Optimal Execution & Algorithmic Trading

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Outline

Reading

There is a wealth of ongoing research and growing body of publications.
For market impact modelling, a good start are two survey papers:

- Lehalle,"Market Microstructure Knowledge Needed for Controlling an Intra-Day Trading Process"
- Gatheral and Schied, "Dynamical models of market impact and algorithms for order execution"

both in 2013 *Handook on Systemic Risk* (ed. Fouque and Langsam) and on arXiv.

For market microstructure, I suggest two review papers and four books:

- Chakraborti et al, "Econophysics review" (parts I and II), in *Quantitative Finance*, 2011
- "How markets slowly digest changes in supply and demand", Bouchaud et al (2009)
- O'Hara, *Market Microstructure Theory*, 1995
- Hasbrouck, *Empirical Market Microstructure: The Institutions, Economics, and Econometrics of Securities Trading*, 2006
- Lehalle and Laruelle, *Market Microstructure in Practice*, 2014.
- Cartea, Jaimungal and Penalva, *Algorithmic and High-Frequency Trading*, 2015.

Outline

Models in Quantitative Finance

- "All models are wrong but some are useful" (G. Box '78)
- Models need to be tailored to
  - the available inputs
  - the intended outputs
- Models need to
  - conform to stylised facts
  - produce reasonably useful and robust outputs
  - avoid creating arbitrage opportunities

Brief history of modelling in QF

- Fair price (fundamental economics)
  model: fundamentals
- Option pricing & optimal investment
  model: the underlying price process (exogenous)
  Samuelson '65, B&S and Merton '73
- Further option pricing: Exotics or FI options
  model: a high- or ∞- dimensional system of underlyings
  e.g.: HJM '92 and LMM '97 in Fixed Income; Market models of Schweizer & Wissel '08, Carmona & Nadtochiy '09

Brief history of modelling in QF – cont.

- Optimal execution of planned trades
  model: impact of trades on price dynamics or
  model: supply & demand dynamics
  Bertsimas & Lo '98, Almgren & Chriss '00;
  Obizhaeva & Wang '13, Alfonsi et al. '08
- Price formation via market microstructure
  model: LOB dynamics (zero intelligence)
  model: Agent trades (agent based)
  Cont et al. '10, Smith et al. '03, Farmer et al. '05;
  Kyle '85, O'Hara '95

... and MANY more references...

Recall: LOBs Impact – pros

The shift from traditional markets to electronic LOB driven markets had many consequences. Some positive:

- competition leading to lower fees and smaller tick sized
- more information available
- democratised trading process
- choice of patient (limit) or impatient (market) orders available to everyone
- computerised/algorithmic trading possible
- high frequency trading possible
  - HFT $\approx$ duration of order of seconds, reaction within milliseconds
  - accounts for $60 - 75\%$ of traded volume
- extra provision of liquidity $\rightsquigarrow$ market efficiency

Recall: LOBs Impact – cons

And some negative:

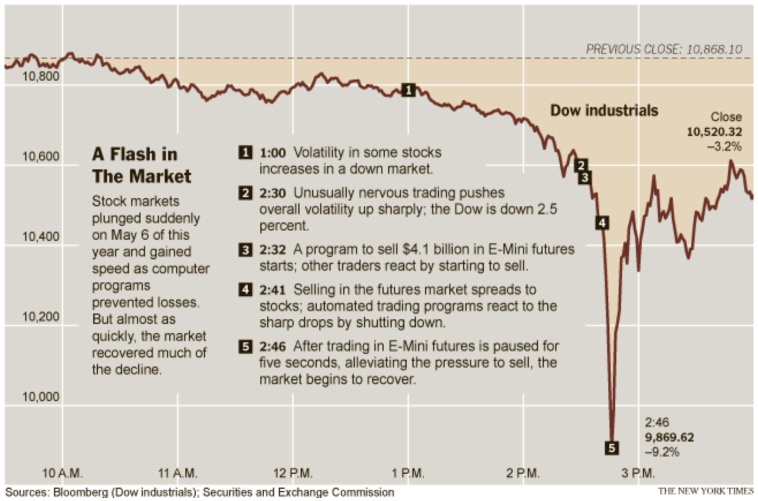
- Technological armsrace
- Little human oversight
- Predatory trading

This led to the infamous Flash Crash of May 6, 2010 when Dow Jones IA (DJIA) dived almost 1000 points (just to recover in minutes).

What do you think caused it?

- A mutual fund activated a program to sell $75,000$ E-Mini S&P 500 contracts (≈ 4.1 billion USD) using VWAP algorithm at 9%
- HFT began to quickly buy and resell these contracts to each other generating more volume: between 2:45:14 and 2:45:27, HFT traded $27,000$ contracts (about 49% of total volume) while buying only 200 contracts net.
- This led the original program to rapidly sell the whole position

Flash Crash of May 6th, 2010



Frictionless modelling setting

The classical modelling framework in mathematical finance, like the one postulated by Black and Scholes '73, assumes infinite liquidity:

- asset traded at uniquely given and known prices
- buying and selling in arbitrary quantities possible
- trading at no cost possible
- trading has no impact on the price

This is unrealistic and unsatisfactory: in reality we have market frictions.

Market Frictions

Market frictions – cont.

Many frictions either part of the game (opportunity cost) or well-defined (taxes). For many traders other frictions satisfactory summarised in

- proportional transaction costs: pay ϵS_t for trading one unit of S_t.

However this is not acceptable for

- large trades (relative to volume & time horizon)
- frequent trading (relative to liquidity)

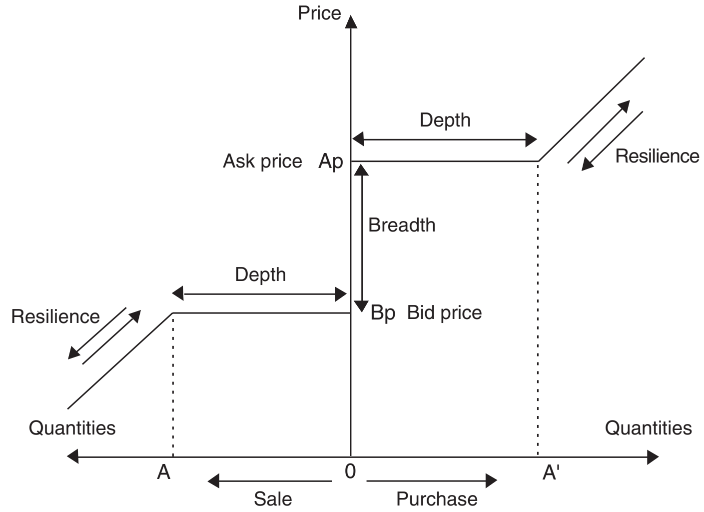
which require understanding of

- liquidity provision and
- price formation.

