Short Course on Behavioral Finance Part II: Portfolio Choice

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Section 1

Portfolio Choice Models: Formulation

Yaari's Dual Theory – Discrete Random Variables

- Given a random payoff $\tilde{X}$, which is a discrete random variable having possible values $\ldots < x_{-n} < \cdots < x_{-1} < 0 < x_1 < \cdots < x_n < \cdots$ and the distribution $\mathbb{P}(\tilde{X} = x_i) = p_i$.
- Evaluation of $\tilde{X}$ (Yaari 1987):

$$V(\tilde{X}) = \sum_{n=1}^{\infty} x_n \left(w(\sum_{j=n}^{\infty} p_j) - w(\sum_{j=n+1}^{\infty} p_j) \right) - \sum_{n=1}^{\infty} (-x_{-n}) \left(w(\sum_{j=n}^{\infty} p_{-j}) - w(\sum_{j=n+1}^{\infty} p_{-j}) \right)$$

where *probability weighting* (or *distortion*) $w : [0,1] \to [0,1]$, $\uparrow$, $w(0) = 0$, $w(1) = 1$

Yaari's Dual Theory – General Random Variables

If $\tilde{X} \geq 0$ is a general random variable:

$$\begin{aligned} V(\tilde{X}) &= \int \tilde{X} d(w \circ \mathbb{P}) \\ &:= \int_0^{\infty} x d[-w(\mathbb{P}(\tilde{X} > x))] \\ &= \int_0^{\infty} w\left(\mathbb{P}(\tilde{X} > x)\right) dx \end{aligned}$$

Rank Dependence

Assuming w is differentiable:
$V(\tilde{X}) = \int_0^\infty x d[-w(1 - F_{\tilde{X}}(x))] = \int_0^\infty x w'(1 - F_{\tilde{X}}(x)) dF_{\tilde{X}}(x)$
where $F_{\tilde{X}}$ is CDF of $\tilde{X}$

- $1 - F_{\tilde{X}}(x) \equiv \mathbb{P}(\tilde{X} > x)$ is *rank* of outcome x of $\tilde{X}$ (the smaller the rank the more favourable the outcome)
- For example, ranks of supremium, median, and infimum of $\tilde{X}$: 0, 1/2, and 1 respectively
- $V(\tilde{X})$ depends on ranks of random outcomes

Evaluation Dictated by Weighting

$V(\tilde{X}) = \int_0^\infty x w'(1 - F_{\tilde{X}}(x)) dF_{\tilde{X}}(x)$

- Risk averse when $w(\cdot)$ is convex (overweighing unfavourable payoffs and underweighing favourable payoffs)
- Risk seeking when $w(\cdot)$ is concave
- Simultaneous risk averse and risk seeking when $w(\cdot)$ is inverse-S shaped

Probability Weighting Functions

- Kahneman and Tversky (1992) weighting

$$w(p) = \frac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{1/\gamma}},$$

- Tversky and Fox (1995) weighting

$$w(p) = \frac{\delta p^\gamma}{\delta p^\gamma + (1-p)^\gamma},$$

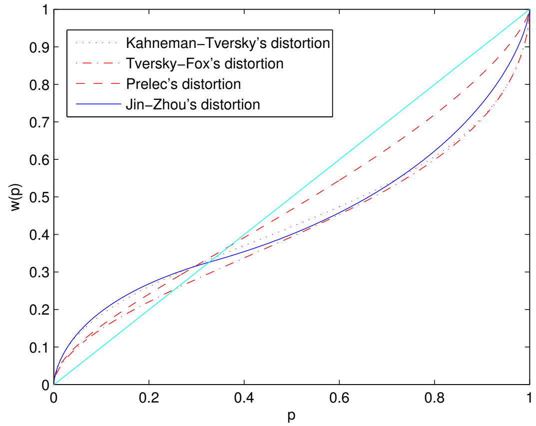
- Prelec (1998) weighting

$$w(p) = e^{-\delta(-\ln p)^\gamma}$$

- Jin and Zhou (2008) weighting

$$w(z) = \begin{cases} y_0^{b-a} k e^{a\mu + \frac{(a\sigma)^2}{2}} \Phi\left(\Phi^{-1}(z) - a\sigma\right) & z \le 1 - z_0, \\ C + k e^{b\mu + \frac{(b\sigma)^2}{2}} \Phi\left(\Phi^{-1}(z) - b\sigma\right) & z \ge 1 - z_0 \end{cases}$$

Inverse-S Shaped Functions



Quiggin's Rank-Dependent Utility Theory

- Rank-dependent utility theory (RDUT): Quiggin (1982), Schmeidler (1989)
- Preference of $\tilde{X} \geq 0$ dictated by an RDUT pair (u, w)

$$\int u(\tilde{X}) d(w \circ \mathbb{P}) \equiv \int_0^\infty w\left(\mathbb{P}(u(\tilde{X}) > x)\right) dx$$

- Two components
  - A concave (outcome) utility function: individuals dislike mean-preserving spread
  - A (usually assumed) inverse-S shaped (probability) weighting function: individuals overweight tails

Primitives

- Present date $t = 0$ and a future date $t = 1$
- Randomness described by $(\Omega, \mathcal{F}, \mathbb{P})$ at $t = 1$
- An atomless *pricing kernel* (or *state-price density* or *stochastic discount factor*) $\tilde{\rho}$ so that any future payoff $\tilde{X}$ is evaluated as $\mathbb{E}[\tilde{\rho}\tilde{X}]$ at present
- An agent with
  - initial endowment $x_0 > 0$ at $t = 0$
  - preference specified by RDUT pair (u, w)

  ... wants to choose future consumption (wealth) $\tilde{c}$

Portfolio/Consumption Choice Model under RDUT

The model

$$\begin{array}{ll} \underset{\tilde{c}}{\text{Max}} & V(\tilde{c}) = \int_0^\infty w\left(\mathbb{P}\left(u(\tilde{c}) > x\right)\right) dx \\ \text{subject to} & \mathbb{E}[\tilde{\rho}\tilde{c}] \leq x_0,\ \tilde{c} \geq 0 \end{array} \qquad \text{(RDUT)}$$

