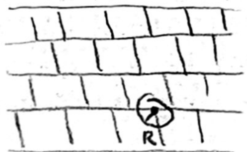
# Projection complexes & Applications to Mapping Class Groups

## Asymptotic dimension

**Definition:** A metric space $X$ has asymptotic dimension at most $n$ if for all $R > 0$ there exists a cover $\mathcal{U}$ of $X$ s.t.
 i) every $R$-ball intersects at most $n+1$ elements of $\mathcal{U}$;
 ii) $\operatorname{mesh} \mathcal{U} := \sup_{U \in \mathcal{U}} \operatorname{diam}(U) < \infty$.

Example: $\operatorname{asdim}(\mathbb{R}^2) \leq 2$ by brick-cover with side-lengths $\gg R$.
Similarly, $\operatorname{asdim}(\mathbb{R}^n) \leq n$.




Fact: $\operatorname{asdim}(\mathbb{R}^n) = n$.

Proof. Let $\mathcal{U}$ be an open cover of $\mathbb{R}^n$ by uniformly open sets. ~~We are~~ Let $N(\mathcal{U}) =: N$ be the nerve of the cover. Using a partition of unity one gets a map $\mathbb{R}^n \xrightarrow{f} N$. For each $U \in \mathcal{U}$ picking a point $p_U \in U$, and extending linearly gives a map $N \xrightarrow{g} \mathbb{R}^n$. Since $\operatorname{mesh} \mathcal{U} < \infty$, $g \circ f$ has bounded distance from $id_{\mathbb{R}^n}$. Since $H_n^{\text{loc.fin.}}(\mathbb{R}^n) \neq 0$, also $H_n^{\text{loc.fin.}}(N) \neq 0$. So $\dim(N) \geq n$, and hence multiplicity $\geq n+1$. $\square$

If $X \overset{q.i.}{\sim} Y$, then $\operatorname{asdim}(X) = \operatorname{asdim}(Y)$. also true for maps generated by coboundeed inclusion + injection that sends uniformly bounded collection of sets on one side to the same on the other side instead of biLip.
 So fin. generated groups have a well-defined asdim.
Also, if $H \leq G$, then $\operatorname{asdim}(H) \leq \operatorname{asdim}(G)$.

Examples: i) $\operatorname{asdim}(\mathbb{Z}^n) = n$.
ii) $\operatorname{asdim}(\ell^2) = \infty$.
iii) $\operatorname{asdim}(\text{bounded space}) = 0$.

More generally, a space has $\operatorname{asdim} = 0$ if it has the "archipelago-structure", i.e., there is a sequence of Partitions $\mathcal{P}_1, \mathcal{P}_2, \ldots$ s.t. (choose $n>0$ s.t. $a_n \gg R$)
- $\operatorname{mesh} \mathcal{P}_n < \infty$
- distance between disjoint elements in $\mathcal{P}_n \geq a_n \longrightarrow \infty$.

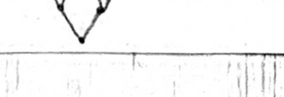
Example: Consider $\mathbb{Q}$ with the $p$-norm $\| p^n \frac{a}{b} \| := p^{-n}$ ($p \nmid a, b$). This is an ultra-norm, i.e., $\|a+b\| \leq \max\{\|a\|, \|b\|\}$. Note $B_R(x_1) = B_R(x_2)$ if $d(x_1, x_2) \leq R$, and $\operatorname{dist}(B_R(x_1), B_R(x_2)) \geq R$ if $d(x_1, x_2) > R$. So $\mathcal{P}_n := \{n\text{-balls}\}$ is an archipelago-structure. So $\operatorname{asdim}(\mathbb{Q}_p) = \operatorname{asdim}(\mathbb{Q}) = 0$ (with $p$-norm).
 Note that for $\mathbb{Q} \subseteq \mathbb{R}$, $\operatorname{asdim}(\mathbb{Q}) = 1$ (usual metric).