Aspects of liquidity (Kyle '85)

- **Tightness** (Breadth): measures how wide the bid-ask is, i.e. measures the cost of a position reversal at a short notice for a standard amount
- **Market depth**: corresponds to the volume which may be bought/sold without immediately affecting the price
- **Market resilience**: describes the speed at which prices revert to previous level (equilibrium) after a random shock in the order flow
- **Time delay**: measures the delay between processing and executing an order

Aspects of liquidity (Kyle '85)



Source: Bervas '06

Summary so far

- Models are build taking into account available inputs and desirable outputs
- In QF models postulate exogenous dynamics for different underlyings depending on what is traded and what one wants to price
- Traditional models assume a frictionless setting with ∞ liquidity
- In practice this fails. A lot can be accounted for using proportional transactions costs.
- Large and/or frequent trading requires modelling of liquidity and/or price impact.
- Electronic markets operate without designated market maker.
- Instead, the Limit Order Book (LOB) holds all active buy and sell orders

Outline

Price impact modelling

We saw that large and/or frequent trades may affect the price. We may need to split and spread large orders in practice. To answer how to do it we need to understand:

- how to model/quantify the impact of trading on the price?
- what are the desirable/undesirable properties of such models?
- how to compute optimal execution trading strategies?

There are two natural approaches to model price impact:

I: postulate fair price dynamics and the price impact of trading

II: be serious about modelling Market Microstructure, i.e. model supply and demand and their interaction.

We focus first on **I**. Then we use the LOB discussion to tackle **II**.

Trade execution setup

Goal: buy/sell x_0 shares by time T.

Trade execution strategy:

- $X = (X_t)_{t \leq T}$, where X_t is the number of shares held at time t
- The initial position $X_0 = x_0$ is positive for a sell strategy and negative for a buy strategy
- The final condition $X_T = 0$ indicates the position is liquidated at T
- The path will be monotone for a pure buy or pure sell strategy. In general it is of finite variation.

We think of T as around $5 - 10$, and up to 30, minutes.
For now, we are ignoring problems from higher(+) or lower(-) levels:

+ How a large desired trade position is split into chunks allocated their time horizons.
- What orders (market vs limit) are used and to which venues these are routed.

Price impact model

Price impact model quantifies the feedback effect of trading strategy X on the asset price. A typical setup is:

- Exogenously specified price process $S^0 = (S^0_t : t \leq T)$ for fair (unaffected) price dynamics.
  S is a semimartingale (usually a martingale) on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and we assume X is predictible
- Given X, a model prescribes S^X the price process realised when implementing trading strategy X.
- Typically, a buy strategy increases the prices and a sell strategy decreases the prices: if $X'(t) \geq 0$ for all $t \leq T$ then $S^X_t \geq S^0_t$, $t \leq T$. However this is not necessarily true for a fixed t since S^X_t may be affected by all of $(X_u : u \leq t)$.

Revenues and costs

Suppose X_t is differentiable in time and S^X_t depends continuously on X, then at time t, the infinitesimal amount of $-dX_t$ shares is sold at price S^X_t. Thus

$$\text{revenues from strategy } X \text{ are} \qquad \mathcal{R}(X) = -\int_0^T S^X_t \, dX_t$$

(when X is not absolutely continuous adjustments are necessary)

Objective: Maximise some performance functional of $\mathcal{R}(X)$.
For example:

- maximise the expected value $\mathbb{E}[\mathcal{R}(X)]$
- maximise a mean-variance criterion $\mathbb{E}[\mathcal{R}(X)] - \lambda \operatorname{var}(\mathcal{R}(X))$
- maximise the expected utility $\mathbb{E}[U(\mathcal{R}(X))]$
- ...

Revenues and costs – cont.

Alternatively: Minimise functional of implementation shortfall (i.e. cost of liquidation), which is the difference between the book value $X_0 S^0_0$ and the revenues (or the capture):

$$\text{liquidation cost of } X \text{ is} \qquad \mathcal{C}(X) = X_0 S^0_0 - \mathcal{R}_T(X).$$

If we write $S^X_t = S^0_t + I^X_t$ then

$$\begin{aligned} \mathcal{R}(X) &= -\int_0^T S^X_t \, dX_t = -\int_0^T S^0_t \, dX_t - \int_0^T I^X_t \, dX_t \\ &= S^0_0 X_0 + \underbrace{\int_0^T X_t \, dS^0_t}_{=-\mathcal{C}^{vol}(X)} - \underbrace{\int_0^T I^X_t \dot{X}_t \, dt}_{=\mathcal{C}^{exec}(X)} \end{aligned}$$

The total liquidation cost $\mathcal{C}(X)$ has two components:

- $\mathcal{C}^{vol}$ expresses the volatility risk of trading over time instead of instantly
- $\mathcal{C}^{exec}$ expresses the effect of price impact

Almgren–Chriss type price impact

The unaffected price follows a Brownian motion:

$$S_t^0 = S_0^0 + \sigma W_t.$$

Then, the price impact has two components:

- permanent impact: $\int_0^t g(\dot{X}_s)ds$
- temporary impact: $h(\dot{X}_t)$

for nondecreasing functions $g, h : \mathbb{R} \to \mathbb{R}$ and $\dot{X}_t = \frac{dX_t}{dt}$ the trading speed. The affected price is given by

$$S_t^X = S_t^0 + \int_0^t g(\dot{X}_s)ds + h(\dot{X}_t).$$

In the special case of linear impacts: $g(x) = \gamma x$ and $h(x) = \eta x$

$$S_t^X = S_t^0 + \gamma \int_0^t dX_s + \eta \dot{X}_t = S_t^0 + \gamma(X_t - X_0) + \eta \dot{X}_t.$$

A–C model with linear price impact

In the special case of linear impacts: $g(x) = \gamma x$ and $h(x) = \eta x$

$$S_t^X = S_t^0 + \gamma \int_0^t dX_s + \eta \dot{X}_t = S_t^0 + \underbrace{\gamma(X_t - X_0) + \eta \dot{X}_t}_{=I_t^X}.$$

The revenues are then given by

$$\begin{aligned}\mathcal{R}(X) &= -\int_0^T S_t^X dX_t = S_0^0 X_0 + \int_0^T X_t dS_t^0 - \int_0^T I_t^X \dot{X}_t dt \\ &= S_0^0 x_0 + \sigma \int_0^T X_t dW_t - \frac{\gamma}{2} x_0^2 - \eta \int_0^T \dot{X}_t^2 dt,\end{aligned}$$

since $X_T = 0$.

A–C model with linear price impact (cont.)