Portfolio/Consumption Choice Model under CPT

- The model

$$\begin{array}{ll} \underset{\tilde{c}}{\text{Max}} & V(\tilde{c}) = \int_0^\infty w_+\left(\mathbb{P}\left(u_+\left((\tilde{c} - \tilde{B})^+\right) > x\right)\right) dx \\ & \quad - \int_0^\infty w_-\left(\mathbb{P}\left(u_-\left((\tilde{c} - \tilde{B})^-\right) > x\right)\right) dx \\ \text{subject to} & \mathbb{E}[\tilde{\rho}\tilde{c}] \leq x_0,\ \tilde{c} \text{ is bounded below} \end{array} \qquad \text{(CPT)}$$

- $u_\pm$ is assumed to be concave so overall value function $u_+(x)\mathbf{1}_{x\geq 0} - u_-(x)\mathbf{1}_{x<0}$ is S-shaped; $u_\pm(0) = 0$
- $w_\pm$ is in general non-convex/non-concave
- $\tilde{B}$ is reference point

Section 2

Portfolio Choice Models: Solutions

Issues Related to the Model

- *Feasibility*: whether there is at least one solution satisfying all the constraints
- *Well-posedness*: whether the supremum value of the problem with a non-empty feasible set is finite (in which case the problem is called *well-posed*) or $+\infty$ (*ill-posed*)
- *Attainability*: whether a well-posed problem admits an optimal solution
- *Uniqueness*: whether an attainable problem has a unique optimal solution

EUT Model Revisited

- The EUT model

$$\begin{array}{ll} \underset{\tilde{c}}{\text{Max}} & V(\tilde{c}) = \int_0^\infty \mathbb{P}\left(u(\tilde{c}) > x\right) dx \equiv \mathbb{E}[u(\tilde{c})] \\ \text{subject to} & \mathbb{E}[\tilde{\rho}\tilde{c}] \le x_0,\ \tilde{c} \ge 0 \end{array} \quad \text{(EUT)}$$

- Lagrange: $\text{Max}_{\tilde{c}}\ \mathbb{E}[u(\tilde{c}) - \lambda\tilde{\rho}\tilde{c}]$
- First-order condition: $\tilde{c}^* = (u')^{-1}(\lambda\tilde{\rho})$
- Determine λ: $\mathbb{E}[\tilde{\rho}(u')^{-1}(\lambda\tilde{\rho})] = x_0$
- Karatzas and Shreve (1998), Jin, Xu and Zhou (2008)

Properties of EUT Solution

- $\tilde{c}^* = (u')^{-1}(\lambda\tilde{\rho})$
- Assume *Inada condition*: $u'(0+) = \infty$, $u'(\infty) = 0$
- $\tilde{c}^* \in (0, +\infty)$
- $\tilde{c}^*$ is a non-increasing function of $\tilde{\rho}$ – *anti-comonotonic* with $\tilde{\rho}$
- Random variables $\tilde{X}$ and $\tilde{Y}$ are called *comonotonic* if

$$\left(\tilde{X}(\omega_1) - \tilde{X}(\omega_2)\right)\left(\tilde{Y}(\omega_1) - \tilde{Y}(\omega_2)\right) \ge 0 \quad a.s.$$

- Random variables $\tilde{X}$ and $\tilde{Y}$ are called *anti-comonotonic* if $\tilde{X}$ and $-\tilde{Y}$ are comonotonic

Solving RDUT Portfolio Choice Model

- The model

$$\begin{array}{ll} \underset{\tilde{c}}{\text{Max}} & V(\tilde{c}) = \int_0^\infty w\left(\mathbb{P}\left(u(\tilde{c}) > x\right)\right) dx \\ \text{subject to} & \mathbb{E}[\tilde{\rho}\tilde{c}] \le x_0, \ \tilde{c} \ge 0 \end{array} \qquad \text{(RDUT)}$$

- u is assumed to be concave
- w is in general non-convex/non-concave
- **Difficulty**: due to **nonlinear** weighting function w, (RDUT) is **not** a concave maximisation problem even though u is concave, and the objective is not an expectation

Literature

- Very little ...
- Shefrin (2008): finite probability space; informal and preliminary
- Carlier and Dana (2008): necessary conditions; no explicit solution

Standing Assumptions

- $\tilde{\rho} > 0$ a.s., **atomless**, with $\mathbb{E}[\tilde{\rho}] < +\infty$
- $u : [0, \infty) \to \mathbb{R}$ is strictly increasing, strictly concave, continuously differentiable on $(0, \infty)$, and satisfies the **Inada** condition: $u'(0+) = \infty$, $u'(\infty) = 0$
- $w : [0, 1] \to [0, 1]$ is strictly increasing and continuously differentiable, and satisfies $w(0) = 0$, $w(1) = 1$

Quantile (Function)

- Given random variable $\tilde{X}$ and its CDF $F_{\tilde{X}} : (-\infty, \infty) \to [0, 1]$
- The (upper) *quantile* $G_{\tilde{X}} : [0, 1) \to [-\infty, \infty]$ is defined as

$$G_{\tilde{X}}(p) := \inf\{x \in \mathbb{R} : F_{\tilde{X}}(x) > p\}, \quad p \in [0, 1)$$

- $G_{\tilde{X}}$ is non-decreasing and right-continuous

The RDUT Model Again

$$\begin{array}{ll} \underset{\tilde{c}}{\text{Max}} & V(\tilde{c}) := \int_0^\infty w\left(\mathbb{P}\left(u(\tilde{c}) > x\right)\right) dx \\ \text{subject to} & \mathbb{E}[\tilde{\rho}\tilde{c}] \le x_0,\ \tilde{c} \ge 0 \end{array} \qquad \text{(RDUT)}$$

