Interfaces in a Random Environment

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Overview

Random vs. periodic

- Introduction
  - Perturbed gradient flow
  - Forced mean curvature flow
- Results for minimizers of random energy
- The random obstacle model
- Results for the random obstacle model
  - Pinning (stationary solutions block interface motion)
  - Motion
  - Bounds on velocity
- Work in progress: Multiscale (sub)solutions
- Open Problems

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Key features of Models:

- Evolution decreases free energy
  - Free energy is surface energy, i.e. area of interface
- Heterogeneities influence free energy **locally** (on small scale)

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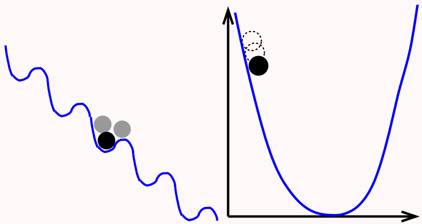

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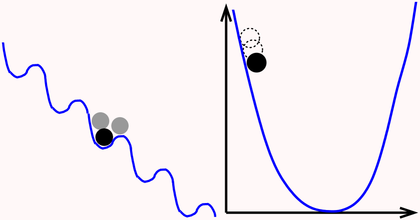

Zoom in on scale of heterogeneities

Perturbed Area Functional/Forced MCF

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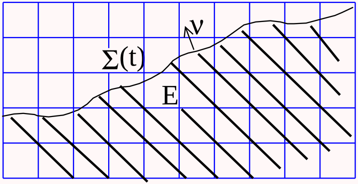
Liapunov functional (formal):

$$\mathrm{Area}(\Sigma) + \int_{\mathbb{R}^{n+1}\cap E} f(X)dX \quad \text{where } \Sigma = \partial E.$$



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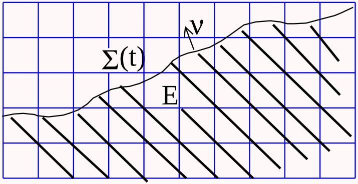

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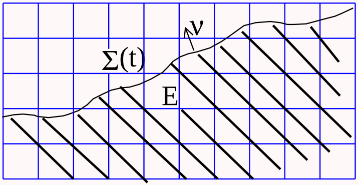

Gradient flow:

$$V_X = \kappa_X + f(X), \; X \in \Sigma(t) \subset \mathbb{R}^{n+1}$$

κ_X mean curvature of Σ at point X, V_X normal velocity at point X.

Minimisers of random of energy (with E. Orlandi)

$$\mathrm{Area}(\Sigma \cap \Lambda) + \int_{\Lambda \cap E} f(X)dX \quad \text{where } \Sigma = \partial E.$$

$$F_\epsilon(u) = \int_\Lambda \left(\frac{\epsilon}{2} |\nabla u(x)|^2 + \frac{1}{\epsilon} W(u(x)) + \frac{\alpha_\epsilon}{\epsilon} h\left(\frac{x}{\epsilon}, \omega\right) u(x) \right) dx$$

h bounded random field, short correlation length
W double-well potential, two minimizers ± 1.

- Idea: u^ϵ minimiser $\Rightarrow u^\epsilon \to \pm 1$ on $\mathbb{R}^{n+1} \setminus \Sigma$ as $\epsilon \to 0$, F_ϵ converges to (possibly anisotropic) area functional.

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- Replace gradient term by nonlocal term (Dirichlet form of fractional Laplacian) (D.-Orlandi in progress)

Forced MCF

$$V_X = \kappa_X + f(X, \omega), \; X \in \Sigma(t) \subset \mathbb{R}^{n+1}$$

κ_X mean curvature of Σ at point X, V_X normal velocity at point X.

