NK
OXPDE

Marshall Slemrod

Convex integration, Nash-Kuiper

see

math.univ-lyon1.fr/homes-www.borrelli/Recherche.html

Example: Let $f_0 : [0,1] \to \mathbb{R}^3$, $t \to (0,0,t)$

Problem. Find $f : [0,1] \xrightarrow{C^1} \mathbb{R}^3$ so that

(i) $\forall t \in [0,1] \quad \left| \dfrac{f'(t) \cdot e_3}{(\|f'(t)\|^2 + 1)^{1/2}} \right| < \varepsilon$

$\left(f = (f_1, f_2, f_3) \quad \dfrac{|f_3'(t)|}{(|f'(t)|^2 + 1)^{1/2}} < \varepsilon \right)$

(Direction cosine $<\ \varepsilon$)

(ii) $\| f - f_0 \|_{C^0} \leq \delta \implies$

$(\| f_1(t) - 0 \| \leq \delta, \quad \| f_3(t) - t \| \leq \delta$

$\| f_2(t) - 0 \| \leq \delta$

(2) $\Rightarrow$ Slope of f_3 is small, so how can

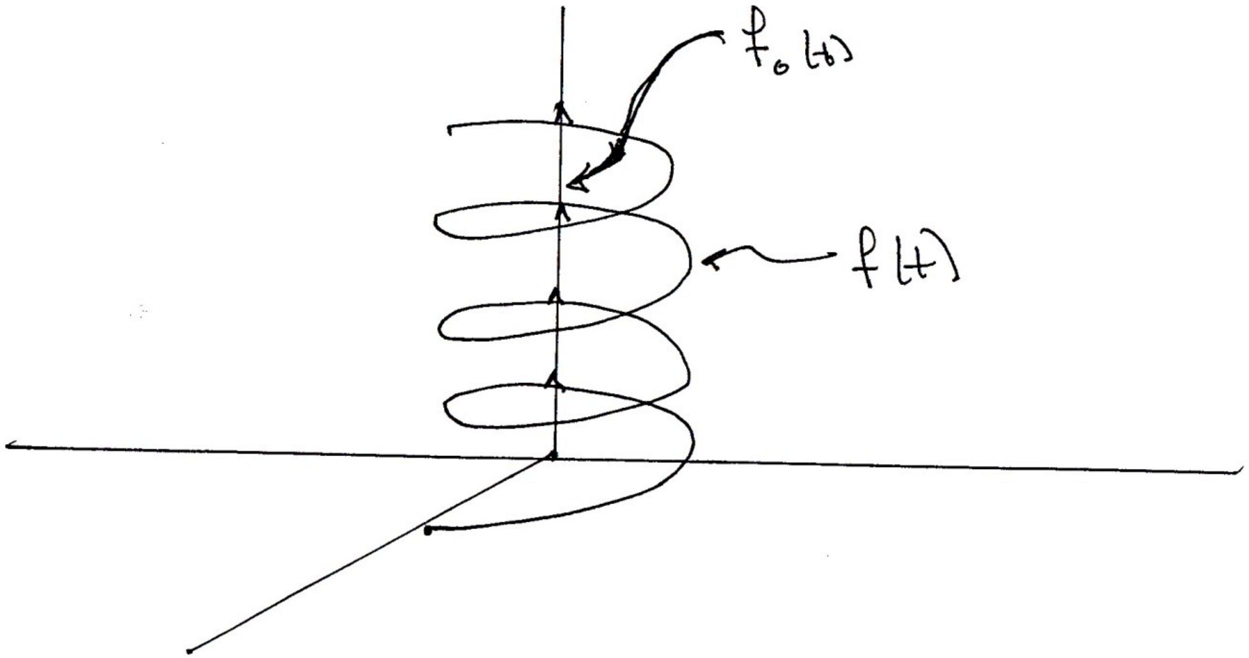
f_3 stay close to t (which is ii) ?