Preference and Cost

- The supremium of (RDUT), as a function of x_0, is strictly increasing in x_0 ("the more money the better")
- V is *law-invariant*: $V(\tilde{c}) = V(\tilde{c}')$ whenever $\tilde{c} \sim \tilde{c}'$
- One may substitute $\tilde{c}$ in V by **any** r.v. $\tilde{c}'$ without changing its value – so long as the distribution remains unchanged
- ... which $\tilde{c}'$ is the **cheapest**?
- Consider $\min_{\tilde{c}' \sim \tilde{c}} \mathbb{E}\left[\tilde{\rho}\tilde{c}'\right]$

Hardy–Littlewood Inequality

Lemma

(Jin and Zhou 2008) *We have that $\tilde{c}^* := G(1 - F_{\tilde{\rho}}(\tilde{\rho}))$ solves $\min_{\tilde{c}' \sim \tilde{c}} \mathbb{E}\left[\tilde{\rho}\tilde{c}'\right]$, where G is quantile of $\tilde{c}$. If in addition $-\infty < \mathbb{E}[\tilde{\rho}\tilde{c}^*] < +\infty$, then $\tilde{c}^*$ is the unique optimal solution.*

Hardy, Littlewood and Pòlya (1952), Dybvig (1988)

Changing Decision Variable

- We only need to consider consumption class of the form $\tilde{c} = G(\tilde{Z})$ where G is quantile of $\tilde{c}$ and $\tilde{Z} := 1 - F_{\tilde{\rho}}(\tilde{\rho}) \sim U(0,1)$
- Budget constraint rewritten

$$\mathbb{E}[\tilde{\rho}\tilde{c}] \le x_0 \Leftrightarrow \mathbb{E}\left[F_{\tilde{\rho}}^{-1}(1-\tilde{Z})G(\tilde{Z})\right] \le x_0 \Leftrightarrow \int_0^1 F_{\tilde{\rho}}^{-1}(1-z)G(z)dz \le x_0$$

- Preference measure rewritten

$$\int_0^\infty w\left(\mathbb{P}\left(u(\tilde{c}) > x\right)\right) dx = \int_0^\infty u(x) d\bar{w}(F_{\tilde{c}}(x)) = \int_0^1 u(G(z)) d\bar{w}(z),$$

where $\bar{w}(p) = 1 - w(1-p)$ (*dual* of w)
- Decision variable is now changed from $\tilde{c}$ to its quantile G!

Original RDUT Model

$$\begin{array}{ll} \underset{\tilde{c}}{\text{Max}} & \int_0^\infty w\left(\mathbb{P}\left(u(\tilde{c}) > x\right)\right) dx \\ \text{subject to} & \mathbb{E}[\tilde{\rho}\tilde{c}] \leq x_0,\ \tilde{c} \geq 0 \end{array} \quad \text{(RDUT)}$$

Quantile Formulation

The *quantile formulation* of (RDUT) is:

$$\begin{array}{ll} \underset{G \in \mathbb{G}}{\text{Max}} & U(G(\cdot)) := \int_0^1 u(G(z)) w'(1-z) dz \\ \text{subject to} & \int_0^1 F_{\tilde{\rho}}^{-1}(1-z) G(z) dz \leq x_0 \end{array} \quad \text{(Q)}$$

where

$$\mathbb{G} = \{G : [0,1) \to [0,\infty] \text{ non-decreasing and right-continuous}\},$$

is the set of quantile functions of nonnegative random variables

A concave maximisation problem!

Lagrange Method

- Apply a multiplier λ to the initial budget constraint
- For each λ, we solve the unconstrained problem and derive the optimal solution G^*_λ
- Find λ^* such that $G^*_{\lambda^*}$ binds the initial budget constraint, i.e.,

$$\int_0^1 F_{\tilde{\rho}}^{-1}(1-z) G^*_{\lambda^*}(z) dz = x_0.$$

  Then $G^*_{\lambda^*}$ is optimal to (Q)
- $\tilde{c}^* := G^*_{\lambda^*}(1 - F_{\tilde{\rho}}(\tilde{\rho}))$ is optimal to (RDUT)

Anti-Comonotonicty

- $\tilde{c}^* = G^*_{\lambda^*}(1 - F_{\tilde{\rho}}(\tilde{\rho}))$
- $\tilde{c}^*$ is a non-increasing function of $\tilde{\rho}$
- $\tilde{c}^*$ is **anti-comonotonic** with $\tilde{\rho}$

Unconstrained Problem

- The quantile problem is to solve

$$\begin{array}{ll} \underset{G \in \mathbb{G}}{\text{Max}} & U(G) = \int_0^1 u(G(z)) w'(1-z) dz \\ \text{subject to} & \int_0^1 F_{\tilde{\rho}}^{-1}(1-z) G(z) dz \le x_0 \end{array} \tag{Q}$$

- Given λ, consider

$$\underset{G \in \mathbb{G}}{\text{Max}} \quad U_\lambda(G) = \int_0^1 \left[u(G(z)) w'(1-z) - \lambda F_{\tilde{\rho}}^{-1}(1-z) G(z) \right] dz \tag{Q$_\lambda$}$$

"Brute Force" Solution

- Maximise the integrand over $G(z)$ **pointwisely**
- First-order condition: $u'(G(z))w'(1-z) - \lambda F_{\tilde{\rho}}^{-1}(1-z) = 0$
- $\bar{G}(z) = (u')^{-1}\left(\frac{\lambda F_{\tilde{\rho}}^{-1}(1-z)}{w'(1-z)}\right)$ would solve the quantile formulation ...
- ... provided that $\frac{F_{\tilde{\rho}}^{-1}(1-z)}{w'(1-z)}$ is non-increasing, or $M(z) := \frac{w'(1-z)}{F_{\tilde{\rho}}^{-1}(1-z)}$ is non-decreasing!