Related Homogenization Problem

Interface as level set: $\Sigma(t) = \{x \in \mathbb{R}^{n+1} : \; w(x,t) = 0\}$

$$V = \epsilon\kappa + f\left(\frac{x}{\epsilon}\right) \Rightarrow \; w_t = \epsilon \,\mathrm{tr}\left[\left(I - \frac{1}{|\nabla w|^2}\nabla w \otimes \nabla w\right) D^2 w\right] + f\left(\frac{x}{\epsilon}\right)|\nabla u|$$

$$V = c(\nu) \; \Rightarrow \; \bar{w}_t = c\left(\frac{\nabla \bar{w}}{|\nabla \bar{w}|}\right)|\nabla \bar{w}|$$

“Singular” Homogenization: Averaging *and* singular limit.

- Degenerate, nonlinear
- $f(x)$ may change sign

Forcing $f(x)$ strictly positive (+additional conditions), not random:
P.-L. Lions, P.E. Souganidis, (2005),
Additional conditions: Caffarelli, Monneau
Connection: Level sets evolve by (forced) MCF (Chen-Giga-Goto)
Random case mostly open! Look for simplified model:
Random Obstacle Model.

Related work

- Physics: QEW ($\partial_t u = \Delta u + f(x, u, \omega)$)
  - S. Brazovsii, Th. Nattermann (et al.) (FRG)
- Material Science
  - K. Bhattacharya
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- Material Science
  - K. Bhattacharya
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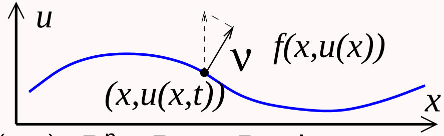
Forced Mean Curvature Flow/Semilinear "Approx."

Forced MCF (Gradient flow of perturbed surface energy):

$V_{x,u} = \kappa_{x,u} + f(x,u) + F$



If surface is graph $(x, u(x,t))$ then $u(x,t) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ solves

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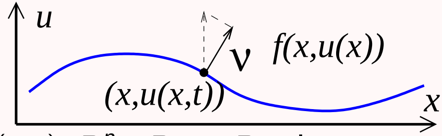

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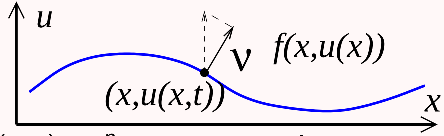

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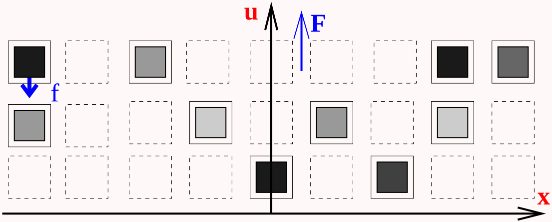

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Quenched Edwards-Wilkinson Model (QEW)

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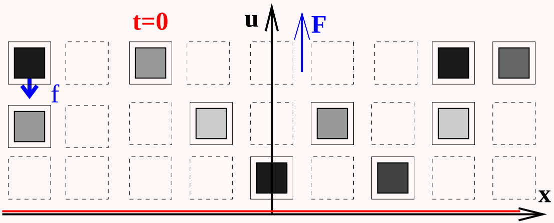

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Important: Comparison Principle.

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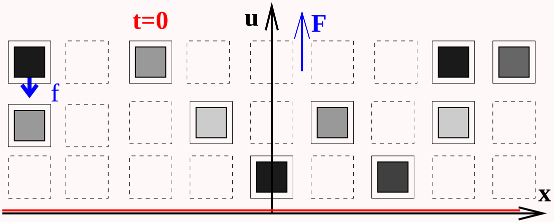

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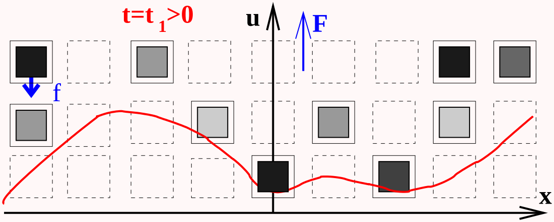

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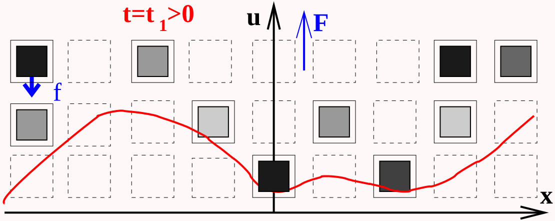