**Theorem** (classical Hurewicz): Let $X, Y$ be compact metric spaces, and $f: X \to Y$ s.t. $\dim f^{-1}(y) \leq n$ for all $y \in Y$. Then $\dim X \leq \dim Y + n$ (top).

**Theorem** (Ball - Hurewicz theorem): Let $f: X \to Y$ be a Lip. map between metric spaces. If for all $R \geq 0$ the collection $\{f^{-1}(B_R)\}$ has $\operatorname{asdim} \leq n$ uniformly, then
$$\operatorname{asdim}(X) \leq \operatorname{asdim}(Y) + n.$$

Examples: i) If $1 \to A \to B \to C$, then $\operatorname{asdim} B \leq \operatorname{asdim} A + \operatorname{asdim} C$.
ii) $\operatorname{asdim}(\mathbb{H}^2) \leq 2$ by taking a Busemann function $b_\xi : \mathbb{H}^2 \to \mathbb{R}$.

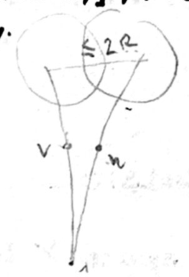

iii) $G = KAN$, then $K \backslash G = AN \longrightarrow N \rightsquigarrow \operatorname{asdim} = \dim$
iv) Let $T$ be a tree with all side lengths $= 1$. Then $\operatorname{asdim}(T) \leq 1$.
 Set $h := d(\cdot, o): T \to \mathbb{R}$. One can show that $\{h^{-1}([a,b]) \mid b - a \leq R\}$ has a uniform archipelago-structure.

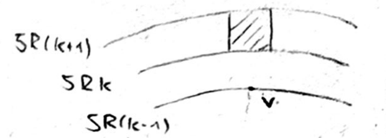


**Theorem** (Gromov): If $G$ is a hyperbolic group, then $\operatorname{asdim}(G) < \infty$.

or $G_v \cap B_R(x)$ and $G_w \cap B_R(x) \neq \emptyset$ — same geod. from 1 to $g$

**Proof.** For $v \in \partial B_{SR(k-1)}(1)$ define $G_v := \{ g \mid |g| \in [SRk, SR(k+1)],\ v \in [1,g] \}$.
If $G_v \cap G_w \neq \emptyset$, then $d(v,w) \leq 2\delta$ (if $R \gg \delta$) by hyperbolicity.
So multiplicity $\leq 2 \# B(2\delta, 1) - 1$. ← this also gives $\operatorname{asdim} \leq 1$ for trees. □




# Projection complexes

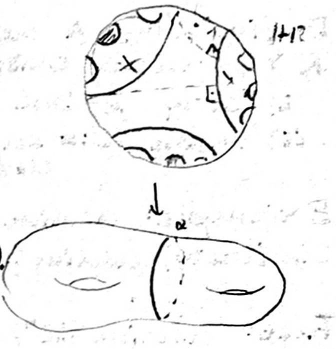
## Motivating example:

Fix $\alpha \subseteq \Sigma_{g \geq 2}$ simple closed curve, and $\mathcal{Y} := \pi^{-1}(\alpha) \subseteq 2^{\mathbb{H}^2}$ (collection of lifts).
For $X \neq Y \in \mathcal{Y}$ there is a nearest point projection $\pi_Y(X)$.
There exists $\Theta = \Theta(\alpha)$ s.t. "think of bounded sets as points"

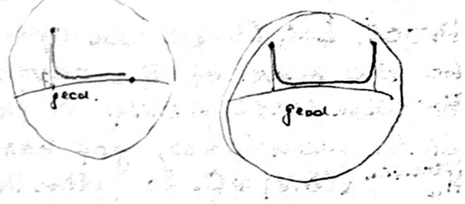
(P1) $\operatorname{diam} \pi_Y(X) \leq \Theta$ for all $X \neq Y$;

(P2) If $d_Y(X,Z) > \Theta$, then $d_X(Y,Z) \leq \Theta$, where $d_Y(X,Z) := \operatorname{diam}(\pi_Y(X) \cup \pi_Y(Z))$.