Integrability Condition

- We impose the following condition as in classical EUT model to ensure that the optimal value is finite and the optimal solution exists

$$\mathbb{E}\left[u\left((u')^{-1} \left(\frac{\lambda \tilde{\rho}}{w'(F_{\tilde{\rho}}(\tilde{\rho}))} \right) \right) \right] < +\infty, \quad \text{for any } \lambda > 0$$

- In the following, we always assume the integrability condition holds

Solution under Monotonicity Condition

Theorem

(Jin and Zhou 2008) *If $M(z)$ is non-decreasing on $z \in (0,1)$, then the unique optimal solution to (RDUT) is given as*

$$\boxed{\tilde{c}^* = (u')^{-1}\left(\frac{\lambda^* \tilde{\rho}}{w'(F_{\tilde{\rho}}(\tilde{\rho}))} \right)}$$

where λ^* is determined by $E(\tilde{\rho}\tilde{c}^*) = x_0$.

Remark

When there is no probability weighting, it reduces to the classical EUT result.

The Monotonicity Condition

- $M(z) = \frac{w'(1-z)}{F_{\tilde{\rho}}^{-1}(1-z)}$ is automatically non-decreasing if w is concave (risk-seeking)
- If $w \in C^2$ and $G_{\tilde{\rho}} \in C^1$, then M is non-decreasing *iff*

$$\frac{w''(z)}{w'(z)} \leq \frac{G'_{\tilde{\rho}}(z)}{G_{\tilde{\rho}}(z)}, \quad 0 < z < 1$$

where $G_{\tilde{\rho}}$ is the quantile of $\tilde{\rho}$
- However: The condition is **violated** for many known weighting functions and a lognormal pricing kernel

Violation of Monotonicity Condition

Proposition

(He and Zhou 2012) *Suppose $\tilde{\rho}$ is lognormally distributed, i.e.,*

$$F_{\tilde{\rho}}(x) = \Phi\left(\frac{\ln x - \mu}{\sigma}\right)$$

for some μ and $\sigma > 0$, where $\Phi(\cdot)$ is the CDF of standard Normal. For any weighting function in K-T, T-F, P with $0 < \gamma < 1$, there exists $\varepsilon > 0$ such that

$$\frac{w''(z)}{w'(z)} > \frac{G'_{\tilde{\rho}}(z)}{G_{\tilde{\rho}}(z)}, \quad 1 - \varepsilon < z < 1.$$

Probability Weighting Functions

- Kahneman and Tversky (1992) weighting

$$w(p) = \frac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{1/\gamma}},$$

- Tversky and Fox (1995) weighting

$$w(p) = \frac{\delta p^\gamma}{\delta p^\gamma + (1-p)^\gamma},$$

- Prelec (1998) weighting

$$w(p) = e^{-\delta(-\ln p)^\gamma}$$

- Jin and Zhou (2008) weighting

$$w(z) = \begin{cases} y_0^{b-a} k e^{a\mu + \frac{(a\sigma)^2}{2}} \Phi\left(\Phi^{-1}(z) - a\sigma\right) & z \leq 1 - z_0, \\ C + k e^{b\mu + \frac{(b\sigma)^2}{2}} \Phi\left(\Phi^{-1}(z) - b\sigma\right) & z \geq 1 - z_0 \end{cases}$$

Endogenous Portfolio Insurance

Theorem

(He and Zhou 2012) *If there exists $\varepsilon > 0$ such that*

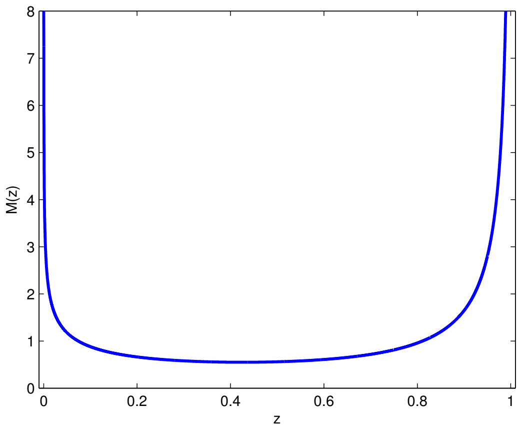
$$\frac{w''(z)}{w'(z)} > \frac{G'_{\tilde{\rho}}(z)}{G_{\tilde{\rho}}(z)}, \quad 1 - \varepsilon < z < 1,$$

then for any optimal solution $\tilde{c}^*$ to (RDUT), we have $\operatorname{essinf}\ \tilde{c}^* > 0$.

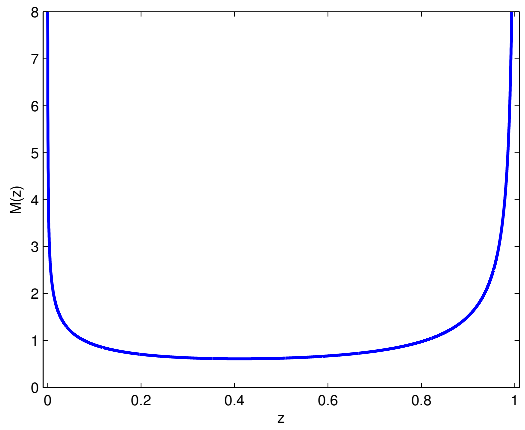
Remark

- Agent will set a positive floor (portfolio/consumption insurance) **endogenously** if $\frac{w''(z)}{w'(z)}$ is sufficiently large when z is near 1
- *Fear index*: $\frac{w''(z)}{w'(z)}$ when z is near 1