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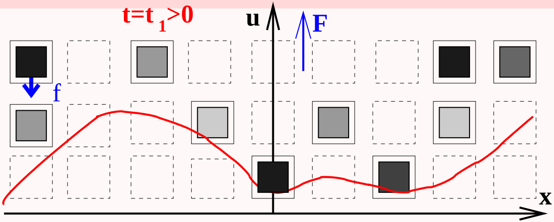

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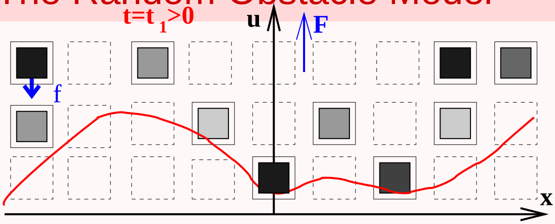

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- Periodic: Such F_* exists, velocity $\sim \sqrt{F - F_*}$
- $F = F^*$? Periodic: Stationary solution due to compactness (D.-Yip)

Random vs. Periodic: Loss of compactness and behaviour at critical forcing in zero dim.

What happens at $F = F*$?

• Periodic environment (compactness): Stationary solution exists as u.c. limit of stationary solutions for $F < F_*$.

• Random environment: Zero Velocity **AND** non-existence of stationary solution possible

$$\dot{X} = F + \sin(2\pi x)$$

$F_* = 1$

χ cut-off, $\chi = 1$ near $x = 0$, $\chi = 0$ on $\mathbb{R} \setminus [-1/8, 1/8]$.

Z_i i.i.d., $Z_i > 0$ a.s., $\mathbb{E}Z_0 = \infty$. (square-root behavior)

Time to cross obstacle at i: $\sim Z_i$

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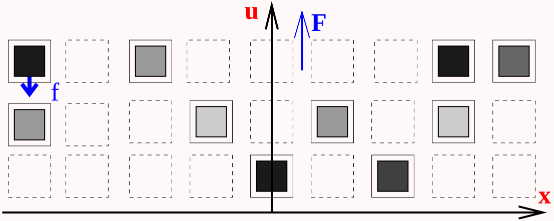
Random Obstacle Model: Precise Setting



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$F \geq 0$, (driving force), ϕ mollifier of $1_{[-\delta,\delta]^{n+1}}(x,u)$,

$$f(x,u) = \sum\nolimits_{(i,j)\in\mathbb{Z}^n\times(\mathbb{Z}+\frac{1}{2})} \left(\mathbb{E}(\ell_{ij}) - \ell_{i,j}(\omega)\right) \phi(x-i,u-j)$$

$(\ell_{i,j}(\omega))_{(i,j)\in\mathbb{Z}^n\times(\mathbb{Z}+\frac{1}{2})}$ are a family of i.i.d. exponential random variables.

Nonnegative Solutions for R. O. M.

$$0 = \Delta u(x,\omega) + f(x, u(x,\omega),\omega) + F \quad \text{on } \mathbb{R}^n, \qquad u(x) \geq 0 \quad (*)$$

Theorem (N.D., J. Coville, S. Luckhaus)

Let $n = 1$ and u solve $(*)$ Then there exist $F_0 > 0$ such that for $F > F_0$ there is almost surely no solution of $(*)$.

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Let $n = 1, 2$. There ex. $0 < F_1$ such that for $0 < F < F_1$, $(*)$ has almost surely a solution with $\mathbb{E}[u(x,\omega)] = c < \infty$ for all $x \in \mathbb{R}^n$.

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Barrier for/limit of

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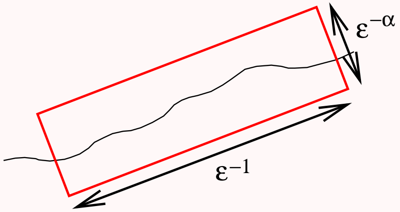
Positive speed and oscillation

• Curve **oscillates sublinearly** in moving frame (kinetic scaling $t = \epsilon^{-1}T,\ r = \epsilon^{-1}x$.)