Try

$f_1(t) = \delta \cos(2\pi N t)$, $f_2(t) = \delta \sin(2\pi N t)$

$f_3(t) = t$

N integer



Introduce oscillations!

<u>Check:</u>

$f_1'(t) = -2\pi\delta N \sin(2\pi N t)$, $f_2'(t) = 2\pi\delta N \cos(2\pi N t)$

$f_3'(t) = 1$

$|f'(t)|^2 = 1 + (2\pi\delta N)^2$

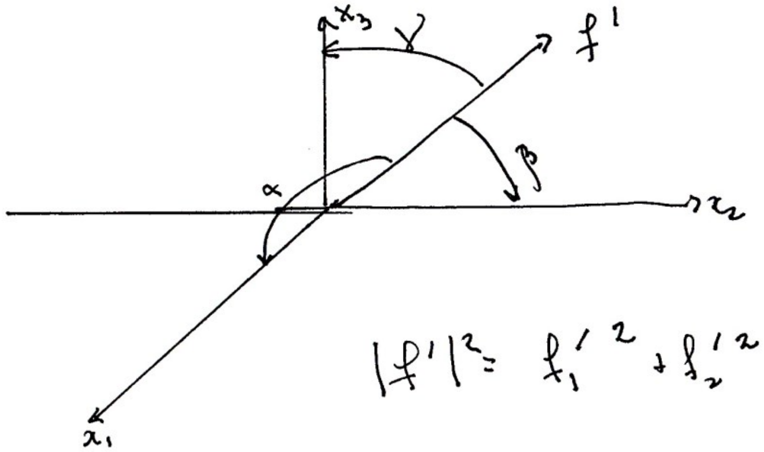
(i) $\dfrac{1}{(1+(2\pi\delta N)^2)^{1/2}} < \varepsilon$ Choose N sufficiently large!

(ii) $|\delta \cos(2\pi N t)| \le \delta$

$|\delta \sin(2\pi N t)| \le \delta$

$|f_3'(t) - t'| = |1-1| = 0 < \delta$

Introduce direction cosines and rephrase problem



$$|f'|^2 = f_1'^2 + f_2'^2 + f_3'^2 \doteq r^2$$

$$f_1' = r\cos\alpha$$
$$f_2' = r\cos\beta$$
$$f_3' = r\cos\gamma$$

(2) becomes $|\cos\gamma| < \varepsilon$

i.e. γ near $\pi/2$

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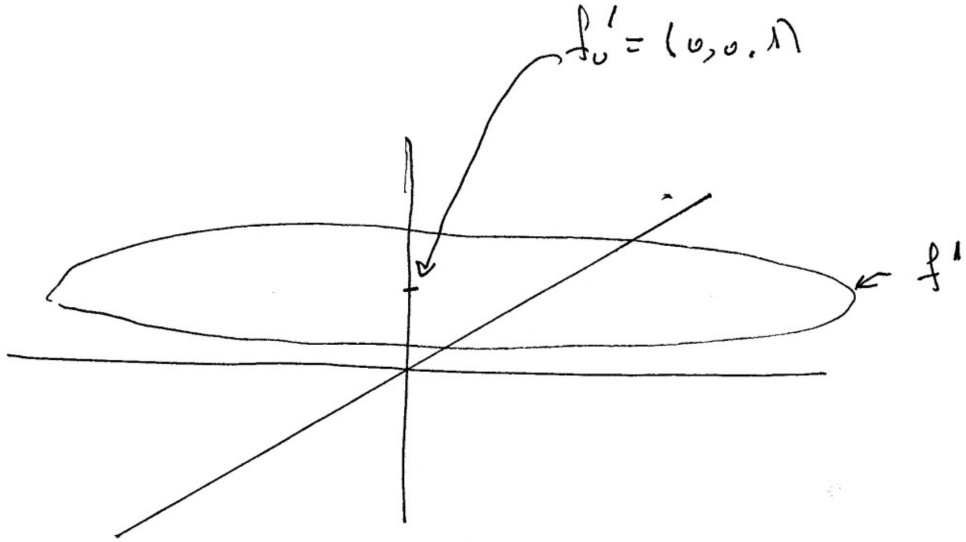