(P3) For all $X, Z$ the set $\{ Y \mid d_Y(X,Z) > \Theta \}$ is finite.




See the picture on the right to why this holds in $\mathbb{H}^2$ & (or Gromov hyp spaces).

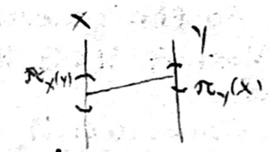



The *projection data* is a
- a collection $\mathcal{Y}$ of metric spaces;
- for all $X \neq Y \in \mathcal{Y}$ a non-empty projection $\pi_Y(X) \subseteq Y$

s.t. for some $\Theta \geq 0$ (P1)–(P3) hold.

Goal: Build an ambient space $\mathbb{Y}$.

Basic strategy: Consider $\coprod_{X \in \mathcal{Y}} X$, and decide when to connect $X$ to $Y$.
Crushing every $X \in \mathcal{Y}$ to a point results in a graph.




"projection complex"

**Theorem** (Bestvina–Bromberg–Fujiwara): This graph is quasi-isometric to a tree. e.g. subtrees in a big tree


The *plan* is to prove the thm under the stronger assumption

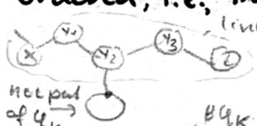
(P2++) If $d_Y(X,Z) > \Theta$, then $\pi_X(Y) = \pi_X(Z)$.

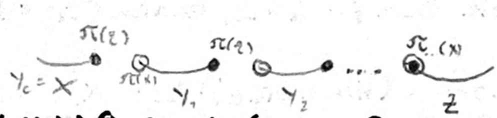
The *technical part* (which is omitted) is that if (P1)–(P3) are satisfied, one can perturb by a bounded amount to satisfy (P1), (P2++), (P3).

## Construction

Choose $K \geq 2\Theta$ and define $\mathcal{Y}_K(X,Z) := \{ Y \in \mathcal{Y} \setminus \{X,Z\} \mid d_Y(X,Z) > K \} \cup \{X, Z\}$ (finite by (P3)). This is naturally linearly ordered, i.e., there ex. an enumeration $X = Y_0, \dots, Y_n = Z$ s.t.

$$\pi_{Y_j}(Y_i) = \begin{cases} \pi_{Y_j}(X), & i < j \\ \pi_{Y_j}(Z), & i > j \end{cases}$$




Indeed, if $k=2$, there is nothing to prove. For $k=3$, either $d_{Y_1}(X, Y_2) > \Theta$ or $d_{Y_2}(Z, Y_1) > \Theta$ since $K \geq 2\Theta$. Say $d_{Y_1}(X, Y_2) > \Theta$. Then by (P2++) for $X, Y_1, Y_2$, yields $\pi_{Y_2}(X) = \pi_{Y_2}(Y_1)$. Similarly, (P2++) for $Y_1, Y_2, Z$, $\pi_{Y_1}(Y_2) = \pi_{Y_1}(Z)$. The case $k > 3$ is done by induction (using case $k=3$).

Define a graph $\mathcal{P}_K(\mathcal{Y})$ with vertices the elements of $\mathcal{Y}$, and connect $X, Z$ by an edge if $d_Y(X,Z) \leq K$ for all $Y$.