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Idea: Fastest plane below and slowest plane above graph (in ϵ^{-1}-box) have same average speed, which is deterministic (Obstacles i.i.d.)

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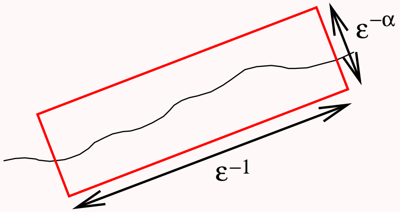

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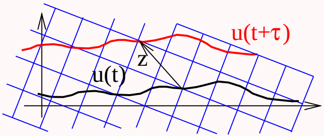
Mean Curvature Flow with periodic forcing



• Effective velocity: Bounds in frame moving with velocity $c(\nu)$, continuous in ν

• Special solutions (Pulsating wave)

N. D., G. Karali, N.K. Yip, Pulsating Wave for Mean Curvature Flow in Inhomogeneous Medium, EJAM 19 (2008) , 661-699.

Assumption: Forcing small in C^1

• Uniform motion

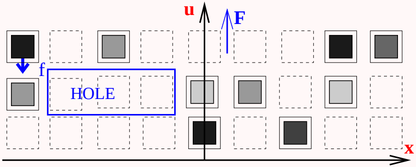
• Flat on $O(1)$ scale (L^∞ bound in moving frame)

Subsolution via Allen-Cahn kinks/bubbles

Random nonlinearity bounded but changes sign

$$\partial_t u(x,t,\omega) = \Delta u - \underbrace{W'(u) + \hat{F}}_{\leq f(x,u,\omega)+F}$$

$f(u) = -\|f(x,u,\omega)\|_\infty$ on height of obstacles except on "holes", smooth by suitable interpolation :



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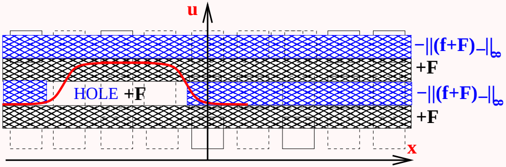

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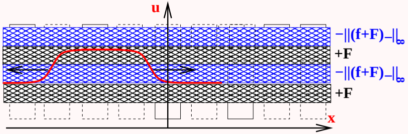

Radially symmetric "bubble-like" initial data: $u(x) > 1/2$ for $|x| < R$, $u(x) < -1/2$ for $|x| > R + K$, then for some K (depending on $W(\cdot)$) and R, F large, the bubble grows linearly in time. ("$F > 1/R = \kappa$")

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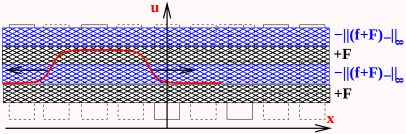

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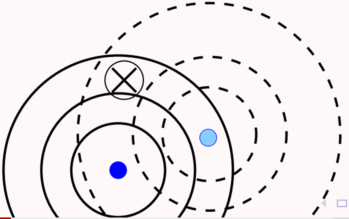
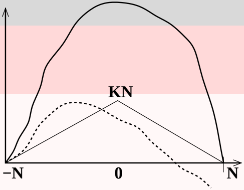
Speed: $\sim$ expectation of distance of z-axis to holes (Obstacles iid!)



Sketch of techniques



Non-Existence

- Consider Dirichlet-problem on $[-N, N]$, discretize
- "bad" paths: $\frac{1}{N}\sum_i \Delta u(i)$ large $\Rightarrow$ sum of iid r.v. large
- Bound on probability of this event (large deviations for sum of iid r.v.)
- Borel-Cantelli Lemma

Existence

Essential probabilistic step: Lipschitz percolation, i.e. a discrete 1-Lipschitz graph $w : \mathbb{Z}^n \to \mathbb{N}$ ex. with $(z, w(z))$ good for all $z \in \mathbb{Z}^n$ if $\mathbb{P}(bad)$ small, (D., Dondl, Grimmett, Holroyd, Scheutzow, Electr. J. of Prob.,)