f' is a circle with f_0' at its center

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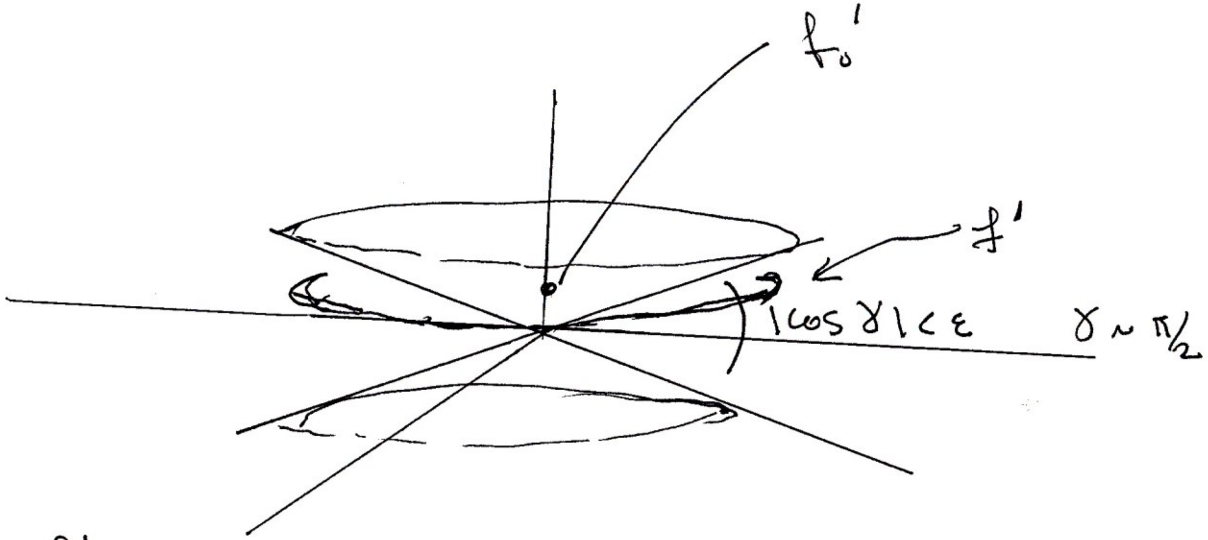

f' lies in the region R $\quad |\cos\gamma| < \varepsilon$

"cone" (or exterior cone)

f_0' does not lie in R but lies in convex hull of R

This accounts for (i)

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To account for (ii)

$$g'(t) = f'(t) - f_0'(t) \qquad \frac{k}{N} \le t \le \frac{k}{N} + \frac{1}{N}$$

$$f(t) - f_0(t) = f(0) - f_0(0) + \boxed{N k \underbrace{\int_0^{\frac{1}{N}} g'(u)\, du}_{=0}}$$

$$+ \int_{\frac{k}{N}}^{t} g'(u)\, du$$

$$f(t) - f_0(t) = \underbrace{f(0) - f_0(0)}_{\text{small}} + \underbrace{\int_{\frac{k}{N}}^{t} g'(u)\, du}_{\text{small for } N \text{ large}}$$

where we used

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$$f_0'(t) = N \int_{k/N}^{k/N + \frac{1}{N}} f'(u)\,du$$

$$I_k = \left[\frac{k}{N},\ \frac{k}{N} + \frac{1}{N}\right] \qquad \text{length of } I_k = \frac{1}{N}$$

i.e. $\boxed{f_0'(t) = \text{Average of } f'(t) \text{ over one loop.}}$

(in convex hull of R) (in R)

Hence the above identity is true for our example

It is the key for convex integration

Fundamental lemma of convex integration.