Assuming X is bounded, the expected revenues are

$$\mathbb{E}[\mathcal{R}(X)] = S_0^0 x_0 - \frac{\gamma}{2} x_0^2 - \eta \,\mathbb{E}\left[\int_0^T \dot{X}_t^2 dt\right].$$

The last term is an integral w.r.t. $\mathbb{P}(d\omega) \otimes dt$ of the square of $\dot{X}_t(\omega)$. It follows that it is minimised, and hence $\mathbb{E}[\mathcal{R}(X)]$ is maximised, by the strategy

$$\dot{X}_t^* = -\frac{x_0}{T}$$

which sells (or buys) the shares at constant speed (to see this simply apply Jensen's inequality). In particular the solution is independent of the volatility! (Bertsimas & Lo '98)

The resulting expected liquidation cost of x_0 shares is

$$\mathbb{E}[\mathcal{C}(X)] = \left(\frac{\gamma}{2} + \eta\right) x_0^2$$

quadratic in number of shares and independent of volatility σ.

A–C model so far – summary

Proposition

In the Almgren–Chriss price impact model with linear permanent impact, $g(x) = \gamma x$, and $xh(x)$ convex, for any given $x_0 \in \mathbb{R}$ the strategy

$$X_t^* = \frac{X_0(T-t)}{T}, \quad t \leq T,$$

maximises the expected revenues $\mathbb{E}[\mathcal{R}(X)]$ in the class of all adapted and bounded trade execution strategies X.

The strategy X^* spreads the execution evenly over the time horizon $t \in [0, T]$. It is often referred to as the time-weighted average price strategy or TWAP. When the time is relative and t corresponds to traded volume the X^* is called volume-weighted average price strategy or VWAP. Both are used as industry benchmarks.

Almgren et al. '05 argued these assumptions are consistent with empirical observations and suggested $xh(x) \approx |x|^{1.6}$.

A–C model with mean-variance criterion

So far we only looked at expected revenues. Almgren and Chriss '00 propose to consider

$$\max_X \mathbb{E}[\mathcal{R}(X)] \quad \text{subject to } \operatorname{var}(\mathcal{R}(X)) \leq v_*$$

which, introducing a Langrange multiplier, turns into an unconstrained problem

$$\max_X \left(\mathbb{E}[\mathcal{R}(X)] - \lambda \operatorname{var}(\mathcal{R}(X))\right).$$

This is a **hard** problem. However assuming X is deterministic it turns into

$$\max_X \left(x_0 S_0^0 - \frac{\gamma}{2} x_0^2 - \int_0^T \left(\frac{\lambda \sigma^2}{2} X_t^2 + \eta \dot{X}_t^2 \right) dt \right)$$

which can be solved explicitly as a standard variational calculus problem.

A–C model with mean-variance criterion

Indeed, the problem is equivalent to

$$\min_X \int_0^T \left(\frac{\lambda \sigma^2}{2} X(t)^2 + \eta X'(t)^2 \right) dt$$

Setting the first variation to zero:

$$0 = \int_0^T \left(g(t) \lambda \sigma^2 X(t) + 2g'(t) \eta X'(t) \right) dt, \quad \forall g \in C^1 : g(0) = g(T) = 0.$$

Integrating by parts:

$$0 = \int_0^T g(t) \left(\lambda \sigma^2 X(t) - 2\eta X''(t) \right) dt, \quad \forall g \in C^1 : g(0) = g(T) = 0$$

which gives the Euler-Lagrange equation

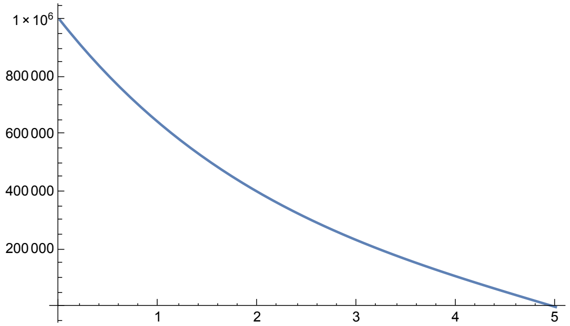
$$X''(t) = \frac{\lambda \sigma^2}{2\eta} X(t), \quad \text{s.t. } X(0) = x_0, X(T) = 0.$$

Solving the ODE we obtain

A–C model with mean-variance criterion

The solution is given by

$$X_t^* = x_0 \frac{\sinh(\kappa(T-t))}{\sinh \kappa T} \quad \text{for } \kappa = \sqrt{\frac{\lambda\sigma^2}{2\eta}}.$$



Optimal liquidation strategy of 10^6 shares over 5 days under 30% annual vol and impact 1% of daily volume = bid-ask. Moderate λ.

A–C model with mean-variance criterion

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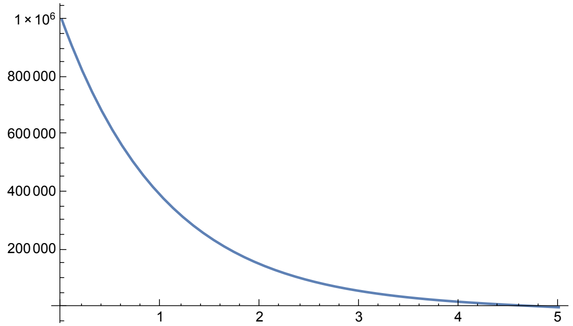


Optimal liquidation strategy of 10^6 shares over 5 days under 30% annual vol and impact 1% of daily volume = bid-ask. High λ.

A–C model with other criteria

Mean-variance is **not** amenable to dynamic programming and leads to time-inconsistent strategies. In analogy to optimal investment, other criteria are natural:

- Maximise expected utility: $\max_X \mathbb{E}[U(\mathcal{R}(X))]$
  The problem can be reformulated as a stochastic control problem with non-standard (finite fuel) constraint: $X_0 = x_0$ and $X_T = 0$. Leads to an **HJB** equation. Solution known for $U(x) = -\exp(-\lambda x)$ ... the same as for mean-variance! (Schied, Schöneborn & Tehranchi '10).
- Maximise

$$\mathbb{E}\left[\mathcal{R}(X) - \lambda \int_0^T X_t S_t^X dt\right]$$

  Gatheral & Schied '11

Criticism of A–C setting

- Price process can go negative; impact additive & in absolute terms. Bertsimas & Lo '98 suggest

$$S_t^X = S_t^0 \exp\left(\int_0^t g(\dot{X}_s)ds + h(\dot{X}_t)\right), \quad S_t^0 = S_0^0 \exp\left(\sigma W_t - \frac{\sigma^2}{2}t\right)$$

  but computing optimal strategies more involved.
- Price impact simplistic, in reality transient effect, see Moro et al. '09 (cf. resilience)
- Computed optimal strategies are deterministic and do not react to price changes
- No modelling of feedback effects between the seller and the market (e.g. Flash Crash 06/05/10)

$\Longrightarrow$ Need to understand price formation better!