Analyst's approach: Fixed point iteration

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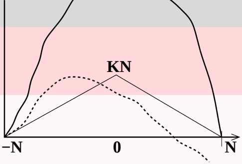

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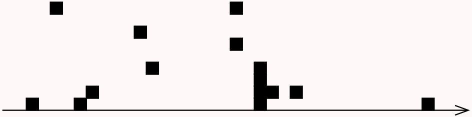

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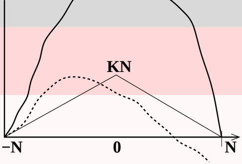

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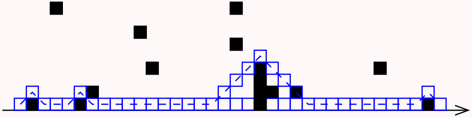

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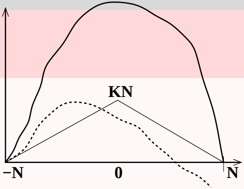

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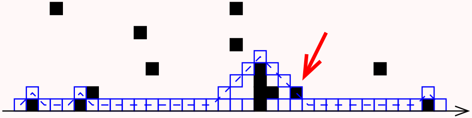

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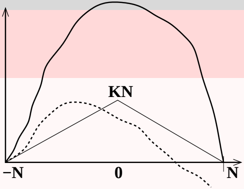

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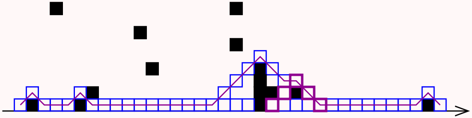

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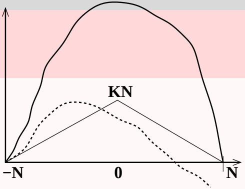

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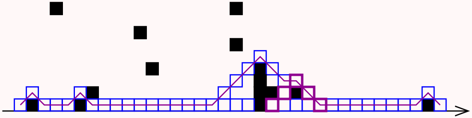

- Approximate Lipschitz graph by function s.t. $\Delta u + f(x, u, \omega) + F \leq 0$.

("convex" corners at obstacles)

Renormalization (multiscale)

Motivation I Elliptic Homogenization: Corrector
$-\mathrm{div}(a(x)(\nabla u_0(x)+\nabla\phi(x))=0$ on one scale. Here oscillations on all scales, so recursive construction of "correctors."

Motivation II: Rigorous renormalization group for random field Ising model (3 d phase transition, 2 d not) by Bricmont-Kupiainen and Bovier-Külske

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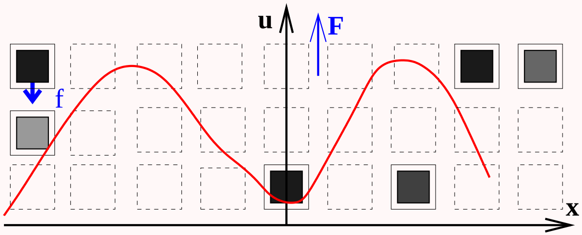
Then very fast motion of minimum (slip)
Of interest: *Bound on total time to cross obstacle*



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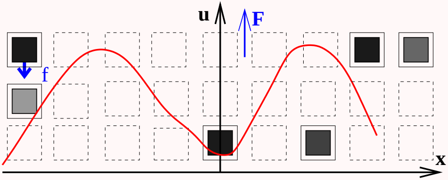

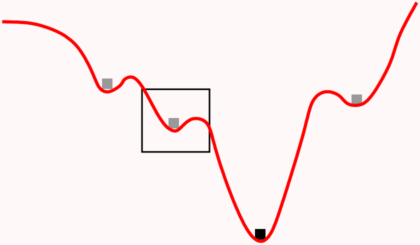

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- Information on oscillations
- Homogenization
- Dependence of average speed on F?
- Effective variational models? $d(u_{k+1}, u_k) + E_{\mathrm{eff}}(F, u_{k+1}) \to \min$

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Thank you for your attention!