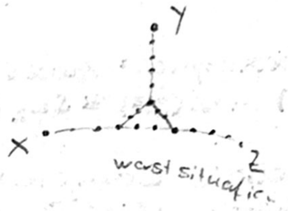
Claim: $\mathcal{P}_K(\mathcal{Y})$ is connected, and $\mathcal{Y}_K(X,Z)$ is a path from $X$ to $Z$. "standard path"

Proof. Suppose not, so there is some $i$ and some $Y$ (outside $\mathcal{Y}_K$) s.t. $d_Y(Y_i, Y_{i+1}) > K$. Applying (P2++) to $Y, Y_i, Y_{i+1}$ yields $\pi_{Y_i}(Y) = \pi_{Y_i}(Y_{i+1})$ and $\pi_{Y_{i+1}}(Y) = \pi_{Y_{i+1}}(Y_i)$. Applying (P2++) to $X, Y_i, Y$ yields $\pi_Y(X) = \pi_Y(Y_i)$, and similarly $\pi_Y(Z) = \pi_Y(Y_{i+1})$. So $Y \in \mathcal{Y}_K(X,Z)$. □

**Theorem** (Manning's Bottleneck criterion): Suppose $X$ is a connected graph, and $\Delta \geq 0$ s.t. the following holds. Assume that for any vertices $v, w$ there ex. a path $p_{v,w}$ from $v$ to $w$ s.t. for any path $q$ from $v$ to $w$ $p_{v,w} \subseteq N_\Delta(q)$. Then $X$ is a quasi-tree.

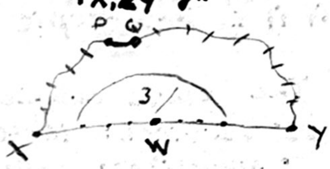
**Exercise**: Show that the Farey graph is a quasi-tree.

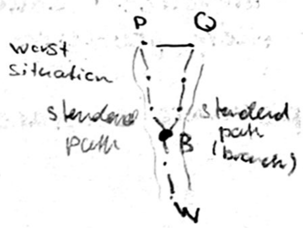
**Exercise**: For all $X, Y, Z \in \mathcal{Y}$ the union $\mathcal{Y}_K(X,Y) \cup \mathcal{Y}_K(Y,Z)$ contains all but at most two vertices of $\mathcal{Y}_K(X,Z)$, and if two, they are adjacent. Hint: Apply (P2+?)




**Theorem**: $\mathcal{P}_K(\mathcal{Y})$ is a quasi-tree.

*Proof.* We show that the bottleneck criterion is satisfied with $\Delta = 3$, and $p_{X,Y} = \mathcal{Y}_K(X,Y)$. Assume not, i.e., there ex. $X, Y$, $W \in \mathcal{Y}_K(X,Y)$ and a path $q$ from $X$ to $Y$ s.t. $q \cap B_3(W) = \emptyset$. It suffices to show that for $P, Q \in q$ with $d_{\mathcal{P}}(P,Q) = 1$, it holds $\pi_W(P) = \pi_W(Q)$ (this will lead to a contraction since by going along $q$ would show $\pi_W(X) = \pi_W(Y)$, but $d_W(X,Y) \geq K \geq 2\Theta$ since $W \in \mathcal{Y}_K(X,Y)$. By the above exercise, the worst situation that can happen is as in the picture. But then $\pi_W(P) = \pi_W(B) = \pi_W(Y)$. $\square$



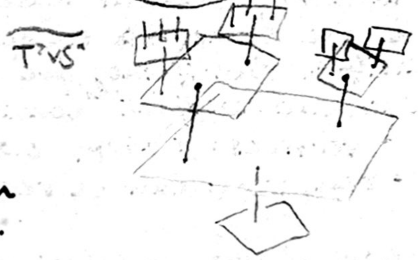

**Exercise**: Set $\ell :=$ length of standard path from $X$ to $Y$. Then
$$\left\lfloor \frac{\ell}{2} \right\rfloor \pm 1 \leq d_{\mathcal{P}}(X,Y) \leq \ell.$$
standard paths are quasi-geod.

## Examples where (P1)–(P3) hold

1) Let $G$ be a non-elementary hyperbolic group. For $g$ of infinite order there exists a quasi-axis $\ell$ of $g$, i.e., $g(\ell) = \ell$, $g|_\ell$ is a translation. Set $\mathcal{Y} := \{$translates of $\ell$ by $G\}$ (and assume that no two distinct elements of $\mathcal{Y}$ are parallel). Then (P1)–(P3) hold as in the surface case.