Summary of A–Ch-type market impact modelling

- Revenues from a large sell/buy order may depend crucially on its execution
- The optimal execution strategy in turn may depend crucially on the criterion
- Almgren–Chriss models involve permament and temporary impact of trades on prices
- Under linear impacts and maximising revenues, it is optimal to sell at a constant speed
- Under linear impacts and among deterministic strategies, optimising mean-variance criterion, it is optimal to use a specific convex programme.

Outline

Types of price impact

So far we have modelled:

- permanent price impact
- temporary price impact

In reality, transactions interact with the LOB. Market orders will eat into the book but new liquidity will then come as markets are resilient.

We need to model

- transient price impact

Modelling transient price impact

Idea: model transient price impact by:

- stochastic dynamics of LOB
  $\rightsquigarrow$ e.g. constant depth λ, model only bid B_t & ask A_t
- a buy (market) order eats into the ask side of the book
  $\rightsquigarrow$ a buy order of $\Delta X_t > 0$ moves ask $A_{t+} = A_t + \Delta X_t/\lambda$
- book then reverts back at some speed
  $\rightsquigarrow$ according to a decay kernel $G(\text{delay})$, e.g. $\mathrm{e}^{-\rho t}$, $(1+t)^{-\alpha}$

Obizhaeva & Wang '13, Alfonsi et al. '08, Gatheral '10, Gatheral et al. '12...

Simple transient price impact (Obizhaeva & Wang '13)

- Assume no bid-ask spread, $S^0_t = B_t = A_t$ is a martingale
- Constant book depth of $\lambda = 1/G(0)$
- A discrete order $X_{t+} - X_t =: \Delta X_t$ moves price

$$S^X_{t+} = S^X_t + \Delta X_t G(0)$$

  and is executed at cost of (= - expected revenue of)

$$\frac{1}{G(0)} \int_{S^X_t}^{S^X_{t+}} v dv = \frac{1}{2G(0)} \left((S^X_{t+})^2 - (S^X_t)^2 \right) = \frac{G(0)}{2} (\Delta X_t)^2 + \Delta X_t S^X_t .$$

- The market is resilient and trade impact wanes away. So that

$$S^X_t = S^0_t + \sum_{s<t:\Delta X_s>0} G(t-s)\Delta X_s$$

Simple transient price impact – cont.

- Assume now trading is only possible at some give time points: $0 = t_0 < t_1 < \ldots < t_n = T$, X_0 given, $X_T = 0$ and
$$X_t = X_0 + \sum_{i: t_i < t} \Delta_i, \quad \text{where } \Delta_i := X_{t_i+} - X_{t_i}$$
- The mid-price resulting from strategy X is
$$S_t^X = S_t^0 + \sum_{i: t_i < t} G(t - t_i)\Delta_i$$
- The total cost of executing X is
$$\mathcal{C}(X) = S_0^0 X_0 - \mathcal{R}(X) = S_0^0 X_0 + \sum_{i=0}^{n} \left(\frac{G(0)}{2}\Delta_i^2 + \Delta_i S_t^X \right)$$
$$= S_0^0 X_0 + \sum_{i=0}^{n} S_{t_i}^0 \Delta_i + \sum_{i=0}^{n} \left(\frac{G(0)}{2}\Delta_i^2 + \Delta_i \sum_{j<i} G(t_i - t_j)\Delta_j \right)$$

Simple transient price impact – cont.

$$S_0^0 X_0 + \sum_{i=0}^{n} S_{t_i}^0 \Delta_i = S_0^0 X_0 + \int_0^t S_t^0 dX_t = -\int_0^t X_{t-} dS_t^0$$

which has zero expectation (assuming Δ_i bounded). Further,

$$\sum_{i=0}^{n} \left(\frac{G(0)}{2}\Delta_i^2 + \Delta_i \sum_{j<i} G(t_i - t_j)\Delta_j \right)$$
$$= \sum_i \frac{G(0)}{2}\Delta_i^2 + \sum_i \sum_{j<i} G(t_i - t_j)\Delta_i\Delta_j$$
$$= \frac{1}{2}\sum_i \sum_j G(|t_i - t_j|)\Delta_i\Delta_j$$

In consequence, the total expected cost of liquidation following X is

$$\mathbb{E}[\mathcal{C}(X)] = \frac{1}{2}\sum_i \sum_j G(|t_i - t_j|)\,\mathbb{E}\left[\Delta_i\Delta_j\right]$$

Simple transient price impact – solution

It is then enough to look for X among deterministic strategies:

$$\text{minimise} \sum_i \sum_j G(|t_i - t_j|)\Delta_i\Delta_j \quad \text{over } \Delta \in \mathbb{R}^{n+1} : \Delta^{\mathrm{T}}\mathbf{1} = -x_0$$

Rk: value invariant under $\Delta \to -\Delta \implies$ Optimal Buy = - Optimal Sell.

If G is strictly positive definite then the optimal solution Δ^* is

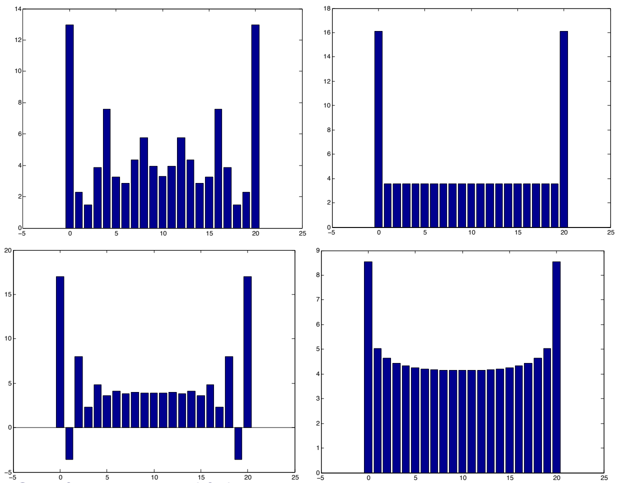
$$\Delta^* = \text{const} \cdot \Gamma^{-1}\mathbf{1}, \quad \text{where } \Gamma_{ij} = G(|t_i - t_j|).$$

Let us take equidistant steps: $t_{i+1} - t_i = \frac{T}{N}$ and look at different examples of G.