Let $\mathcal{R} \subset \mathbb{R}^n$ be an arc-wise connected subset, $f_0 \in C^\infty(I, \mathbb{R}^n)$ be a map such that $f_0'(I) \subset \operatorname{Int} \operatorname{Conv}(\mathcal{R})$. Then there exists a C^∞ map $h : I \times E/Z \to \mathcal{R}$ such that for all $t \in I$

$$f_0'(t) = \int_0^1 h(t,u)\, du .$$

in convex hull   in $\mathcal{R}$

Set $F(t) = f_0(0) + \int_0^t h(s, Ns)\, ds$, $N \in \mathbb{N}^*$ (pos integers)

We say $F \in C^\infty(I, \mathbb{R}^n)$ is obtained from f_0 by a convex integration process

$$\boxed{F(t) = f_0(0) + \int_0^t h(s, Ns)\,ds}$$

Note F depends on N.

(2) satisfied
$F'(t) = h(t, Nt) \in \mathbb{R}$

Proposition

$$\|F - f_0\|_{C^0} \leq \frac{1}{N}\left(2\|h\|_{C^0} + \left\|\frac{\partial h}{\partial t}\right\|_{C^0}\right)$$

Large N allows to make F close to f_0 in the space C^0.

(ai) satisfied
$\|F - f_0\|_{C^0} \leq \delta$

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Summary

Convex integration $F(t) = f_0(0) + \int_0^t h(u, N(u))\, du$

resolves the problem

$\mathcal{R} \subset \mathbb{R}^3$ path-wise connected subset

$f_0'(t) \in \mathrm{Int\ Conv}(\mathcal{R})$

Find $f : [0,1] \xrightarrow{C^1} \mathbb{R}^3$ so that

(i) $\forall t \in [0,1]$, $f'(t) \in \mathcal{R}$

(ii) $\| f - f_0 \|_{C^0} \leq \delta$

with $\delta > 0$ given.

<u>One small problem</u> : If $f_0(t)$ defines a closed curve $f_0(0) = f_0(1)$,

$$F(t) = f_0(0) + \int_0^t h(s, N(s))\,ds$$

does <u>not</u> necessarily define a closed curve $F(0) = F(1)$.

Problem rectified with

$$\boxed{\tilde{F}(t) = F(t) - t(F(1) - F(0))}$$

$\tilde{F}(0) = F(0) = f_0(0)$

$\tilde{F}(1) = F(0) = f_0(0)$

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$$\tilde{F}(t) = F(t) - t(F(1) - F(0))$$

Proposition :

$$\| \tilde{F} - f_0 \|_{C^0} \leq \frac{2}{N} \left(2 \|h\|_{C^0} + \left\| \frac{\partial h}{\partial t} \right\|_{C^0} \right)$$

and

$$\tilde{F}'(E/\mathbb{Z}) \subset \mathbb{R}$$

i.e. The new $\tilde{F}$ satisfies (i), (ii) and $\tilde{F}(1) = \tilde{F}(0)$.

Question: Can we use the Fundamental Lemma of Convex Integration to solve our original Example?

Answer: Yes and No

Yes, it gives a solution

No, it may not give the solution

$$f_1(t) = \delta \cos(2\pi N t), \quad f_2(t) = \delta \sin(2\pi N t),$$
$$f_3(t) = t;$$