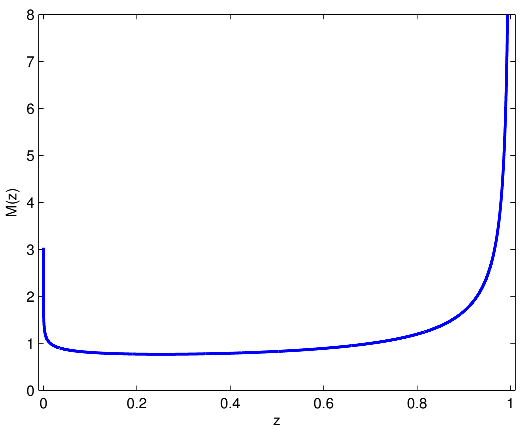
Tversky and Kahneman 1992



Tversky and Fox 1995



Prelec 1998



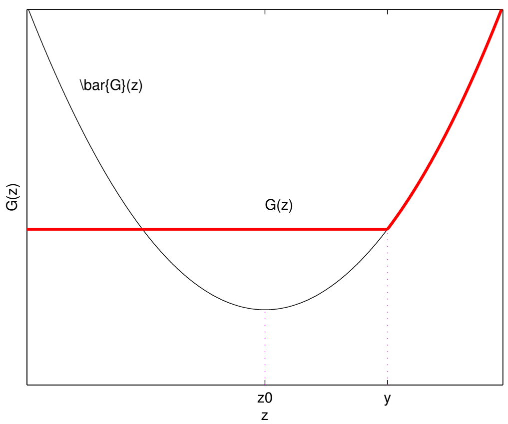
Monotonicity Condition

Assumption

$M(\cdot)$ is continuously differentiable on $(0,1)$ and there exists $0 < z_0 < 1$ such that $M(\cdot)$ is strictly decreasing on $(0, z_0)$ and strictly increasing on $(z_0, 1)$. Furthermore, $\lim_{z\uparrow 1} M(z) = +\infty$.

- Under this assumption, $\bar{G}(z) = (u')^{-1}\left(\frac{\lambda F_{\tilde{\rho}}^{-1}(1-z)}{w'(1-z)}\right) \equiv (u')^{-1}(\lambda/M(z))$ is no longer non-decreasing, so the brutal force (point-wise maximization) fails

Way Out: An Illustration



One Dimensional Optimisation

- We only need to consider quantiles in the form of

$$G(z) := \bar{G}(y)\mathbf{1}_{0<z\leq y} + \bar{G}(z)\mathbf{1}_{y<z<1}$$

  for $z_0 \leq y < 1$
- Substitute above G into

$$U_\lambda(G) = \int_0^1 \left[u(G(z))w'(1-z) - \lambda F_{\tilde{\rho}}^{-1}(1-z)G(z)\right] dz$$

  and find optimal y!
- Optimal y exists and is unique, and independent of λ
- Denote optimal y by z^*, which is shown to be the unique root of

$$\varphi(y) = \int_0^y w'(1-z)dz - M(y)\int_0^y F_{\tilde{\rho}}^{-1}(1-z)dz, \quad z_0 \leq y < 1$$

Solution under Two-Piece Monotonicity Condition

Theorem

(He and Zhou 2012) *Under the specified condition on M, (RDUT) has a unique optimal solution*

$$\boxed{\tilde{c}^* = (u')^{-1}\left(\frac{\lambda^*\tilde{\rho}}{w'(F_{\tilde{\rho}}(\tilde{\rho}))}\right)\mathbf{1}_{(\tilde{\rho}\leq a^*)} + (u')^{-1}\left(\frac{\lambda^* a^*}{w'(F_{\tilde{\rho}}(a^*))}\right)\mathbf{1}_{(\tilde{\rho}>a^*)}}$$

where $a^* > 0$ is the root of

$$\varphi(x) := x(1 - w(F_{\tilde{\rho}}(x))) - w'(F_{\tilde{\rho}}(x))\int_x^\infty s dF_{\tilde{\rho}}(x)$$

on $(F_{\tilde{\rho}}^{-1}(z_0), +\infty)$, and $\lambda^* > 0$ is such that $E(\tilde{\rho}\tilde{c}^*) = x_0$.

Section 3

Quantile Formulation as a General Approach

A Generic Model

$$\begin{array}{ll} \underset{\tilde{c}}{\text{Max}} & V(\tilde{c}) \\ \text{subject to} & E[\tilde{\rho}\tilde{c}] \le x_0, \ \tilde{c} \ge 0 \end{array} \qquad \text{(P)}$$

Basic Assumptions

- V is law invariant
- "The more money the better": $v(x_0) > v(x_0')$ whenever $x_0 > x_0'$, where $v(x_0)$ is the supremum of (P)
- $\tilde{\rho}$ is atomless

Quantile Formulation

- Quantile formulation

$$\begin{array}{ll} \underset{G \in \mathbb{G}}{\text{Max}} & V(G(\tilde{Z})) \\ \text{subject to} & E[F_{\tilde{\rho}}^{-1}(1-\tilde{Z})G(\tilde{Z})] \le x_0 \end{array} \qquad \text{(Q)}$$

  where $\tilde{Z} \sim U(0,1)$
- If G^* is optimal to (Q) then $\tilde{c}^* := G^*(1 - F_{\tilde{\rho}}(\tilde{\rho}))$ is optimal to (P)
- So $\tilde{c}^*$ is **always** anti-comonotonic with $\tilde{\rho}$

Goal Achieving

$$\begin{array}{ll} \underset{\tilde{c}}{\text{Max}} & \mathbb{P}(\tilde{c} \ge b) \\ \text{subject to} & E[\tilde{\rho}\tilde{c}] \le x_0, \ \ \tilde{c} \ge 0 \end{array}$$

where b: the *goal*

Kulldorff (1993), Heath (1993), Browne (1999), Föllmer and Leukert (1999), Spivak and Cvitanić (1999), etc.

Quantile Formulation

- $\mathbb{P}(\tilde{c} \geq b) = \int_0^\infty \mathbf{1}_{(x \geq b)} dF_{\tilde{c}}(x) = \int_0^1 \mathbf{1}_{(F_{\tilde{c}}^{-1}(z) \geq b)} dz$
- Quantile formulation

$$\begin{array}{ll} \underset{G \in \mathbb{G}}{\text{Max}} & U(G) = \int_0^1 \mathbf{1}_{(G(z) \geq b)} dz \\ \text{Subject to} & \int_0^1 F_{\tilde{\rho}}^{-1}(1-z) G(z) dz \leq x_0 \end{array}$$

Solution

Theorem

(He and Zhou 2009) *The unique optimal solution to goal-achieving problem is* $\tilde{c}^* = b\mathbf{1}_{(\tilde{\rho} \leq a)}$ *where* $a > 0$ *is such that* $E[\mathbf{1}_{(\tilde{\rho} \leq a)} \tilde{\rho}] = x_0 / b$*. The optimal value is* $F_{\tilde{\rho}}(a)$*.*

Proof.