Optimal strategy – examples

Optimal Δ_i^* for $t \in [0,1]$, $N = 20$, $X_0 = -100$ and four decay kernels:

$$G_1(t) = \mathrm{e}^{-5t},\; G_2(t) = (0.5 - 2.7t)^+,\; G_3(t) = \frac{1}{(1+10t)^2},\; G_4(t) = \frac{1}{1+(10t)^2}.$$

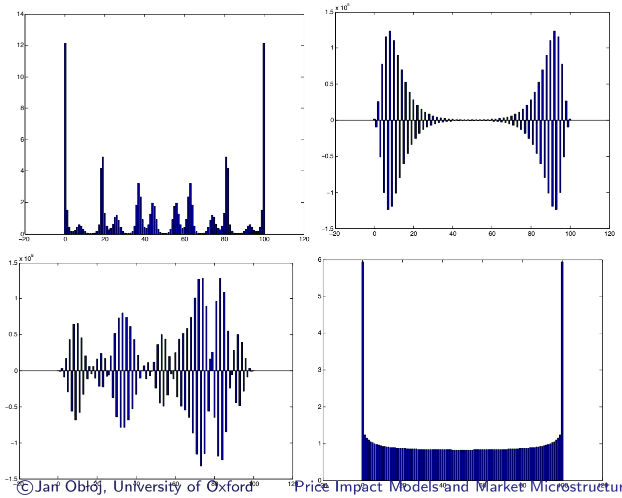


Which one is which?

Optimal strategy – examples

Optimal Δ_i^* for $t \in [0,1]$, $N = 100$, $X_0 = -100$ and four decay kernels:

$$G_1(t) = (0.5{-}2.7t)^+,\; G_2(t) = \frac{1}{(1+5t)^2},\; G_3(t) = \frac{1}{1+(10t)^2}\; G_4(t) = \frac{1}{1+(7t)^2}.$$

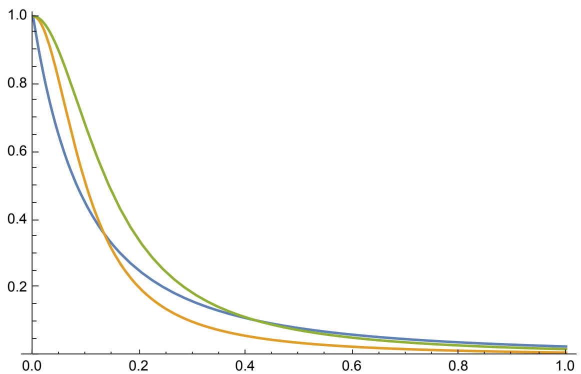


Which one is which?

Non-robustness w.r.t. decay kernel

The optimal Δ_i^* for $t \in [0,1]$, $N = 100$, $X_0 = -100$ and three decay kernels:

$$G_2(t) = \frac{1}{(1+5t)^2},\; G_3(t) = \frac{1}{1+(10t)^2}\; G_4(t) = \frac{1}{1+(7t)^2}.$$



differ dramatically...

Notion of "price manipulation strategy"

We saw that very similar decay functions may lead to drastically different optimal portfolios, including round-trip-taking trading. Clearly requires further studies.

Definition

A round trip strategy X, $X_0 = X_T = 0$ with strictly negative expected cost $\mathbb{E}[\mathcal{C}(X)] < 0$ is called a price manipulation strategy.

Note that this is not the usual arbitrage since profit is not a.s. but in expectation. However in some models rescaling and repeating price manipulation leads to (weak) arbitrage.

We first extend our previous analysis to arbitrary strategies X.

Transient price impact with arbitrary strategies

With discrete X, the impacted price process was

$$S_t^X = S_t^0 + \sum_{i:t_i<t} G(t-t_i)\Delta_i = S_t^0 + \int_{s<t} G(t-s)dX_s$$

and the last term extends to arbitrary X (predictable, left-continuous, of bounded variation). The revenues of a continuous strategy are given as previously

$$-\int_0^T S_t^X dX_t = -\int_0^T S_t^0 dX_t - \int_0^T \int_{s<t} G(t-s)dX_s dX_t.$$

In the case of discrete X we had

$$\begin{aligned}&-\sum_{i=0}^n S_{t_i}^0\Delta_i - \frac{1}{2}\sum_i\sum_j G(|t_i-t_j|)\Delta_i\Delta_j \\ &= -\int_0^T S_t^0 dX_t - \frac{1}{2}\int\int G(|t-s|)dX_s dX_t\end{aligned}$$

Transient price impact with arbitrary strategies

Combining, the execution cost of X are

$$\mathcal{C}(X) = S_0^0 X_0 - \mathcal{R}(X) = \int_0^T X_{t-} dS_t^0 + \frac{1}{2}\int_0^T\int_0^T G(|t-s|)dX_s dX_t.$$

composed of volatility risk and price impact cost

$$\mathcal{C}^{exec}(X) = \frac{1}{2}\int_0^T\int_0^T G(|t-s|)dX_s dX_t.$$

Price manipulation $\iff \mathbb{E}[\mathcal{C}^{exec}(X)] < 0$.

Let's start with understanding when $\mathcal{C}^{exec}(X) \geq 0$ a.s.

Bochner's theorem and positive costs

Proposition

We have $\mathcal{C}^{exec}(X) \geq 0$ for all strategies X iff G is positive definite, i.e. can be represented as the Fourier transform of a positive finite Borel measure μ on $\mathbb{R}$. Further, if G is strictly positive definite (μ is not discrete) then $\mathcal{C}^{exec}(X) > 0$ for all nonzero X.

We may also formalise the case of deterministic discrete strategies.

Proposition (Gatheral, Schied and Slynko '12)

Suppose G is positive definite. Then among deterministic strategies trading at given times (t_i), an optimal one X^* satisfies a generalised Freedholm integral equation

$$\int G(|t_i - s|)dX_s^* = \lambda, \quad i = 0, 1, \ldots, N$$

for some constant λ.

Rk.: We wrote this equation as $\Gamma\Delta = \text{const} \cdot \mathbf{1}$ before.

Is absence of price manipulation enough?

We have

positive definite $G \Longrightarrow$ no price manipulation strategy.