Nash-Kuiper Theorem for Curves

One dimensional isometric embedding problem

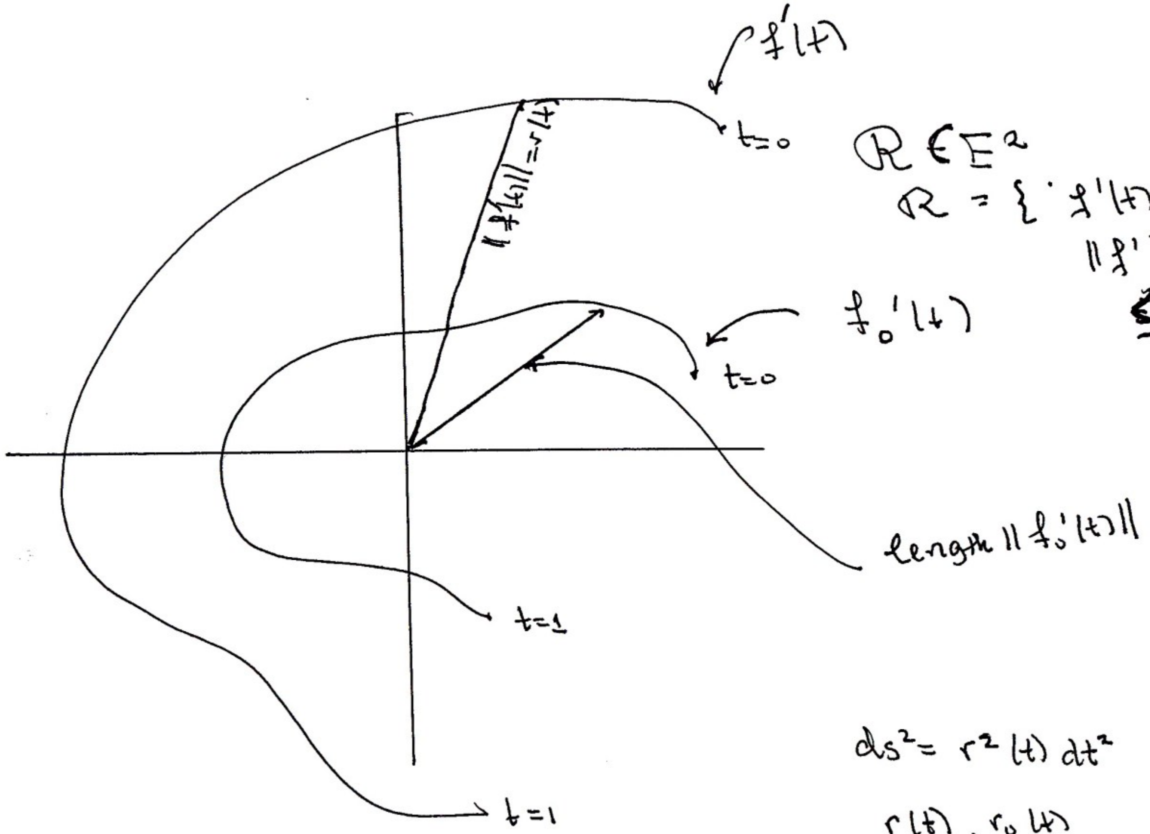
$f_0 : [0,1] \xrightarrow{C^\infty} E^2$ be a given regular curve

$r : [0,1] \xrightarrow{C^\infty} E$ so that $r(t) > \|f_0'(t)\|$.

Build $f : [0,1] \xrightarrow{C^\infty} E^2$ so that

i) $\forall t \in [0,1]$, $\|f'(t)\| = r(t)$

ii) $\|f - f_0\|_{C^0} < \delta$ for given δ.



$\mathcal{R} \in E^2$

$\mathcal{R} = \{ f'(t) ; \|f'(t)\| \leq r(t) \}$

$ds^2 = r^2(t)\, dt^2$

$r(t)$, $r_0(t)$ are metrics: $f(t)$, $f_0(t)$

This picture satisfies (i) but not necessarily (ii).

Fundamental lemma of convex integration:

It would resolve (ii) but not (i)

(i) is an additional constraint

So we will satisfy (i) by an explicit construction of h that works.

Constrained problem

Find parametrized map $h(t,\cdot):[0,1]\to\mathbb{E}^2$

in C^∞ so that

$$\int_0^1 h(t,u)\,du = f_0'(t)$$

and $\forall t\in[0,1]$, $\forall u\in\mathbb{E}/\mathbb{Z}$

$$\|h(t,u)\| = r(t).$$

How does h solve our problem?

Define f by convex integration process:

$$\forall t \in [0,1] \qquad f(t) = f_0(0) + \int_0^t h(s, \{Ns\})\, ds$$

where $\{Ns\}$ denotes fractional part of Ns.