Lagrange – pointwise maximisation – binding budget constraint □

SP/A Portfolio Choice Model

$$\begin{array}{ll} \underset{\tilde{c}}{\text{Max}} & V(\tilde{c}) = \int_0^\infty w\left(\mathbb{P}\left(u(\tilde{c}) > x\right)\right) dx \\ \text{subject to} & E[\tilde{\rho}\tilde{c}] \leq x_0,\ \tilde{c} \geq 0, \\ & \mathbb{P}(\tilde{c} \geq A) \geq \alpha \end{array} \quad \text{(SPA)}$$

where

- $A \geq 0$: aspiration level
- α: confidence level

Quantile Formulation

$$\begin{array}{ll} \underset{G \in \mathbb{G}}{\text{Max}} & U(G) := \int_0^1 u(G(z)) w'(1-z) dz \\ \text{Subject to} & \int_0^1 F_{\tilde{\rho}}^{-1}(1-z) G(z) dz \leq x_0, \quad G(1-\alpha) \geq A \end{array} \quad \text{(Q)}$$

Solution

Theorem

(He and Zhou 2012) *Assume that* $x_0 \geq AE\left[\tilde{\rho}\mathbf{1}_{(\tilde{\rho} \leq F_{\tilde{\rho}}^{-1}(\alpha))}\right]$, *and* M *is non-decreasing on* $(0,1)$. *Then the unique optimal solution to (SPA) is given as*

$$\tilde{c}^* = (u')^{-1}\left(\frac{\lambda^*\tilde{\rho}}{w'(F_{\tilde{\rho}}(\tilde{\rho}))}\right)\mathbf{1}_{(\tilde{\rho} \geq F_{\tilde{\rho}}^{-1}(\alpha))} + \left[(u')^{-1}\left(\frac{\lambda^*\tilde{\rho}}{w'(F_{\tilde{\rho}}(\tilde{\rho}))}\right) \vee A\right]\mathbf{1}_{(\tilde{\rho} < F_{\tilde{\rho}}^{-1}(\alpha))}$$

where λ^* *is the one binding the initial budget constraint, i.e.,* $E(\tilde{\rho}\tilde{c}^*) = x_0$.

CPT Portfolio Choice Model

- The model

$$\begin{aligned}\underset{\tilde{c}}{\text{Max}} \quad & V(\tilde{c}) = \int_0^\infty w_+\left(\mathbb{P}\left(u_+\left((\tilde{c}-\tilde{B})^+\right) > x\right)\right)dx \\ & \quad - \int_0^\infty w_-\left(\mathbb{P}\left(u_-\left((\tilde{c}-\tilde{B})^-\right) > x\right)\right)dx \\ \text{subject to} \quad & E[\tilde{\rho}\tilde{c}] \leq x_0, \ \tilde{c} \text{ is bounded below}\end{aligned} \tag{CPT}$$

- $u_\pm$ is assumed to be concave so overall value function $u_+(x)\mathbf{1}_{x\geq 0} - u_-(x)\mathbf{1}_{x<0}$ is S-shaped; $u_\pm(0) = 0$
- $w_\pm$ is in general non-convex/non-concave
- $\tilde{B} = 0$ without loss of generality

Approaches

- Quantile formulation to deal with probability weighting
- A "divide-and-conquer" approach to deal with S-shaped utility function
- Need to solve a **minimisation** problem of a **concave** functional in the quantile space: a combinatorial optimisation in infinite dimension
- Explicit solution; anti-comonotonicity; gambling strategies; leverage: Jin and Zhou (2008)

A Mathematical Programme

Consider a mathematical programme in (a, x_+):

$$\begin{aligned}\text{Max}_{(a,x_+)} \quad & E\left[u_+\left((u'_+)^{-1}\left(\frac{\lambda(a,x_+)\tilde{\rho}}{w'_+(F_{\tilde{\rho}}(\tilde{\rho}))}\right)\right)w'_+(F_{\tilde{\rho}}(\tilde(\rho))\mathbf{1}_{(\tilde{\rho}\leq a)}\right] \\ & -u_-\left(\frac{x_+ - x_0}{E[\tilde{\rho}\mathbf{1}_{\tilde{\rho}>a}]}\right)w_-(1-F(a)) \\ \text{subject to} \quad & \begin{cases} \text{essinf } \tilde{\rho} \leq a \leq \text{esssup } \tilde{\rho}, \ \ x_+ \geq x_0^+, \\ x_+ = 0 \text{ when } a = \text{essinf } \tilde{\rho}, \ \ x_+ = x_0 \text{ when } a = \text{esssup } \tilde{\rho}, \end{cases}\end{aligned} \tag{MP}$$

where $\lambda(a, x_+)$ satisfies $E\left[(u'_+)^{-1}\left(\frac{\lambda(a,x_+)\tilde{\rho}}{w'_+(F_{\tilde{\rho}}(\tilde{\rho}))}\right)\tilde{\rho}\mathbf{1}_{(\tilde{\rho}\leq a)}\right] = x_+$

Grand Solution

Theorem

(Jin and Zhou 2008) *Assume $u_-(\cdot)$ is strictly concave at 0 and M is non-decreasing. Let (a^*, x_+^*) solves (MP). Then the optimal solution to (CPT) is*

$$\tilde{c}^* = \left[(u'_+)^{-1} \left(\frac{\lambda \tilde{\rho}}{w'_+(F_{\tilde{\rho}}(\tilde{\rho}))} \right) \right] \mathbf{1}_{(\tilde{\rho} \le a^*)} - \left[\frac{x_+^* - x_0}{E[\tilde{\rho} \mathbf{1}_{(\tilde{\rho} > a^*)}]} \right] \mathbf{1}_{(\tilde{\rho} > a^*)}.$$