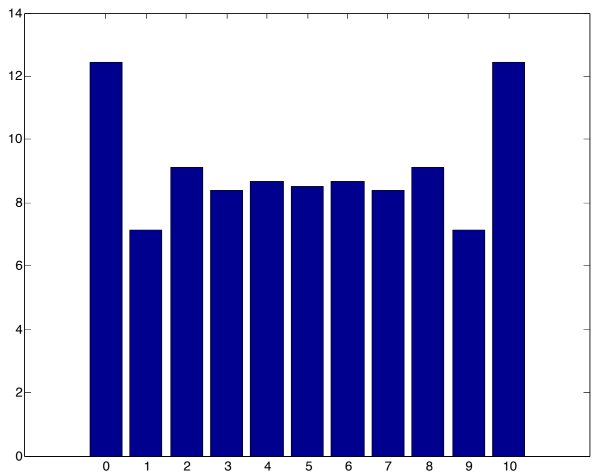
Is this enough? Take

$$G(t) = \mathrm{e}^{-t^2}$$

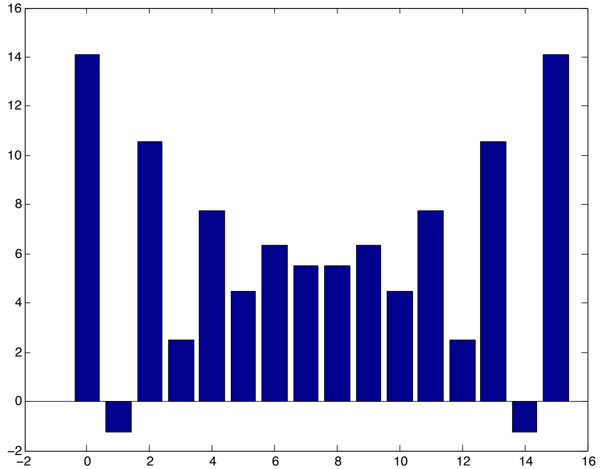
which, up to scaling, is its own Fourier transform and hence positive definite.

Let's look at the optimal strategy for $T = 10$, $X_0 = -100$ and vary N.

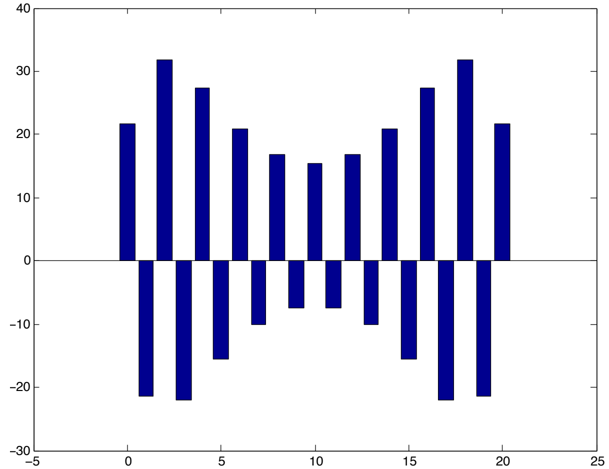
Optimal trading with Gaussian decay $G(t) = \exp(-t^2)$, $T = 10$, $X_0 = -100$, $N = 10$



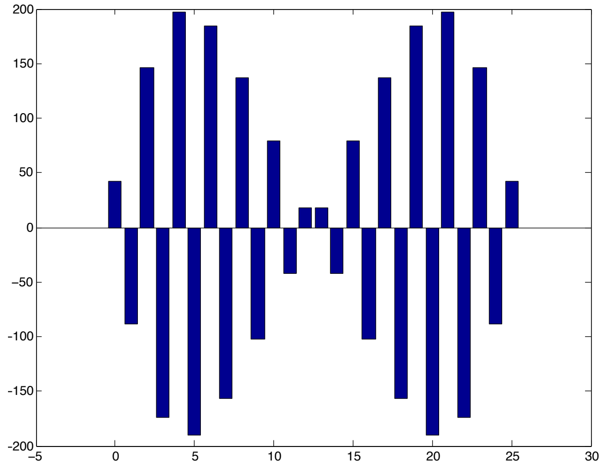
Optimal trading with Gaussian decay $G(t) = \exp(-t^2)$, $T = 10$, $X_0 = -100$, $N = 15$



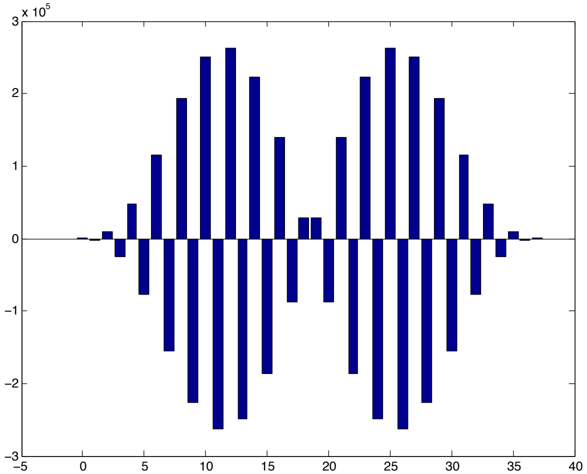
Optimal trading with Gaussian decay $G(t) = \exp(-t^2)$, $T = 10$, $X_0 = -100$, $N = 20$



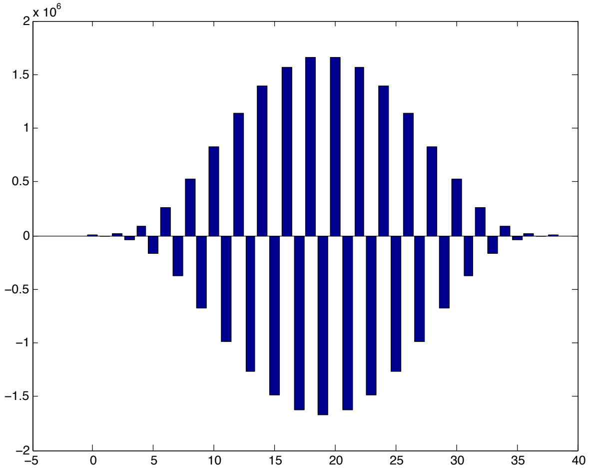
Optimal trading with Gaussian decay $G(t) = \exp(-t^2)$, $T = 10$, $X_0 = -100$, $N = 25$



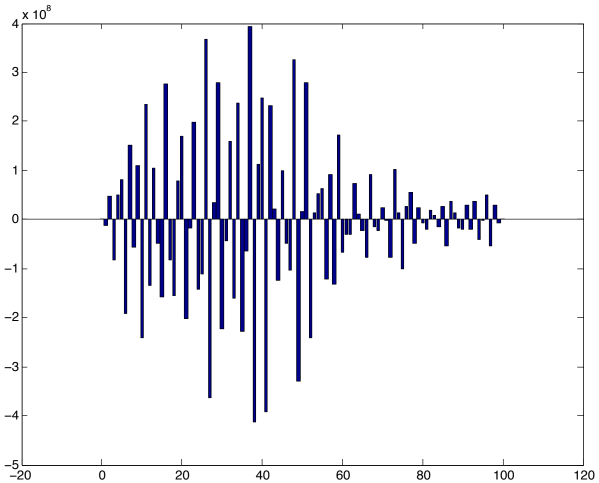
Optimal trading with Gaussian decay $G(t) = \exp(-t^2)$, $T = 10$, $X_0 = -100$, $N = 37$



Optimal trading with Gaussian decay $G(t) = \exp(-t^2)$, $T = 10$, $X_0 = -100$, $N = 38$



Optimal trading with Gaussian decay $G(t) = \exp(-t^2)$, $T = 10$, $X_0 = -100$, $N = 100$



Clearly excluding price manipulation strategies is not enough...