(a) Clearly $f'(t) = h(t, \{Nt\}) \implies$

$$\|f'(t)\| = r(t)$$

and (b) C^0 density property (Since f defined by conv. integ)

$$\|f - f_0\|_{C^0} = O\left(\frac{1}{N}\right) < \delta$$

for N large enough.

We want to solve

$$\| h(t,u) \| = r(t)$$

$$\int_0^1 h(t,u)\, du = f_0'(t)$$

Try $$h(t,u) = r(t) e^{i\psi(t,u)} \frac{f_0'(t)}{\| f_0'(t) \|}$$

$$\| h(t,u) \| = r(t)$$

$$\int_0^1 e^{i\psi(t,u)} du \cdot \frac{r(t) f_0'(t)}{\| f_0'(t) \|} = f_0'(t)$$

$$\boxed{\int_0^1 e^{i\psi(t,u)} du = \frac{\| f_0'(t) \|}{r(t)}}$$

$$\int_0^1 e^{i\psi(t,u)}\,du = \frac{\| f_0'(t) \|}{r(t)}$$

Try $\psi(t,u) = \alpha(t)\cos 2\pi u$

$$\int_0^1 e^{i\alpha(t)\cos 2\pi u}\,du =$$

$$\int_0^1 \cos(\alpha(t)\cos 2\pi u)\,du$$

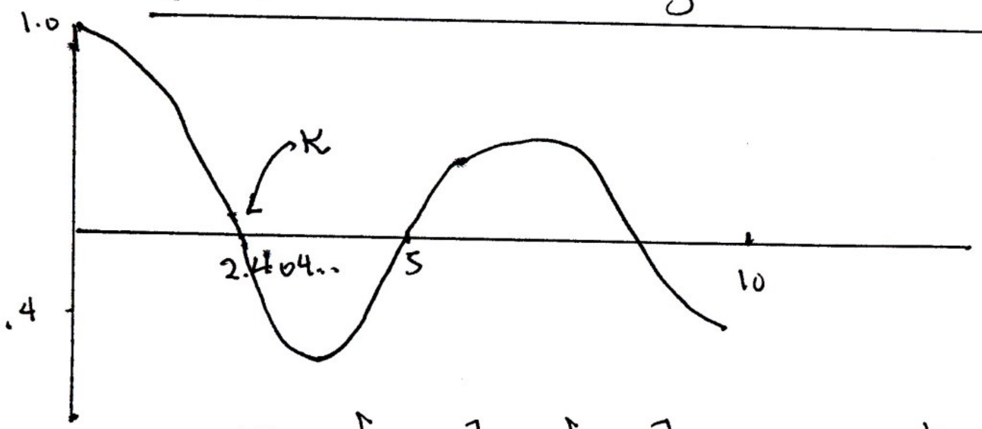
$$+ i \underbrace{\int_0^1 \sin(\alpha(t)\cos 2\pi u)\,du}_{\overset{\|}{0} \text{ by odd-even}}$$

$$\int_0^1 \cos(\alpha(t)\cos 2\pi u)\,du = \frac{\|f_0'(t)\|}{r(t)}$$

$$\|$$

$$J_0(\alpha(t))$$

$J_0(\alpha)$ Modified Bessel function of first kind of order 0.



$J_0 : [0, K] \to [0, 1]$ one to one.

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$$J_0(\alpha(t)) = \|f_0'(t)\| / r(t) \qquad \|f_0'(t)\| < r(t)$$