So $G \curvearrowright \mathcal{P}_K(\mathcal{Y})$, i.e., every non-elementary hyperbolic group $G$ acts non-elementary on a quasi-tree.
need to prove something

2) Consider the picture and $\mathcal{Y} := \{$planes in $\widetilde{T^2 \vee S^1}\}$. Then (P1)–(P3) hold for $\Theta = 0$, and for $K = 0$ $\mathcal{P}_K(\mathcal{Y})$ is obtained by collapsing the planes.


## Quasi-trees of metric spaces

assume all spaces in $\mathcal{Y}$ are graphs, and projections are subgraphs

Define $\mathcal{C}_K(\mathcal{Y})$ as follows. Start with $\coprod_{X \in \mathcal{Y}} X$, and if $d_{\mathcal{P}}(X,Y) = 1$ then attach edges from every vertex in $\pi_Y(X)$ to every vertex in $\pi_X(Y)$.

## Properties

i) If each attached edge is assigned the length $2K$, then
- the inclusion $X \hookrightarrow \mathcal{C}_K(\mathcal{Y})$ is convex and isometric;
- the nearest proj. of $X \subseteq \mathcal{C}_K(\mathcal{Y})$ to $Y \subseteq \mathcal{C}_K(\mathcal{Y})$ is within uniform distance from $\pi_Y(X)$.

ii) If all $Y \in \mathcal{Y}$ have asdim $\leq n$ uniformly, then asdim $\mathcal{C}_K(\mathcal{Y}) \leq n+1$ (Hurewicz).

iii) If all $Y \in \mathcal{Y}$ are quasi-trees with uniform $\Delta$, then $\mathcal{C}_K(\mathcal{Y})$ is a quasi-tree.

iv) If all $Y \in \mathcal{Y}$ are $\delta$-hyp., then $\mathcal{C}_K(\mathcal{Y})$ is hyperbolic.

**Exercise**: In the situation of example 1) above, $G \curvearrowright \mathcal{C}_K(\mathcal{Y})$ quasi-tree is non-trivial since $g$ is a translation along $\ell$.

# Mapping Class Group

Let $S$ be a closed orientable surface with a finite number of points removed. The *mapping class group* of $S$ is

$$\mathrm{Mod}(S) := \mathrm{Homeo}_+(S)/\text{isotopy}.$$

Examples: $\mathrm{Mod}(S^2) = \{1\}$.
ii) $\mathrm{Mod}(T^2) \cong SL_2(\mathbb{Z})$.

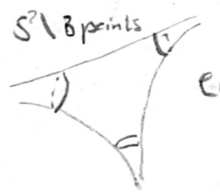
## Curve complex and graph

When $\chi(S) < 0$, then $S$ admits a complete hyp. metric of finite area.
A simple closed curve is essential if it does not bound $D^2$ or $D^2 \setminus pt$, or equivalently, if it is homotopic (isotopic) to a geodesic.

The curve complex $\mathcal{C}(S)$ is a simplicial complex with
- vertices: isotopy classes of essential simple closed curves
- simplices: collections of essential s.c.c. that can be realized disjointly, or equivalently, their geodesic representatives are disjoint.

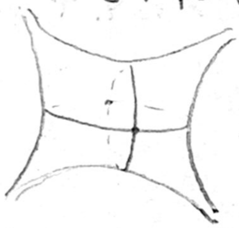
Sporadic surfaces:

$S^2 \setminus 3$ points



$\mathcal{C}(S) := \emptyset$

$S^2 \setminus 4$ points



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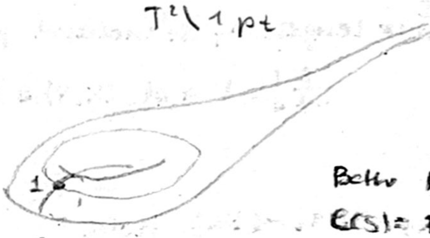
$T^2 \setminus 1$ pt



Better here have $\mathcal{C}(S) =$ Farey graph

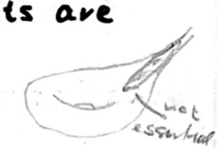
in these examples replace "disjoint" by "intersect minimally" (i.e., 2 or 1)

The curve complex is always connected and has infinite diameter.

