Mladen
Bestvina

Projection complexes + mapping class groups   18/06/22

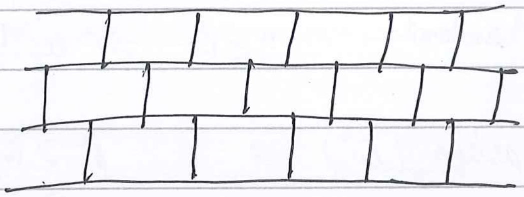
Asymptotic dimension

Motivation: X compact metric space. dim $X \leq n \iff \forall \varepsilon > 0$ $\exists$ open cover $\mathcal{U}$ of X with multiplicity $\leq n+1$.
(mesh $\mathcal{U} < \varepsilon$)   intersection of n+2 sets is $\emptyset$
sup diam U
$U \in \mathcal{U}$

(covering / Lebesgue dim)

Def X metric space. asdim $X \leq n \iff \forall R > 0$ $\exists$ (open) cover $\mathcal{U}$ s.t. 1) every metric R-ball intersects $\leq n+1$ elements of $\mathcal{U}$.
2) mesh $\mathcal{U} < \infty$

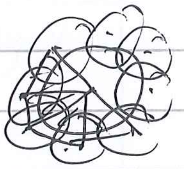
Ex asdim $\mathbb{R}^2 \leq 2$



Also asdim $\mathbb{R}^n \leq n$ by a similar argument

Fact asdim $\mathbb{R}^n = n$.

Suppose $\mathcal{U}$ is an open cover of $\mathbb{R}^n$ by uniformly bounded sets. Need to show multiplicity is $\geq n+1$.



Let N = nerve of the cover
(points in elements, edges if intersection, fill in triangles)

$\exists$ map $f: \mathbb{R}^n \to N$ induced by a partition of unity subordinate to $\mathcal{U}$
gives barycentric coordinates of f.

as a vertex of N
$\exists$ map $g: N \to \mathbb{R}^n$, $g(U) \subseteq U$, linear on simplices.

$\mathcal{U}$ uniformly bounded $\Rightarrow$ $\underset{\text{bndd}}{gf \sim 1}$

locally finite

$$\begin{tikzcd} 0 \neq H_n^{lf}(\mathbb{R}^n) \arrow[r] \arrow[rr, bend right, "id"] & H_n^{lf}(N) \arrow[r] & H_n^{lf}(\mathbb{R}^n) \end{tikzcd}$$

$\Rightarrow H_n^{lf}(N) \neq 0 \Rightarrow \dim N \geq n$.

If $X \overset{QI}{\sim} Y \Rightarrow \text{asdim} X = \text{asdim} Y$

generated by: biLipschitz homeos + cobounded inclusions
$\downarrow$
can replace by injections that send uniformly bounded collections on one side to the same on the other

Ex f.g. groups have well defined asdim

Ex asdim $\mathbb{Z}^n = n$

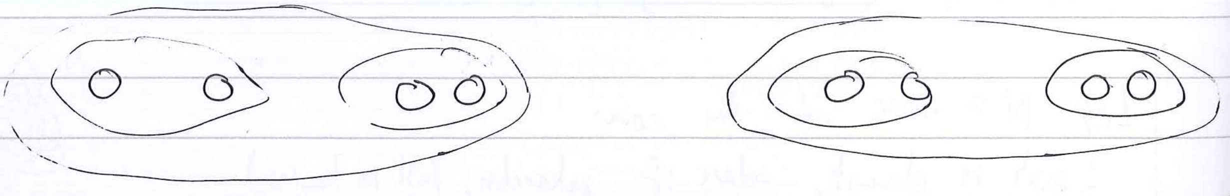
Monotonicity $H < G \Rightarrow \text{asdim } H \leq \text{asdim } G$

asdim $(\ell^2) = \infty$ $\ell^2 \supset \mathbb{R}^n \ \forall n$

Thompson's group has asdim $= \infty$ $(\supset \mathbb{Z}^n \ \forall n)$

Ex Bounded metric spaces has asdim $= 0$

More generally, spaces with the "archipelago structure" have asdim $= 0$



i.e. $\exists$ sequence of partitions of X, $P_1, P_2, P_3 \dots$
- mesh $P_n < \infty$
- the distance between distinct islands in P_n is $\geq a_n$, $a_n \to \infty$.