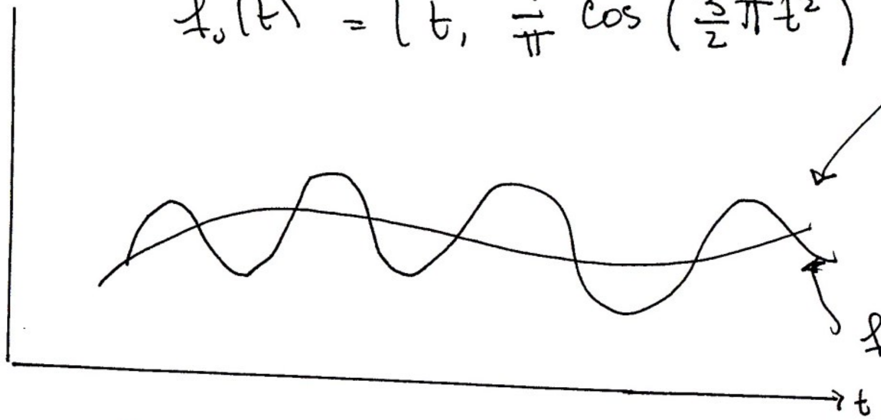
crucial ("short")

$$\alpha(t) = J_0^{-1}\left(\|f_0'(t)\| / r(t) \right)$$

$$f(t) = f_0(0) + \int_0^t r(t)\cos(\alpha(t)\cos 2\pi N u)\, du$$

$$\doteq IC(f_0, h, N) \qquad \underline{\underline{(SOLN)}}$$

$$f_0(t) = (t, \tfrac{1}{\pi}\cos(\tfrac{3}{2}\pi t^2))$$



N=4

f(t)

$f_0(0) = f_0(1)$ closed

Choose (as before)

$$\tilde{F}(t) = f(t) - t(f(1) - f(0))$$

$$= f_0(0) + \int_0^t r(t)\cos(\alpha(t)\cos 2\pi N u)\,du$$

$$- t\left(\int_0^1 r(t)\cos(\alpha(t)\cos 2\pi N u)\,du\right.$$

(i) $\tilde{F}(1) = \tilde{F}(0)$

(ii) $\|\tilde{F} - f_0\| = O\left(\frac{1}{N}\right)$

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New problem (and last)

$$\tilde{F}(t) = f(t) - t\,(f(1) - f(0))$$

$$\tilde{F}'(t) = f'(t) - (f(1) - f(0))$$

$f(0) = f_0(0)$
$= f_0(1)$

$$\| \tilde{F}'(t) - f'(t) \|_{C^0} = \sup_{0 \le t \le 1} \| f(t) - f_0(t) \|_{C^0}$$

$$= O\left(\frac{1}{N}\right)$$

$$\left| \, \| \tilde{F}'(t) \|_{C^0} - \| f'(t) \|_{C^0} \right| = O\left(\frac{1}{N}\right)$$

$$\left| \, \| \tilde{F}'(t) \|_{C^0} - r(x) \right| = O\left(\frac{1}{N}\right)$$

$\tilde{F}$ Approx but <u>Not</u> Exact soln!

Nash-Kuiper iteration:

Approximate the desired metric $r(t)$ from below by an increasing sequence of metrics $\{r_k(t)\}$.

Since each r_k is "short" ($r_k(t) < r(t)$) we can set up iteration process.

$\{\delta_k\}$ $\delta_k \nearrow 1$, $\delta_k > 0$

$$r_k^2(t) = r_0^2(t) + \delta_k\left(r^2(t) - r_0^2(t)\right)$$

$$r_0(t) = \|f_0'(t)\|$$

$$\tilde{F}_k = IC(\tilde{F}_{k-1}, h_k, N_k)$$

$$h_k(t,u) = r_k(t) \cos(\cos \alpha_k(t) \cos 2\pi u)$$

THM. Choose $\{\delta_k\} \ni \sum_k \sqrt{\delta_k - \delta_{k-1}} < \infty$.

Then there exists $\{N_k\} \ni \{\tilde{F}_k\}$

converges in C^1 to C^1 solution $\tilde{F}_\infty$, $\|\tilde{F}_\infty'(t)\| = r(t)$.

