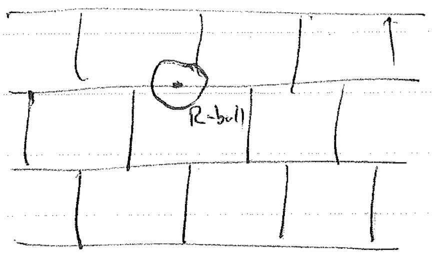
~~Asymptotic dimension of metric spaces~~

Asymptotic dimension

def Let X be a compact metric space. Then ~~[illegible]~~ we say $\dim X \leq n$ if for any $\varepsilon > 0$ there is an open cover $\mathcal{U}$ of X with multiplicity at most $n+1$ and $\operatorname{mesh} \mathcal{U} < \varepsilon$

def: Let X be a metric space. We say asdim $X \leq n$ if for every $R > 0$ there is an (open) cover $\mathcal{U}$ of X satisfying:
(i) every metric R-ball intersects at most $n+1$ elements of $\mathcal{U}$
(ii) $\operatorname{mesh} \mathcal{U} < \infty$

eg
• asdim $\mathbb{R}^2 \leq 2$:



take a 'brick' tiling with bricks much bigger than R

asdim $\mathbb{R}^n \leq n$ in a similar way
• asdim $\mathbb{R}^n = n$
we have the upper bound, so we just need the lower one
Suppose $\mathcal{U}$ is an open cover of $\mathbb{R}^n$ by uniformly bdd sets.
we need to show that the multiplicity is at least $n+1$
let N be the nerve of $\mathcal{U}$
There is a map $f: \mathbb{R}^n \to N$ induced by a partition of unity subordinate to $\mathcal{U}$ (i.e. we get barycentric coordinates for f)
there is also a map $g: N \to \mathbb{R}^n$ that sends $U \in \mathcal{U}$ to a point in U, and extends linearly along simplices
$\mathcal{U}$ is uniformly bounded and gf is uniformly close to the identity
Then we have the sequence on locally finite homology:

$$\begin{tikzcd} H_n^{lf}(\mathbb{R}^n) \arrow[r, "f_*"] \arrow[rr, bend right, "g_* f_*"'] & H_n^{lf}(N) \arrow[r, "g_*"] & H_n^{lf}(\mathbb{R}^n) \end{tikzcd}$$

and $g_* f_*$ is nontrivial, $H_n^{lf}(\mathbb{R}^n) \neq 0$, so $H_n^{lf}(N) \neq 0$
this means $\dim N \geq n$

This is pretty much the only method we have for showing lower bounds on asymptotic dimension.

If $X \overset{q.i}{\sim} Y$, then $\operatorname{asdim} X = \operatorname{asdim} Y$
(all you need is the cobounded inclusion and bijection sending unif. bdd collections to unif. bdd collections (i.e. coarse equivalence))

Fact
f.g. groups have a well defined asymptotic dimension (in fact even countable discrete groups)

eg
• $\operatorname{asdim} \mathbb{Z}^n = n$

Monotonicity holds for asdim: if $H \leq G$, $\operatorname{asdim} H \leq \operatorname{asdim} G$

eg
$\operatorname{asdim}(\ell^2) = \infty$ (it has $\mathbb{R}^n$ for every n as a subsp)
$\operatorname{asdim}(\text{Thompson's gp}) = \infty$ (as it contains $\mathbb{Z}^n$ for all n)

ex
bdd metric spaces have asdim 0
(but the converse is not true!)