Theorem (Mazur-Minsky): $\mathcal{C}(S)$ is $\delta$-hyperbolic.

## The arc complex

Assume $S$ has at least one puncture. An arc is either embedded, or if the endpoints are the same, it is embedded otherwise. An arc is essential if either its endpoints are distinct, and if they are the same it should not be contractible.




The arc complex $\mathcal{A}(S)$ consists of
- vertices: isotopy classes of essential arcs ($\Leftrightarrow$ realized by a geodesic arc).
- simplices: can be realized disjointly (except at endpoints).

Theorem (Mazur-Schleimer): $\mathcal{A}(S)$ is $\delta$-hyperbolic. (same for all $S$)

The main tool is the following.

Lemma (Guessing geodesics): Let $X$ be a graph, and assume that for some $D \geq 0$ for all vertices $x, y \in X$ there exists a subgraph $P(x,y)$ s.t.
i) $P(x,y)$ is connected and contains $x, y$;
ii) If $d(x,y) = 1$, then ~~P(x,y)~~ $\operatorname{diam} P(x,y) \leq D$;
iii) for all $x, y, z$ $\quad P(x,z) \subseteq N_D(P(x,y) \cup P(y,z))$.


Proof ~~of~~ that $\mathcal{A}(S)$ is hyperbolic. Let $X = \mathcal{A}(S)^{(1)}$. For $x, y \in X$ a unicorn-arc is the union of an initial segment of $x$ and a terminal segment of $y$ (for fixed orientation), s.t. the union is an arc (i.e. injective).
Define $P(x,y) :=$ set of all unicorn arcs from $x, y$ with all orientations and orders of $\{x,y\}$.
One can check i)–iii) for $D = 1$. $\square$

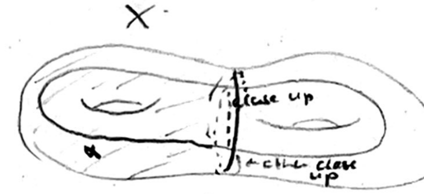
# Subsurface projections

Let $X \subseteq S$ be a subsurface, i.e., connected, $\pi_1$-inj. and not a pair of pants.
There is a partially defined coarse map

$$\mathcal{C}(S) \xrightarrow{\pi_X} \mathcal{C}(X)$$

defined by sending $\alpha$ not disjoint from $X$ to

- $\{\alpha\}$ if $\alpha \subseteq X$;
- if $\alpha \cap X$ is a collection of arcs close them up along $\partial X$ to get the projections. different options




For two subsurfaces $X, Y$ we write $X \pitchfork Y$ if $\partial X \cap Y \neq \emptyset$ and $\partial Y \cap X \neq \emptyset$. Define

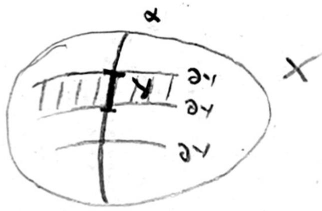
uniformly bounded diameter

$$\pi_X(Y) := \pi_X(\partial Y) \subseteq \mathcal{C}(X).$$

up to uniform distance $\simeq \pi_{\mathcal{C}(Y)}(\mathcal{C}(Y))$ since


**Theorem:** i) Let $X, Y \subseteq S$ be surfaces s.t. $X \pitchfork Y$, and let $\alpha$ be a s.c.c. If $d_X(\alpha, Y) > 50$, then $d_Y(\alpha, X) < 50$. (Behrstock inequality)
ii) If $\alpha, \beta$ are s.c.c.'s in $S$, then $\{Z \mid d_Z(\alpha, \beta) > 50\}$ is finite.
bounded by intersection number