Ex.
Circle

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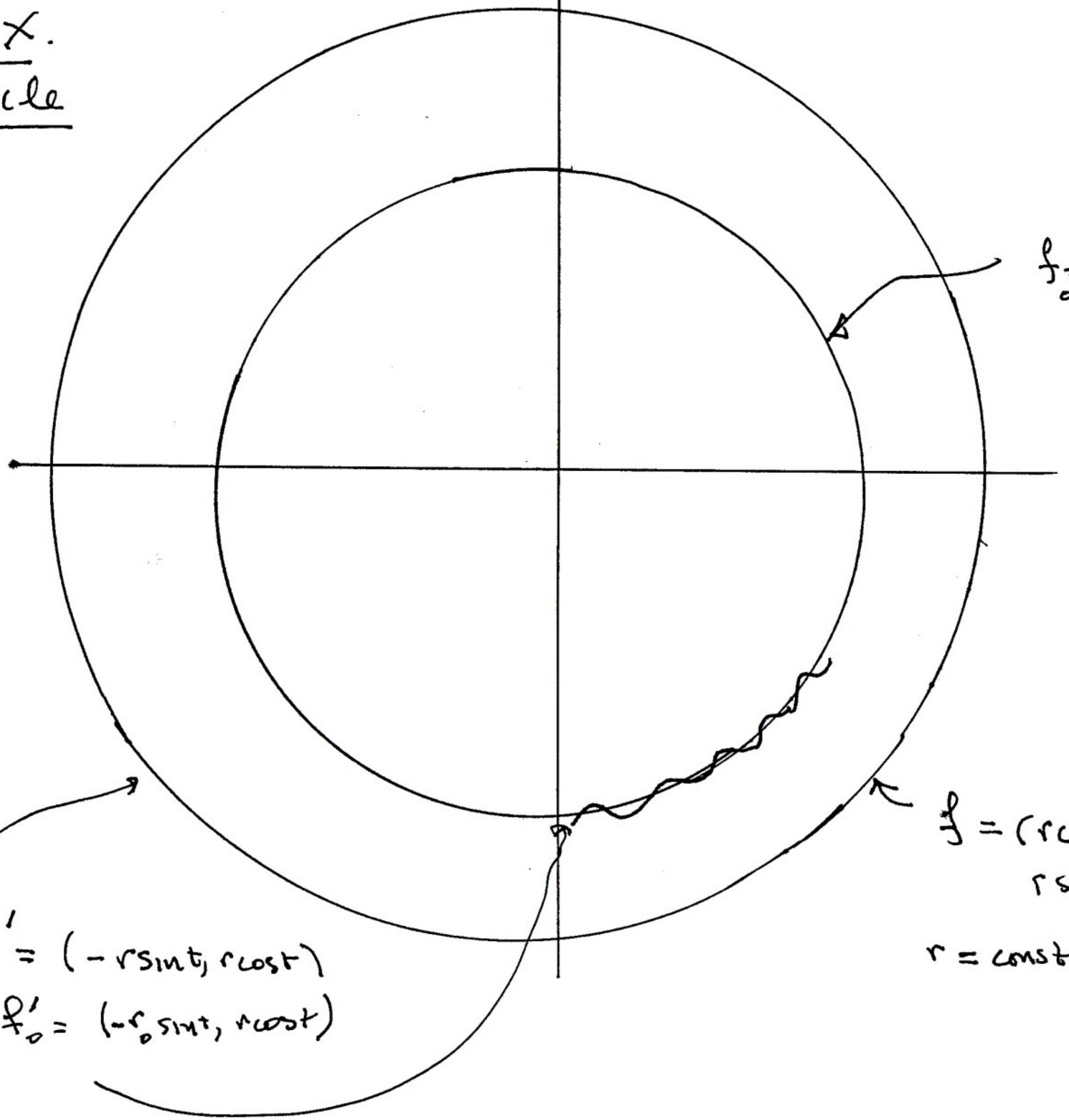

$f_0 = (r_0 \cos t,$
$r_0 \sin t)$

$r_0 < r$

$r_0 = \text{const.}$

$f = (r \cos t,$
$r \sin t)$

$r = \text{const.}$

$f' = (-r \sin t, r \cos t)$

$f_0' = (-r_0 \sin t, r \cos t)$