Interpretations and Implications

$$\tilde{c}^* = \left[(u'_+)^{-1} \left(\frac{\lambda \tilde{\rho}}{w'_+(F_{\tilde{\rho}}(\tilde{\rho}))} \right) \right] \mathbf{1}_{(\tilde{\rho} \le a^*)} - \left[\frac{x_+^* - x_0}{E[\tilde{\rho} \mathbf{1}_{(\tilde{\rho} > a^*)}]} \right] \mathbf{1}_{(\tilde{\rho} > a^*)}$$

- Future world divided by "good" states (where you have gains) and "bad" ones (losses), *completely* determined by whether $\tilde{\rho} \le a^*$ or $\tilde{\rho} > a^*$
- Agent buy claim $\left[(u'_+)^{-1} \left(\frac{\lambda \tilde{\rho}}{w'_+(F_{\tilde{\rho}}(\tilde{\rho}))} \right) \right] \mathbf{1}_{(\tilde{\rho} \le a^*)}$ at cost $x_+^* \ge x_0$ and sell $\left[\frac{x_+^* - x_0}{E[\tilde{\rho} \mathbf{1}_{(\tilde{\rho} > a^*)}]} \right] \mathbf{1}_{(\tilde{\rho} > a^*)}$ to finance shortfall $x_+^* - x_0$
- Agent not only invests in stocks, but also generally takes a **leverage** to do so
- Optimal strategy is a *gambling* policy, betting on the good states while accepting a **known** loss on the bad

Section 4

Continuous Time and Time Inconsistency

A Continuous-Time Economy

- An economy in which $m+1$ securities traded continuously
- Market randomness described by a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \ge 0}, \mathbb{P})$ along with an $\mathbb{R}^m$-valued, $\mathcal{F}_t$-adapted standard Brownian motion $W(t) = (W^1(t), \cdots, W^m(t))'$ with $\{\mathcal{F}_t\}_{t \ge 0}$ generated by $W(\cdot)$
- A bond whose price process $S_0(t)$ satisfies

$$dS_0(t) = r(t) S_0(t) dt; \;\; S_0(0) = s_0$$

- m stocks whose price processes $S_1(t), \cdots S_m(t)$ satisfy stochastic differential equation (SDE)

$$dS_i(t) = S_i(t) \left(\mu_i(t) dt + \sum_{j=1}^{m} \sigma_{ij}(t) dW^j(t) \right); \; S_i(0) = s_i$$

Tame Portfolios

- Let
$$\sigma(t) := (\sigma_{ij}(t))_{m\times m}$$
$$B(t) := (\mu_1(t) - r(t), \cdots, \mu_m(t) - r(t))'$$
- An $\mathcal{F}_t$-progressively measurable process $\pi(t) = (\pi_1(t), \cdots, \pi_m(t))'$ represents a (monetary) *portfolio*, where $\pi_i(t)$ is the capital amount invested in stock i at t
- A portfolio $\pi(\cdot)$ is *admissible* if
$$\int_0^T |\sigma(t)'\pi(t)|^2 dt < +\infty, \int_0^T |B(t)'\pi(t)| dt < +\infty, \text{ a.s.}$$
- An agent has an initial endowment x_0

Wealth Equation

- Wealth process $x(\cdot)$ follows the *wealth equation*
$$\begin{cases} dx(t) & = [r(t)x(t) + B(t)'\pi(t)]dt + \pi(t)'\sigma(t)dW(t) \\ x(0) & = x_0 \end{cases}$$
- An admissible portfolio $\pi(\cdot)$ is called *tame* if the corresponding wealth process $x(\cdot)$ is essentially lower bounded

Market Assumptions

Market assumptions:

(i) There exists $k \in \mathbb{R}$ such that $\int_0^T r(t)dt \geq k$,

(ii) $\int_0^T [\sum_{i=1}^m |b_i(t)| + \sum_{i,j=1}^m |\sigma_{ij}(t)|^2]dt < +\infty$,

(iii) $\mathrm{Rank}\ (\sigma(t)) = m,\ t \in [0, T]$,

(iv) There exists an $\mathbb{R}^m$-valued, uniformly bounded, $\mathcal{F}_t$-progressively measurable process $\theta(\cdot)$ such that $\sigma(t)\theta(t) = B(t)$

Pricing Kernel

- Define
$$\rho(t) := \exp\left\{-\int_0^t \left[r(s) + \frac{1}{2}|\theta(s)|^2\right] ds - \int_0^t \theta(s)'dW(s)\right\}$$
- Denote $\tilde{\rho} := \rho(T)$
- Assume that $\tilde{\rho}$ is atomless

Continuous-Time Portfolio Choice under EUT

$$\begin{array}{ll} \text{Max} & E[u(x(T))] \\ \text{subject to} & (x(\cdot), \pi(\cdot)):\ \text{tame and admissible pair} \end{array} \quad (1)$$

where u is a concave utility function satisfying the usual assumptions

Time Consistency

- *Time (dynamic) consistency*: $E(\tilde{c}|\mathcal{F}_t) = E[E(\tilde{c}|\mathcal{F}_s)|\mathcal{F}_t]$ $\forall t < s$
- *Bellman's principle*: If $(x^*(\cdot), \pi^*(\cdot))$ is optimal for problem starting at $(0, x_0)$, then $(x^*(\cdot), \pi^*(\cdot))|_{[s,T]}$ is also optimal for problem starting at $(s, x^*(s))$, for any $s \in [0, T]$