Price manipulation strategies

Definition

A market model admits price manipulation if there exists a round trip strategy X, $X_0 = X_T = 0$ with strictly positive expected revenues $\mathbb{E}[\mathcal{R}(X)] > 0$.

Definition

We say that a market impact model admits transaction–triggered price manipulation if the expected revenues of a sell (resp. buy) program can be increased by intermediate buy (resp. sell) orders.

Remark: in a sensible model (i.e. if buying increases prices and selling decreases prices) absence of *transaction–triggered price manipulation* implies absence of the usual *price manipulation*.

Regularity of Almgren–Chriss type models

Recall that in A-CH framework, the impacted price is

$$S_t^X = S_t^0 + \int_0^t g(\dot{X}_s)ds + h(\dot{X}_t).$$

Proposition (Huberman & Stanzl '04, Gatheral '10)

If the model above does NOT admit price manipulation for all $T > 0$ then $g(x) = \gamma x$ for some $\gamma \geq 0$.
Further, if g is linear and $x \to xh(x)$ is convex than the model does NOT admit transaction–triggered price manipulation.

Rk: the second part is clear since in this setting the optimal X^* is linear.

Regularity of Obizhaeva–Wang type models

Proposition (Alfonsi, Schied & Slynko '12)

A transient price impact model with decay kernel G s.t.

$$G(0) - G(s) < G(t) - G(t+s), \quad \textit{for some } s \neq t,$$

admits transaction-triggered price manipulation trading at $\{0, s, t+s\}$.

In particular, it is enough that G is NOT convex for small t.

Proposition (Alfonsi et al. '12, Gatheral et al. '12)

A transient price impact model with convex, decreasing, non-negative decay kernel G admits a unique optimal X^* which is monotone in time. In particular the setup does NOT admit transaction-triggered price manipulation.

Other developments

- Non-linear transient price impact models: the book has varying depth according to a given shape f, see Alfonsi & Schied '10
- A combination of impacts, e.g. Gatheral '10

$$S_t^X = S_t^0 + \int_0^t h(-\dot{X}_t)G(t-s)ds$$

- Stochastic models of LOB where the shape f is a stochastic process in space of curves and/or stochastic resilience, see Alfonsi & Infante Acevedo '12, Klöck '12, Fruth, Schöneborn & Urusov '11, Müller & Keller-Ressel '15.
- ...

Summary of transient market impact models

- Transient price impact models take into account the interaction of orders with the LOB and market resilience
- Under constant LOB depth, discrete trading at (t_i) and maximising expected revenues the optimal strategy explicit for many impact decay kernels G
- More generally the problem quickly becomes very hard...
- Even in simple setting, the optimal strategies may often involve round trips. Solution is non-robust with respect to G.
- Possible to study, and provide sufficient conditions for, the absence of price-triggered manipulation strategies.

Outline

Multi-agent frameworks

- In reality many agents interact in a market.
- Mathematically best modelled as game. When number of players $n \to \infty$, sometimes possible to analyse as a mean field game.
- Interesting as it allows to study
  - Interaction of one large player with n small players (e.g. predatory trading)
  - Global market implications of interactions between small players
  - Properties of markets which facilitate different phenomena

Predatory Trading

Large Trader facing a forced liquidation

$+$

other (HF) traders aware of this fact

$\Downarrow$

Predatory Trading

Examples of "targets":

- Index-replicating funds at rebalancing dates
- Institutional investors subject to regulatory constraints (e.g. when an instrument is downgraded)
- Traders using portfolio insurance or stop-loss strategies
- Hedge funds close to a margin call
- Recalled short-seller

Predatory Trading

"... if lenders know that a hedge fund needs to sell something quickly, they will sell the same asset – driving the price down even faster. Goldman, Sachs & Co. and other counterparties to LTCM did exactly that in 1998."

Business Week, 26 Feb 2001

"When you smell blood in the water, you become a shark ... when you know that one of your number is in trouble ... you try to figure out what he owns and you start shorting those stocks ... "

Cramer, 2002

Predatory Trading – mechanisms

When a need of a large trader (prey) to liquidate is recognised, the strategic traders (predators) might

- first trader **in the same direction**
  - withdraw liquidity instead of providing it
  - market impact is greater leading to price overshooting
  - may further enforce distressed trader's need to liquidate
- then **reverse direction** to profit from the overshoot
- closing the roundtrip at a profit.

However when strategic traders have a longer horizon than the liquidation, their behaviour may depend on market characteristics:

- could act as predators as above $\rightsquigarrow$ large trader tries to keep intentions hidden (stealth trading)
- could act as liquidity providers $\rightsquigarrow$ large trader announces intentions (sunshine trading)

see Brunnermeier & Pedersen '05, Carlin, Lobo & Viswanathan '05, Schied & Schöneborn '08.

One-period game model with A–Ch price impact

- $n+1$ players with portfolios $X_0(t), \dots, X_n(t)$, $t \in [0,T]$, assumed cont. diff. in time
  - one prey (seller): $X_0(0) = x_0 > 0$, $X_0(T) = 0$
  - n predators: $X_i(0) = X_i(T) = 0$, $i = 1, \dots, n$
- and the above is common knowledge
- players are risk-neutral and maximise their expected profit

$$\mathcal{R}^i(X) = -\mathbb{E}\left[\int_0^T S_t dX_i(t)\right]$$

- one risk-free and one risky asset, continuous trading, Almgren–Chriss linear price impact model

$$S(t) = S(0) + \sigma W_t + \gamma \sum_{i=1}^n (X_i(t) - X_i(0)) + \eta \sum_{i=1}^n \dot{X}_i(t)$$

- Solved by searching for Nash equilibrium.