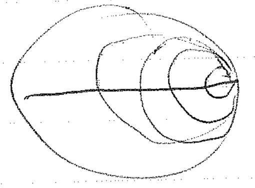
Spaces with an 'archipelago structure' have asdim 0 in general.
These are spaces with a sequence of ~~[illegible]~~ partitions $P_1, P_2, \dots$
such that
- mesh $P_i < \infty$
- the distance between distinct islands in P_n is $\geq a_n$, $a_n \to \infty$

eg
- $\mathbb{Q}$ with the standard metric coming from $\mathbb{R}$ has asdim $= 1$
- $\mathbb{Q}$ with the p-adic ~~metric~~ (ultra)metric has an archipelago structure (indeed as does any ultrametric space), so has asdim $= 0$
  (for example, take the sequence of n-balls as P_n, for $n \to \infty$)

Hurewicz theorem
Let X, Y be compact metric spaces, $f: X \to Y$ a map with $\dim(f^{-1}(y)) \leq n$ for each $y \in Y$.
Then $\dim X \leq \dim Y + n$

Bell-Dranishnikov-~~Hurewicz~~ Hurewicz theorem
Let X, Y be metric spaces, $f: X \to Y$ Lipschitz
Then for all $R \geq 0$ the collection $\{f^{-1}(B_R) \mid B_R \subseteq Y$ is an R-ball$\}$
has asdim $\leq n$ uniformly (the mesh has to be uniformly bdd for each ball)
Then asdim $X \leq$ asdim $Y + n$

eg
- If $1 \to A \to B \to C \to 1$ is a ses, then
  asdim $B \leq$ asdim A + asdim C
- for the hyperbolic plane,



define a projection onto a line via horocycles
this is a Lipschitz map
so asdim $\mathbb{H}^2 \leq 2$
(in fact it is equal to 2)

eg

$G = KAN$, $X = G/K = AN \to A$

(this is really the same as the last example)

Thus asdim X = dim X (we get the lower bound through the same algebraic topological argument)

eg

asdim$(T) \leq 1$ where T is a tree

Let $h: T \to [0, \infty)$ be the map that gives the distance from a marked vertex

We can show that the collection of sets of preimages $\{h^{-1}([a,b]) \mid b-a = R\}$ of R-intervals has a uniform archipelago structure.

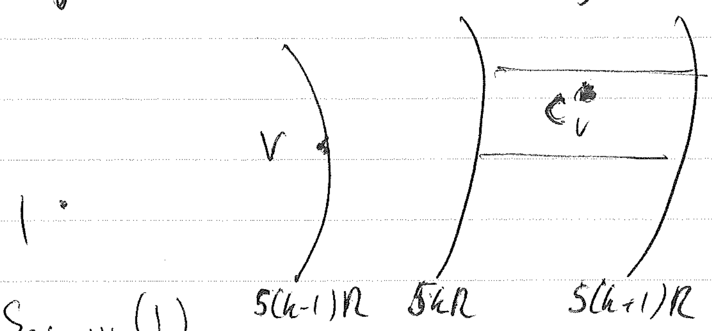
Theorem (Gromov)

Let G be a hyperbolic group. Then asdim $G < \infty$

Proof

Pick R big and an integer (a lot bigger than δ)

Look at large balls about the identity

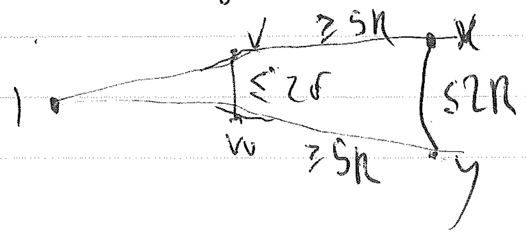


Pick $v \in S_{5(k-1)R}(1)$

C_v^k is the set of points $g \in G$ with $5kR \leq |g| \leq 5(k+1)R$ and there is a geodesic $[1, g]$ containing v.

We try to count how many of these sets a ball of radius R intersects.

Pick $x \in C_v$, $y \in C_w$



v, w must be close because xv and yw are $\geq 5R$ and R is really big

Altogether, this implies that $\operatorname{asdim} G \leq 2|B_{2R}(1)| - 1$
(for a free group $\delta = 0$ so we recover $\operatorname{asdim} G \leq 1$) $\square$

In the next episode we talk about projection complexes, then mapping class groups and the curve complex, subsurface projections.

At the end we prove Bromberg-Fujiwara's theorem that mapping class groups have finite asymptotic dimension.