Forward Approach: Dynamic Programming

- Let v be the *value function* corresponding to (1): $v(t, x)$ is the optimal value of (1) if the initial time is t (instead of 0) and the initial budget is x (instead of x_0)
- v satisfies the *Hamilton–Jacobi–Bellman (HJB) equation*:

$$\begin{cases} v_t + \sup_{\pi \in \mathbb{R}^m} \left(\frac{1}{2}\pi'\sigma\sigma'\pi v_{xx} + B\pi v_x\right) + rxv_x = 0, & (t,x) \in [0,T) \times \mathbb{R}, \\ v(T,x) = u(x) \end{cases} \quad (2)$$

- *Verification theorem*: optimal portfolio

$$\pi^*(t,x) = -(\sigma(t)')^{-1}\theta(t)\frac{v_x(t,x)}{v_{xx}(t,x)} \quad (3)$$

Backward Approach: Replication

- One solves first a static optimization problem in terms of terminal wealth, $\tilde{c}$:

$$\begin{array}{ll} \text{Max} & E[u(\tilde{c})] \\ \text{subject to} & E[\tilde{\rho}\tilde{c}] \le x_0;\ \tilde{c} \text{ is } \mathcal{F}_T\text{-measurable} \end{array} \quad (4)$$

- $\tilde{c}^* = (u')^{-1}(\lambda^*\tilde{\rho})$
- Solve *backward stochastic differential equation (BSDE)* in $(x^*(\cdot), z^*(\cdot))$:

$$dx^*(t) = [r(t)x^*(t) + \theta(t)'z^*(t)]dt + z^*(t)'dW(t);\ x^*(T) = \tilde{c}^* \quad (5)$$

- Setting $\pi^*(t) = (\sigma(t)')^{-1}z^*(t)$ and $(x^*(\cdot), \pi^*(\cdot))$ is optimal pair

Time Inconsistency under Probability Weighting

- *Choquet expectation*: $\hat{E}[\tilde{X}] = \int \tilde{X} d(w \circ \mathbb{P}) = \int_0^\infty w(\mathbb{P}(\tilde{X} > x))dx$
- How to define "conditional Choquet expectation"?
- Even if a conditional Choquet expectation can be defined, it will not satisfy $\hat{E}(\tilde{c}|\mathcal{F}_t) = \hat{E}[\hat{E}(\tilde{c}|\mathcal{F}_s)|\mathcal{F}_t]$
- Dynamic programming falls apart

Replication: Pre-Committed Strategies

- Solve a static optimisation problem (with probability weighting) in terms of terminal wealth
- Such a problem has been solved by our approach developed
- Find a dynamic portfolio replicating the obtained optimal terminal wealth
- Such a portfolio is an optimal *pre-committed* strategy (Jin and Zhou 2008, He and Zhou 2011)

Time Inconsistency and Equilibrium Strategies

- Sources of time inconsistency: probability weighting, variance (mean field), state-dependent preferences, hyperbolic discounting ...
- Pre-committed strategies exercised only in shorter time period, special circumstances, or a select group of people
- Equilibrium strategies: Nash equilibrium strategies where the players are incarnations of oneself at different time periods
- Ekeland and Pirvu (2008), Hu, Jin and Zhou (2012,2015), Bjork, Murgoci and Zhou (2012) ...

Section 7

Summary and References

Summary

- Technical challenge arising from probability weighting: non-convex optimisation in infinite dimension
- Approach – quantile formulation
- Think of **distribution/quantile** of future consumption!
- A monotonicity condition - its economic interpretation
- Quantile formulation can treat a much broader class of problems, including behavioural and neoclassical ones
- Behavioural models are typically time inconsistent due to probability weighting

Summary (Cont'd)

- Conditions on an RDUT economy provided under which the Arrow-Debreu equilibrium exists uniquely
- At equilibrium one cannot distinguish between RDUT and EUT economies; however, representative risk aversion level is (possibly substantially) altered
- Asset prices not only depend upon level of risk aversion and beta, but also upon agents' belief on economic growth
- Probability weighting may offer a new way of thinking in explaining many economic phenomena

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Section 8

Final Words

Two Revolutions in Finance

- Finance ultimately deals with **interplay** between market risk and human judgement
- History of financial theory over the last 50 years characterised by two revolutions
  - Neoclassical (maximising) finance starting 1960s: *Expected utility maximisation, CAPM, efficient market theory, option pricing*
  - Behavioural finance starting 1980s: *Cumulative prospect theory, SP/A theory, regret and self-control, heuristics and biases*

Neoclassical vs Behavioural

- Neoclassical: the world and its participants are rational "wealth maximisers"
- Behavioural: emotion and psychology influence our decisions when faced with uncertainties, causing us to behave in unpredictable, inconsistent, incompetent, and most of all, irrational ways
  - A relatively new field that attempts to explain how and why emotions and cognitive errors influence investors and create stock market anomalies such as bubbles and crashes
  - It seeks to explore the consistency and predictability in human flaws so that such flaws can be avoided or even exploited for profit

Do We Need Both?

- Foundations of the two
  - Neoclassical finance: Rationality (correct beliefs on information, risk aversion) – A **normative** theory
  - Behavioural finance: The lack thereof (experimental evidence, cognitive psychology) – A **descriptive** theory
- Do we need both? *Absolutely yes!*
  - Neoclassical finance tells what people **ought** to do
  - Behavioural finance tells what people **actually** do
  - Robert Shiller (2006), "the two ... have always been interwind, and some of the most important applications of their insights will require the use of both approaches"

Quantitative Behavioural Finance

- "Quantitative behavioural finance" leads to new problems in mathematics, engineering and finance
- But ... is it justified: to **rationally** and **mathematically** account for irrationalities?
- Irrational behaviours are by no means random or arbitrary
- "misguided behaviors ... are systematic and predictable – making us predictably irrational" (Dan Ariely, *Predictably Irrational*, Ariely 2008)
- We use CPT/RDUT/SPA and specific value functions as the carrier for exploring the "predictable irrationalities"
- *Quantitative behavioural finance*: research is in its infancy, yet potential is unlimited – or so we believe