**Proof.** i) Since $d_X(\alpha, Y) > 50$, $\alpha$ and $\partial Y$ intersect a lot. So there is a subarc of $\alpha$ in $Y$ disjoint from $\partial X$. Hence $d_Y(\alpha, X)$ is small.
ii) Exercise. $\square$

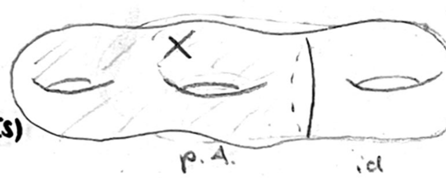



There is an obvious action $Mod(S) \curvearrowright \mathcal{C}(S)$ by $f \cdot \alpha := f(\alpha)$.

**Addendum to Mazur-Minsky:**
$f \in Mod(S)$ is loxodromic iff no s.c.c. is $f$-periodic ($\Leftrightarrow f$ is pseudo-Anosov).

**Proposition:** If $\mathcal{Y}$ is a collection of subsurfaces of $S$ that are pairwise $\pitchfork$, then (P1) – (P5) hold.

**Example:** Let $X \subseteq S$ be a subsurface s.t. for all $f \in Mod(S)$ $f(X) = X$ or $f(X) \pitchfork X$. Consider $\mathcal{Y} := \{f(X) \mid f \in Mod(S)\}$.
Take $f \in Mod(S)$ s.t. $f = $ p.A. on $X$, $f = id$ on $S \setminus X$. So $f \in Mod(S) \curvearrowright \mathcal{C}(S)$ is not loxodromic since $f$ is not p.A. on $S$. But $f \curvearrowright \mathcal{C}_K(\mathcal{Y})$ is loxodromic since it has an axis in $\mathcal{C}(X) \overset{\text{isometric}}{\subseteq} \mathcal{C}_K(\mathcal{Y})$ because $f \curvearrowright \mathcal{C}(X)$ is p.A.




**Theorem:** The asymptotic dimension of $Mod(S)$ is finite.

i.e. subsurfaces from same colour intersect $\pitchfork$

**Outline of proof.** Step 1: There is a finite equivariant partition of the set of subsurfaces of $S$ into pairwise $\pitchfork$ families. ~~For the~~ constructed this out of a coloring of s.c.c.'s s.t. curves of the same color intersect (obtain this by looking at $H_1(S; \mathbb{Z}/2)$) ↳ also go to all 2-covers
Step 2: By a thm of Bell-Fujiwara asdim$(\mathcal{C}) < \infty$. (One can copy the proof of Gromov's thm that asdim(hyp. group) $< \infty$ when replacing 'geodesits' by 'tight geodesics'.)
number of colours
Step 3: For every color construct the quasi-tree of curve complexes $Y_1, \dots, Y_m$. $Y_i = \mathcal{C}_K(\mathcal{Y}_i)$. Each $Y_i$ has finite asdim by Bell-Fujiwara and the properties of $\mathcal{C}_K(\mathcal{Y})$. So $Y_1 \times \dots \times Y_m$ has finite asdim.
coming from the action $Mod(S) \curvearrowright Y_1 \times \dots \times Y_m$
Step 4: The orbit map $Mod(S) \longrightarrow Y_1 \times \dots \times Y_m$ is a q-i-embedding. This follows from the Masur-Minsky distance formula: Let $\{\alpha, \beta\}$ be a finite collection of s.c.c. that fill $S$. Then for all $f, g \in Mod(S)$ $\quad d(f,g) \asymp \sum_{X \subseteq S} [[ d_X(f(\alpha \cup \beta), g(\alpha \cup \beta)) ]]_K$, where $[[x]]_K = \begin{cases} 0, & x < K \\ x, & x \geq K \end{cases}$. $\square$