{L^2} \leq \Lambda.$$

There exists a unique solution of (NLW)

$$u \in L^\infty([0, C^{-1}\Lambda^{-3}], H^1 \times L^2(M))$$

Moreover the solution satisfies

$$\|(u, \partial_t u)\|_{L^\infty([0,C\Lambda^{-3}],H^1 \times L^2)} \leq C\Lambda$$

and $(u, \partial_t u)$ is unique in the class

$$L^\infty([0, C\Lambda^{-3}], H^1 \times L^2)$$

Proof: Fixed point in the ball centered on $S(t)(u_0, u_1)$ in

$$L^\infty((0, T); H^1_x)$$

Use that u satisfies $(\partial_t^2 - \Delta)u = -u^3$, and hence Duhamel formula gives

$$u = S(t)(u_0, u_1) - \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} u^3(s)ds$$
$$= S(t)(u_0, u_1) + K(t)(u)$$

where $K(t)$ satisfies (using Sobolev embeddings $H^1_x \hookrightarrow L^6_x$)

$$\|K(t)(u)\|_{L^\infty(0,T);H^1(M)} \leq \|u^3\|_{L^1(0,T);L^2(M)}$$
$$\leq T\|u\|^3_{L^\infty(0,T);L^6(M)} \leq CT\|u\|^3_{L^\infty(0,T);H^1(M)}$$

Deterministic theory in H^1: a remark

Theorem

Assume that

$$\|u_0\|_{H^1} + \|u_1\|_{L^2} + \|f\|_{L^\infty(\mathbb{R};L^6(M))} \leq \Lambda.$$

There exists a unique solution in $L^\infty([0, C\Lambda^{-3}], H^1 \times L^2)$ *of*

$$(\partial_t^2 - \Delta)u + (f + u)^3 = 0, (u, \partial_t u) \mid_{t=0} = (u_0, u_1)$$

Moreover the solution satisfies

$$\|(u, \partial_t u)\|_{L^\infty([0,\tau],H^1 \times L^2)} \leq C\Lambda.$$

(same proof as before)

A result by Paley and Zygmund (1930)

Consider a sequence $(\alpha_k)_{k\in\mathbb{N}} \in \ell^2$

$$\sum_k |\alpha_k|^2 < +\infty.$$

Let u be the trigonometric series on $\mathbb{T}$

$$u = \sum_k \alpha_k e^{ik\theta}$$

This series is convergent in $L^2(\mathbb{T})$ but "in general" (generically for the ℓ^2 topology), the function u is in

no $L^p(\mathbb{T}), p > 2$ space.

If one changes the signs in front of the coefficients α_k randomly and independently, i.e. if one considers

$$\sum_k \mathbf{g_k} \alpha_k e^{ik\theta} = \mathbf{u}(\theta)$$

where $\mathbf{g_k}$ are Bernouilli *independent* random variables,

$$P(\mathbf{g_k} = \pm 1) = \frac{1}{2}$$

Theorem (Paley-Zygmund 1930-32, also Rademacher, Kolmogorov 30')

For any $p < +\infty$, almost surely, the series $\mathbf{u} = \sum_k \mathbf{g_k} \alpha_k e^{ik\theta}$ is convergent in $L^p(\mathbb{T})$.
Furthermore, large deviation estimate:

$$\mathcal{P}(\{\|\mathbf{u}\|_{L^p(\mathbb{T})} > \lambda\}) \le Ce^{-c\lambda^2}$$

Local existence, $M = \mathbb{T}^3$

We look for the solution $\mathbf{u}$ under the form

$$\mathbf{u} = S(t)(\mathbf{u_0}, \mathbf{u_1}) + \mathbf{v} = \mathbf{u}_f + \mathbf{v}$$

$\mathbf{v}$ is solution of an equation of the form

$$(\partial_t^2 - \Delta)\mathbf{v} + (S(t)(\mathbf{u_0}, \mathbf{u_1}) + \mathbf{v})^3 = 0, \qquad (\mathbf{v}, \partial_t \mathbf{v}) \mid_{t=0} = (0, 0)$$

which is essentially a cubic non linear wave equation with a source term $(S(t)(\mathbf{u_0}, \mathbf{u_1}))^3$. According to Paley-Zygmund, a.s. this source term is admissible, and according to the deterministic H^1 theory, there exists a time $\mathbf{T} > 0$ such that this equation is well posed in H^1: notice that

$$L^{p/3}((0, \mathbf{T}); L^2(M)) \subset L^1_t; L^2_x.$$

In some sense, this result shows that the seemingly super-critical problem is in fact sub-critical