One-period game model with A–Ch price impact

Assuming all X_i are deterministic this can be solved explicitly giving

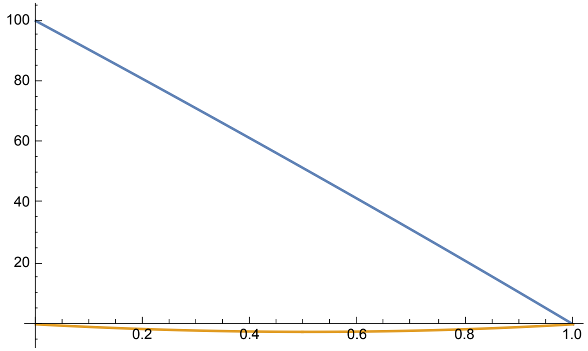
$$\dot{X}_i^*(t) = \alpha \mathrm{e}^{-\frac{n}{n+2}\frac{\gamma}{\eta}t} + \beta_i \mathrm{e}^{\frac{\gamma}{\eta}t},$$

where

$$\alpha = \frac{-n}{n+2}\frac{\gamma}{\eta}\left(1 - \mathrm{e}^{-\frac{n}{n+2}\frac{\gamma}{\eta}T}\right)^{-1}\frac{x_0}{n+1},$$

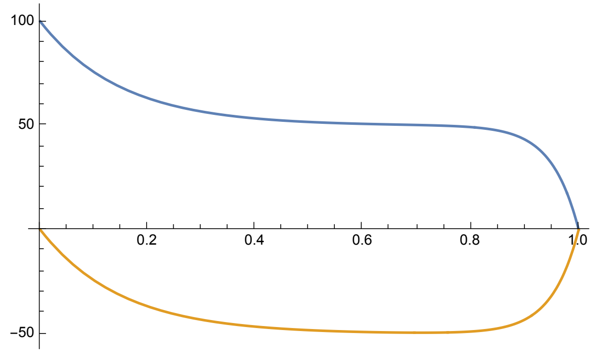
$$\beta_i = \frac{\gamma}{\eta}\left(\mathrm{e}^{\frac{\gamma}{\eta}T} - 1\right)^{-1}\left(X_i(T) - X_i(0) + \frac{x_0}{n+1}\right)$$

Optimal strategies with $n = 1,\ T = 1,\ \frac{\gamma}{\eta} = 0.3$



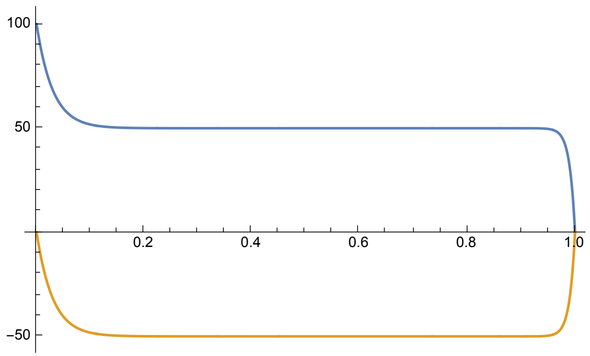
Distressed trader (blue) and one predator in a elastic market
(i.e. temporary impact > permanent impact)

Optimal strategies with $n = 1,\ T = 1,\ \frac{\gamma}{\eta} = 20$



Distressed trader (blue) and one predator in an plastic market
(i.e. permanent impact > temporary impact)

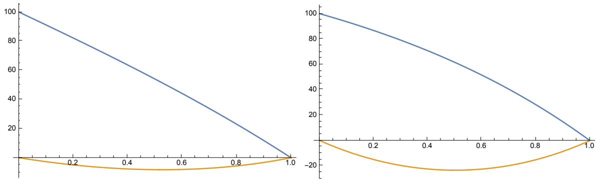
Optimal strategies with $n = 1,\ T = 1,\ \frac{\gamma}{\eta} = 100$



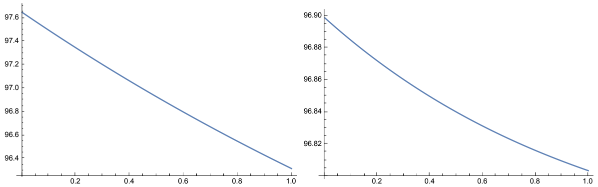
Distressed trader (blue) and one predator in a highly plastic market.

Effect of predators, $T = 1$, $x_0 = 100$, $S_0 = 100$, $\gamma = \eta = 2\%$

Comparison of $n = 1$ and $n = 40$ predators. Aggregated Holdings:



Expected market price:

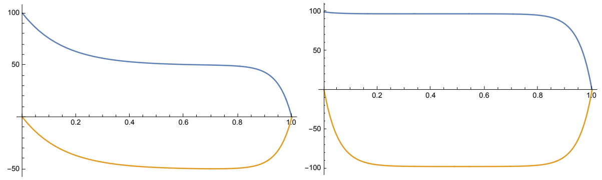


Expected execution cost $\mathbb{E}[\mathcal{C}(X)]$: 3.1% and 3.2% (*compare with* 3% *when* $n = 0$)

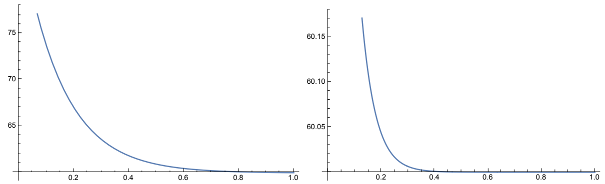
Expected revenue per predator: 7.27 and 0.4.

Effect of predators, $T = 1$, $x_0 = 100$, $S_0 = 100$, $\gamma = 20 * \eta = 2\%$

Comparison of $n = 1$ and $n = 40$ predators. Aggregated Holdings:



Expected market price:



Expected execution cost $\mathbb{E}[\mathcal{C}(X)]$: 33.3% and 40% (*was* 22% *when* $n = 0$)

Expected revenue per predator: 665 and 2.3.

Price and execution costs scale linearly with costs when keeping $\frac{\gamma}{\eta}$ fixed.

HF hot-potato game

Schied & Zhang '13 considered the following setup:

- two HF players X and Y trading in an Obizhaeva & Wang market with $G(t) = \mathrm{e}^{-\rho t}$
- trading at an equidistant discrete time grid
- with opposite initial positions $X_0 = -Y_0$.

Using a Nash equilibrium analysis, they show that

- the optimal behaviour, if trading is frequent enough, involves a highly oscillatory trading
- hot-potato effect with volume passed between traders
- the effect can be eliminated if transaction costs present and high enough compared to LOB depth

Multi-agent setup summary

- Detailed analysis of market behaviour may require models with interacting agents
- Mathematically, often done using game theory and searching for Nash equilibria
- Predatory trading can be described as a game between one large seller (prey) and n strategic traders (predators)
- Both from the theory and practice, we see that predators often first trade in the same direction as the large trader leading to price overshoot of which they then take advantage.
- The optimal behaviour highly dependent on the market characteristic (e.g. which type of price impact dominates)
- More involved situations (e.g. strategic traders having longer trading horizon) may lead to qualitatively different solutions
- Many other situations in which game analysis is interesting, e.g. high trade volume (hot-potato) effect of trading between two agents.