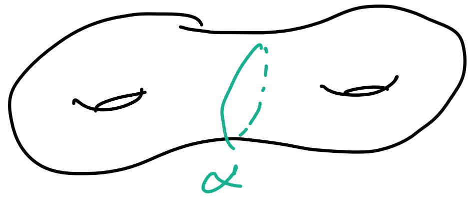
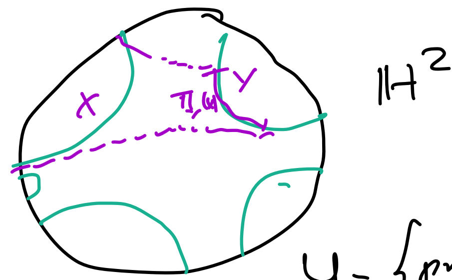
Projection complexes

Let





$\mathcal{Y} = \left\{ \begin{matrix} \text{preimages} \\ \text{of } \alpha \end{matrix} \right\}$

Fact. There is $\Theta \geq 0$ s.t.

$\pi_Y(X)$ = nearest point projection of X to Y

(P1) $\operatorname{diam} \pi_Y(X) \leq \Theta \quad \forall X \neq Y.$

Let $d_Y(X, Z) := \operatorname{diam}\left(\pi_Y(X) \cup \pi_Y(Z)\right)$

Behrstock inequality { (P2) If $d_Y(X,Z) > \Theta$, then $d_X(Y,Z) \leq \Theta$

$d_Z(Y,X) \leq \Theta$

(P3) For all X, Z, $\{Y \mid d_Y(X,Z) > \Theta\}$ is finite.

Projection Data ① Collection $\mathcal{Y} = \{X, Y, Z, \dots\}$ of metric spaces

② Collection of projections $\pi_Y(X) \subseteq Y$ non-empty for $X \neq Y$.

③ Assume we have $\Theta \geq 0$ s.t. P1, P2, P3 hold.

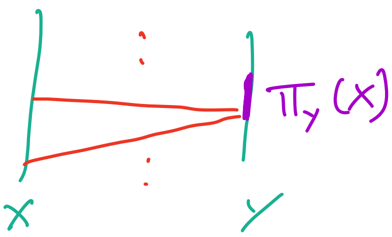
Goal: "reconstruct" ambient space $\mathcal{Y}$

want to do so equivariantly.

Basic strategy for building $\mathcal{Y}$: start with

$$\coprod_{X\in\mathcal{Y}} X$$

Decide when to connect X and Y.



Crushing every $X\in\mathcal{Y}$ to a point results in a graph (of red edges).

← given method of putting edges

Th. This graph is $\underbrace{\text{QI to a tree}}_{\text{is a quasitree}}$. ⇝ get "quasi-tree of metric spaces"

(B.-Bromberg-Fujiwara)
—Sisto

Plan: prove theorem under stronger hypothesis

(P2++) If $d_X(Y,Z) > \Theta$, then $\pi_X(Y) = \pi_X(Z)$

⇝ technical part: prove if P1–P3 hold, then can perturb to get P2++

Construction: why not take $K = 2\Theta$?

Choose $K \geq 2\Theta$. (P3) to finite

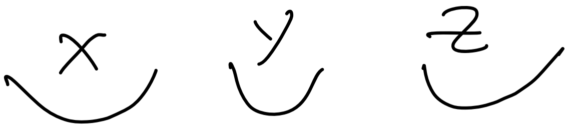
Define $\mathcal{Y}_K(X,Z) = \{Y\in\mathcal{Y}-\{X,Z\} \mid d_Y(X,Z) > K\}$
$\cup \{X,Z\}$

Claim $\mathcal{Y}_K(X,Z)$ is naturally ordered.

property: $\Pi_{Y_j}(Y_i) = \begin{cases} \pi_{Y_j}(X), & i<j \\ \Pi_{Y_j}(Z), & i>j \end{cases}$

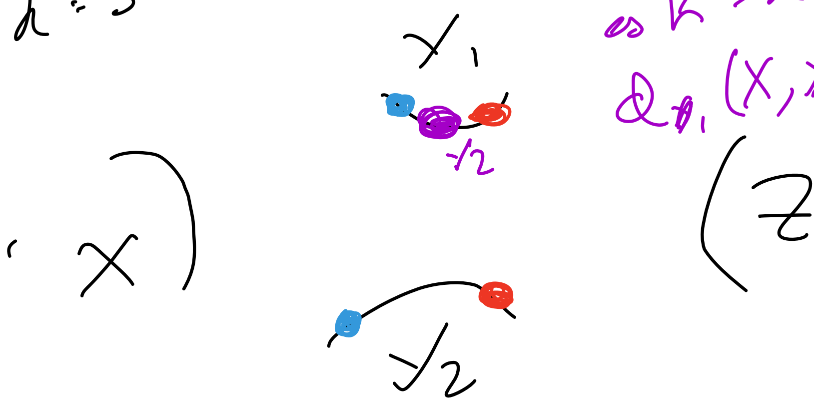
$k := |\mathcal{Y}_K(X,Z)| - 1.$

If $k=2$, nothing to prove.



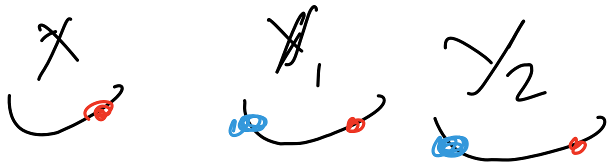
$k=3$

as $K > 2\Theta$, $d_{Y_1}(X, Y_2) > \Theta$ or $d_{Y_1}(Z, Y_2) > \Theta$.

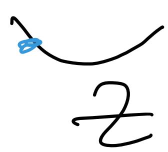


Say $d_{Y_1}(X, Y_2) > \Theta$. Then by P2++, $\Pi_{Y_2}(X) = \Pi_{Y_2}(Y_1)$

so



Then we P2++ for Y_1, Y_2, Z to get $\Pi_{Y_1}(Y_2) = \Pi_{Y_1}(Z)$.



For general case, induct. what to add in Y_i:

$\pi_{Y_i}(Y)$ is $\pi_{Y_i}(X)$ or $\pi_{Y_i}(Z)$.

$\pi_Y(Y_i)$ is -- --

Look for a place where projection of Y jumps from $\pi_{Y_i}(X)$ to $\pi_{Y_{i+1}}(Z)$.

Defn $P_K(\mathcal{Y})$ is a graph.

Vertices are elements of $\mathcal{Y}$.

Connect X to Z by an edge if $d_Y(X,Z) \leq K$ $\forall Y$.

Claim $P_K(\mathcal{Y})$ is connected and $\mathcal{Y}_K(X,Z)$ is a path from X to Z.

Would have issue if for Y

$X \cdot - Y_i \cdot Y_{i+1} -- Z$

have Y s.t. $d_Y(Y_i, Y_{i+1}) > K$.

Manning's Bottleneck Criterion

Suppose X connected graph, $\Delta \geq 0$.

Assume that for any two vertices v, w, have path $p_{v,w}$ s.t. for any $\overset{v}{\cdot} \underline{\quad q \quad} \overset{w}{\cdot}$, $N_\Delta(q) \supseteq p_{v,w}$.

Then X is a quasi-tree.

Ex. Every graph satisfies this.