Global existence $M = \mathbb{T}^3$

Fix $T > 0$. Want to prove almost surely existence up to time T of a solution. Fix $N \gg 1$. Seek $\mathbf{u}$ as

$$\mathbf{u} = \mathbf{w} + \mathbf{v} = S(t)(\mathbf{u}_0, \mathbf{u}_1) + \mathbf{v}$$

$$(\partial_t^2 - \Delta)\mathbf{v} + (S(t)(\mathbf{u}_0, \mathbf{u}_1) + \mathbf{v})^3 = 0,$$
$$(\mathbf{v}, \partial_t \mathbf{v}) \mid_{t=0} = (0, 0)$$

AIM: Prove that $\mathbf{v}$ exists on $[0, T]$ with probability 1.
H^1-norm controls local Cauchy theory, hence enough to prove that H^1 norm of $\mathbf{v}$ remains bounded on $[0, T]$

A priori bound

$$E(v) = \int_M \frac{1}{2}|\partial_t v|^2 + \frac{1}{2}|\nabla_x v|^2 + \frac{1}{4}|v|^4 dx$$

$$\begin{aligned}
&\frac{d}{dt}E(t) \\
&= \int_M \partial_t u(v^3 - (w^\omega + \mathbf{v})^3)dx = \int_M \partial_t u(-3v^2 w - 3w^2 v - w^3)dx \\
&\leq 3\|\partial_t v\|_{L^2_x}\|v\|^2_{L^4_x}\|w\|_{L^\infty_x} + 3\|\partial_t v\|_{L^2_x}\|v\|_{L^4_x}\|w\|^2_{L^\infty_x} + \|\partial_t v\|_{L^2_x}\|w\|^3_{L^\infty_x} \\
&\leq C(E(t)\|w\|_{L^\infty_x} + E(t)^{3/4}\|w\|^2_{L^\infty_x} + E(t)^{1/2}\|w\|^3_{L^\infty_x}) \\
&\leq C(f(t)E(t) + g(t))
\end{aligned}$$

Paley-Zugmund gives

$$\mathcal{P}(\|w\|_{L^p((0,T);L^\infty(M))} > \lambda) \leq \mathcal{P}(\|w\|_{W^{\varepsilon,p}_{t,x}} > \lambda) \leq Ce^{-c\lambda^2}$$

$$\Rightarrow f(t), g(t) \in L^1_{loc}(\mathbf{R}_t)$$

Conclude using Gronwall

General manifolds

In the case of tori, Paley-Zygmund uses in an essential way the trivial estimate for eigenfunctions of the Laplace operator

$$\|e^{in\cdot x}\|_{L^\infty(\mathbb{T}^d} \leq 1$$

In the case of a general manifold, this is no more true. Have to find a substitute, the *precised Weyl's formula*

Theorem (Hörmander, 1968)

$$\sum_{\lambda\leq\lambda_n<\lambda+M} |e_n|(x)^2 \sim \lambda^{d-1}, \qquad \sharp\{n; \lambda \leq \lambda_n < \lambda + M\} \sim \lambda^{d-1}.$$

This results implies $\|e_n\|_{L^\infty} \leq C\lambda_n^{\frac{d-1}{2}}$ and if x is fixed, there is essentially only one eigenfunction which can be this large at x As a consequence, in a "mean-value" meaning, the eigenfunctions of the Laplace operator behave as if they were bounded. Exploit this phenomenon in Paley-Zygmund.

Further developments

- Extends to other non-linearities (but requires the use of Strichartz estimates for the proof)
- Allow correlations in the random variables (using some slack in the arguments)
- Relax the mean equal to 0 assumption on the random variables i.e. perform the randomization around a given solution (e.g. smooth, or given by the preceding procedure) instead of the trivial (vanishing) solution
- Extend to other dispersive equations such as non linear Schrödinger equations with or without harmonic potential (with L. Thomann), see also the work by Yu Deng