Ex $\mathbb{Q}$ with the standard norm (as $\subset \mathbb{R}^1$) asdim $\mathbb{Q} = 1$.

p-adic norm $\|p^n \frac{a}{b}\|_p = p^{-n}$, $p \nmid a,b$

ultra norm $\|\alpha + \beta\| \leq \max\{\|\alpha\|, \|\beta\|\}$.

Any ultra - metric space has an archipelago structure.

R-balls are disjoint > R apart

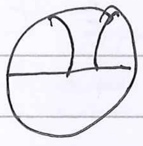
P_n = collection of n-balls in X   asdim $\mathcal{P}_p = 0$.

H ? Thm   X, Y compact metric, $f: X \to Y$ map, $\dim f^{-1}(y) \leq n$
$\forall y \in Y$ $\Rightarrow$ $\dim X \leq \dim Y + n$.

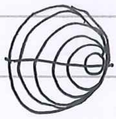
Bell - Dra ? H? Thm X, Y metric spaces, $f: X \to Y$ biLipschitz
if $\forall R$ the collection $\{f^{-1}(B_R) \mid B_R$ R-ball in Y$\}$ has asdim $\leq n$
uniformly $\Rightarrow$ asdim X $\leq$ asdim Y + n.

Ex $1 \to A \to B \to C \to 1$ ses $\Rightarrow$ asdim B $\leq$ asdim A + asdim C

Ex



H^2 nearest point projection doesn't work

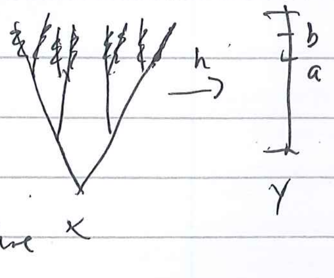


instead use horocycles

Ex $G = KAN$, $X = \frac{G}{K} = AN \to A$ $\Rightarrow$ asdim X = dim Y

Ex   asdim (tree T) ≤ 1

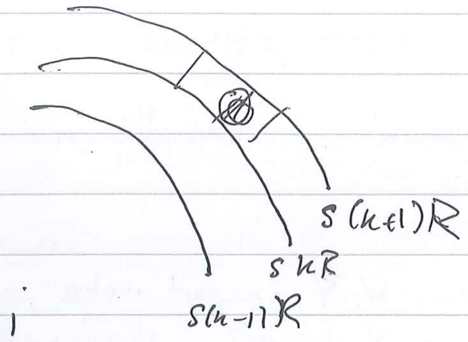
$h: T \to [0, \infty)$ dist from a vertex.



$\{h^{-1}[a,b] \mid b-a = R\}$
has a uniform
archipelago structure

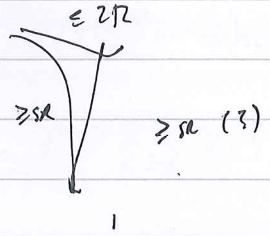
Gromov's thm G word hyp group $\Rightarrow$ asdim $G < \infty$.

R large integer



For $v \in S(s(k-1)R, 1)$ Define $G_v = \{g \in G \mid \|g\| \in [skR, s(k+1)R]$

and $\exists$ geodesic $[1, g] \ni v$



$\Rightarrow d(v, w) \leq 2\delta)$

$\Rightarrow$ asdim $G \leq 2 \# B(2\delta, 1) - 1$

Projection complex (2 lectures)

MCG, curve complex, subsurface projection

Thm (BBF) asdim(MCG) $< \infty$.